Centre Number

Student Number

CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NEW SOUTH WALES

2004
TRIAL HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics

Morning Session
Monday 9 August 2004

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- ~~Write using blue or black pen~~
- Board-approved calculators may be used.
- A table of standard integrals is provided separately
- All necessary working should be shown in every question
- Write your Centre Number and Student Number at the top of this page

Total marks – 120

- Attempt Questions 1-10
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet.

Question 1 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) If $x^5 = 5000$, find x correct to 3 significant figures. **2**

(b) Express $0.3 + 0.\dot{3}$ in the form $\frac{a}{b}$, where a and b are integers. **2**

(c) Solve $\tan\alpha = 3$, for $0^\circ \le \alpha \le 360^\circ$, giving the answers to the nearest degree. **2**

(d) Simplify: $1 - \dfrac{a-b}{a+b}$ **2**

(e) Solve: $8^x = 32$, leaving the answer as a fraction. **2**

(f) Find the integers a and b such that $\dfrac{1}{2-\sqrt{3}} = a + b\sqrt{3}$ **2**

Question 2 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Write down the derivatives of:

(i) $(3x+4)^7$ **2**

(ii) x^3e^x **2**

(iii) $\dfrac{\tan 5x}{5x}$ **2**

(b) (i) Write down the primitive function of $e^{3x}+\sqrt{x}$ **2**

(ii) Find the exact value of $\int_1^2 \dfrac{x^4+1}{x}dx$ **2**

(iii) Given that $\dfrac{dy}{dx}=2x-\sin x$ and $y=2$ when $x=0$, find y in terms of x. **2**

Question 3 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) For what values of a, will ax^2+5x+a be positive definite? **3**

(b) Find the values of k if $\int_1^k (x+1)dx=6$ **2**

~~(c) The points A, B and C have co-ordinates $(1,5)$, $(6,0)$ and $(5,7)$ respectively.~~
Plot these points on a number plane. Hence:

(i) Show that the length of AB is $5\sqrt{2}$. **1**

(ii) Show that the triangle ABC is isosceles by finding the length of BC. **1**

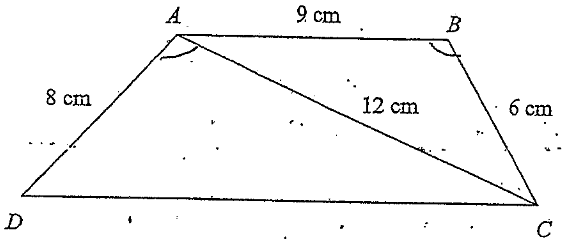
(iii) Find the equation of the line AB. **2**

(iv) BA is produced to meet the line $y=7$ at P; show that P has co-ordinates $(-1,7)$. **1**

(v) Find the area of triangle PAC. **2**

Question 4 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) In the diagram below, $AB = 9$ cm, $BC = 6$ cm, $AD = 8$ cm, $AC = 12$ cm and $\angle ABC = \angle DAC$



NOT TO SCALE

(i) Prove $\Delta ABC \parallel\!| \Delta CAD$, giving clear reasons. 3

(ii) Hence, find the value of side CD. 2

(b) A parabola whose equation is $y = ax^2$, where a is a constant, has the line $y = 12x + 3$ as a tangent.

(i) By equating the two given equations, find a quadratic equation in terms of x and a. 1

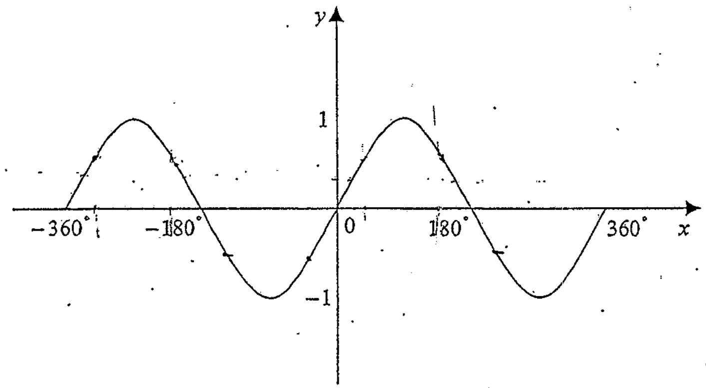
(ii) By using the discriminant of the quadratic equation found, find the value of a. 2

(iii) Find the coordinates of the point of contact between the tangent and the parabola. 2

(iv) Sketch the parabola and the tangent line, showing the co-ordinates of the point of contact. 2

Question 5 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Below is the graph of $y = \sin x$, for $-360^\circ \leq x \leq 360^\circ$.



NOT TO SCALE

One solution of the equation $\sin x = 0.5$ is $x = -210^\circ$. 3
Find the other solutions of this equation for $-360^\circ \leq x \leq 360^\circ$.

(b) (i) On the same graph, sketch the curves $y_1 = 2\sin x$ and $y_2 = -\sin 2x$, for $0 \leq x \leq 2\pi$. 2

(ii) Give three solutions to the equation $2\sin x + \sin 2x = 0$, for $0 \leq x \leq 2\pi$ 1

(c) If α and β are the roots of the quadratic equation $x^2 + 5x - 2 = 0$, find the value of:

(i) $\alpha + \beta$ 1

(ii) $\alpha\beta$ 1

(iii) $\alpha^2 + \beta^2$. 1

(d) Consider the series $\log 3 + \log 6 + \log 12 + \ldots$

(i) Explain why $\log 3 + \log 6 + \log 12 + \ldots$ are the first three terms of an arithmetic series. 1

(ii) Find the ninth term, expressing your answer in the form $\log k$. 2

Question 6 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) (i) The curve $y = x^3 + ax^2 + 7x - 5$ has a stationary point at $x = 1$. Find the value of a, and hence: **2**

(ii) Find the coordinates of all stationary points. **2**

(iii) Determine the nature of the stationary points. **2**

(iv) Sketch the curve and then determine for what values of x the curve is increasing. **2**

(b) Consider the geometric series $\frac{6}{\sqrt{3}} + 2 + \frac{2}{\sqrt{3}} + \ldots$

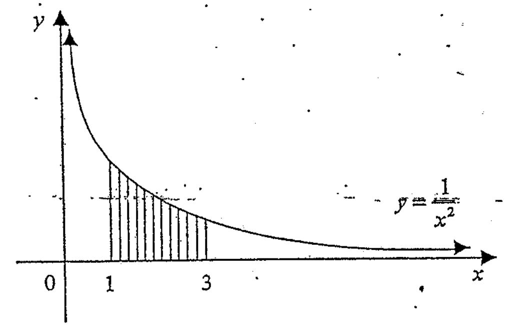
(i) Find the common ratio. **1**

(ii) Explain why the series has a limiting sum. **1**

(iii) Find the exact value of the limiting sum. Write your answer with a rational denominator. **2**

Question 7 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a)



NOT TO SCALE

The diagram above shows the area bounded by the graph $y = \frac{1}{x^2}$, (for $x > 0$), the x – axis and the lines $x = 1$ and $x = 3$.

(i) Find the shaded area. Leave your answer as a fraction. **2**

(ii) Find the volume of the solid formed when the shaded area is rotated about x – axis. Leave your answer in exact form. **3**

(b) (i) Sketch the graph of $f(x) = e^x$ for all values of x in the domain and state its range. **2**

(ii) The curve $f(x) = e^x$ is rotated about the y – axis to give a solid. Show that the volume V_y of the solid formed, from $y = 3$ to $y = 5$, is given by $V_y = \pi \int_3^5 (\ln y)^2 \, dy$. **2**

(iii) Use Simpson's rule with 5 function values to find the volume of this solid, correct to 2 significant figures. **3**

Question 8 (12 marks) Use a SEPARATE writing booklet. **Marks**

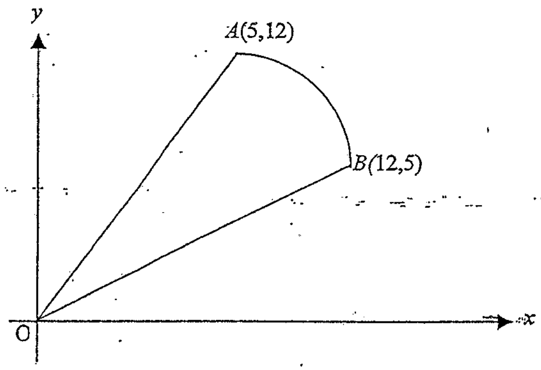
(a) A certain soccer team has a probability of 0.6 of winning a match and a probability of 0.3 of drawing a match.

(i) If this soccer team plays two matches, draw a tree diagram to show all possible outcomes. **2**

(ii) Find the probability of this soccer team winning at least one match out of the two matches. **2**

(iii) Find the probability of this soccer team not winning either of the two matches. **2**

(b)



NOT TO SCALE

The figure above shows a sector of a circle OAB, centre O, with its arc joining the points A (5,12) and B(12,5). Copy this figure into your answer booklet.

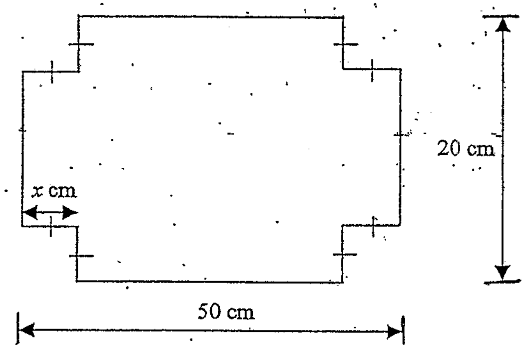
(i) Find the value, in degrees, of one radian, correct to the nearest minute. **1**

(ii) Show that the size of $\angle AOB$ is 0.78 radians, correct to 2 decimal places. **3**

(iii) Calculate the perimeter of sector OAB, correct to 2 decimal places. **2**

Question 9 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) A box is made from a 50 cm by 20 cm rectangle of cardboard by cutting out four equal squares of side x cm from each corner as shown below:



NOT TO SCALE

The edges are turned up to make an open box.

(i) Show that the volume V of this box is given by the equation: **2**

$$V = 4x^3 - 140x^2 + 1000x \text{ (cm}^3\text{)}$$

(ii) Find the value of x, correct to one decimal place, that gives this box its greatest volume. **3**

(iii) Hence, find the maximum volume of this box, correct to 2 decimal places. **1**

(b) Jordan has to pay annual instalments for his superannuation at the beginning of each year according to the formula:

$$M_n = \left(1 + \frac{r}{100}\right) M_{n-1}, \ n \geq 2$$

where $r(\%)$ is the annual rate of interest paid by the fund and M_n is the instalment at the beginning of the n^{th} year.

If the interest rate is 12 % p.a., compounded yearly, and Jordan's first instalment is \$500, find:

(i) How much is his second instalment? **2**

(ii) Find the amount Jordan has to pay into the fund at the beginning of the 20^{th} year. **2**

(iii) Find the total value of his investment after 20 years. **2**

Question 10 (12 marks) Use a SEPARATE writing booklet. **Marks**

(a) Consider the function $y = \sin x + \cos x$ in the domain $0 \le x \le 2\pi$.

(i) Find $\dfrac{dy}{dx}$. 1

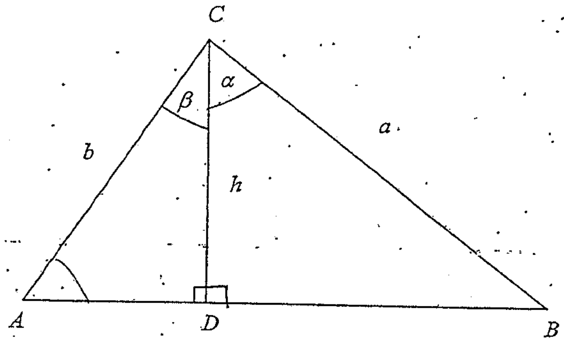
(ii) Find the maximum and minimum values of $\sin x + \cos x$ in the given domain. 3

(iii) Show that the curve cuts the x axis at $x = \dfrac{3\pi}{4}$ and at $x = \dfrac{7\pi}{4}$. 1

(iv) Hence sketch the curve of $y = \sin x + \cos x$ in the domain $0 \le x \le 2\pi$. 1

Question 10 continues on page 12

(b)



The diagram above shows a triangle ABC, and CD is perpendicular to AB. It is given that $BC = a$, $AC = b$, $\angle ACD = \beta$ and $\angle BCD = \alpha$.

(i) By using triangles ACD and BCD, show that $h = b\cos\beta = a\cos\alpha$. 1

(ii) Show that the area of triangle ACD is equal to $\dfrac{1}{2}ab\sin\beta\cos\alpha$ 1

(iii) Find another expression for the area of triangle BCD in terms of a, b, α and β. 1

(iv) Show that the area of triangle ABC is equal to $\dfrac{1}{2}ab\sin(\alpha+\beta)$ 1

(v) Hence, but not otherwise, deduce that: 2

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

End of paper

EXAMINERS

Liviu Spiridon (convenor) LaSalle Catholic College, Bankstown
Magdi Farag LaSalle Catholic College, Bankstown

SUGGESTED SOLUTIONS TO MATHEMATICS CSSA TRIAL 2004

Question 1

(a) $x^5 = 5000 \therefore x = \sqrt[5]{5000} \doteqdot 5.49$

(b) $0.3 + 0.\dot{3} = \frac{3}{10} + \frac{1}{3} = \frac{19}{30}$

(c) $\tan\alpha = 3 \therefore \alpha = 72^\circ$ or $\alpha = 252^\circ$ (to the nearest degree)

(d) $1 - \frac{a-b}{a+b} = \frac{a+b-(a-b)}{a+b} = \frac{2b}{a+b}$

(e) $8^x = 32 \therefore (2^3)^x = 2^5 \therefore 2^{3x} = 2^5 \therefore 3x = 5 \therefore x = \frac{5}{3}$

(f) $\frac{1}{2-\sqrt{3}} = \frac{1}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}} = 2+\sqrt{3} \therefore a = 2$ and $b = 1$

Question 2

(a) (i) $\frac{d}{dx}\left[(3x+4)^7\right] = 7(3x+7)^6 \times 3 = 21(3x+4)^6$

(ii) $\frac{d}{dx}\left(x^3e^x\right) = x^3e^x + 3x^2e^x = x^2e^x(x+3)$

(iii) $\frac{d}{dx}\left(\frac{\tan 5x}{5x}\right) = \frac{5\sec^2 5x \times 5x - \tan 5x \times 5}{25x^2} = \frac{5x\sec^2 5x - \tan 5x}{5x^2}$

(b) (i) $\int\left(e^{3x} + \sqrt{x}\right)dx = \frac{e^{3x}}{3} + \frac{2}{3}x^{\frac{3}{2}} + c$

(ii) $\int_1^2 \frac{x^4+1}{x}dx = \int_1^2\left(x^3 + \frac{1}{x}\right)dx = \left[\frac{x^4}{4} + \ln x\right]_1^2 = \left(\frac{2^4}{4} + \ln 2\right) - \left(\frac{1}{4} + \ln 1\right) = 3\frac{3}{4} + \ln 2$

(iii) $\frac{dy}{dx} = 2x - \sin x \therefore y = x^2 + \cos x + c$

when y=2 , x=0 $\therefore$ c = 1 $\therefore y = x^2 + \cos x + 1$

Question 3

(a) $\Delta < 0$ and $a > 0$

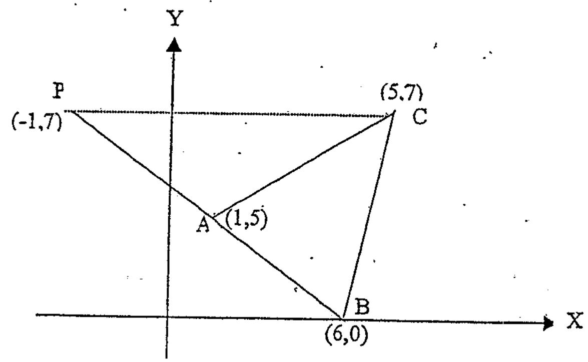
$\therefore 25 - 4a^2 < 0$ i.e. $(5-2a)(5+2a) < 0$

$a > \frac{5}{2}$ or $a < -\frac{5}{2}$, and for positive definite $\therefore a > \frac{5}{2}$

(b) $\int_1^k (x+1)dx = 6 \quad \therefore \left[\frac{x^2}{2} + x\right]_1^k = 6 \quad \therefore \left[\frac{k^2}{2} + k\right] - \left[\frac{1}{2} + 1\right] = 6$

$\therefore k^2 + 2k - 15 = 0 \therefore (k+5)(k-3) = 0 \therefore k = 3$ or $k = -5$

(c)



(i) AB= $\sqrt{(5-0)^2 + (1-6)^2} = \sqrt{50} = 5\sqrt{2}$

(ii) BC == $\sqrt{(7-0)^2 + (5-6)^2} = \sqrt{50} = 5\sqrt{2}$ $\therefore \Delta$ ABC is isosceles.

(iii) Gradient of AB = $\frac{5-0}{1-6} = -1$

$\therefore$ equation of line AB is $y - 0 = -1(x-6) \therefore x + y = 6$ (1)

(iv) Substitute y=7 into (1) $\therefore x = -1$ $\therefore$ P is (- 1,7)

(v) PC = 5+1=6 units and the perpendicular distance from A to PC = 7-5=2 units.

$\therefore$ Area of Δ PAC = ½ × 6 × 2 = 6 units^2

Question 4

(a) (i) In Δ's ABC and CAD:

$\frac{AB}{AC}=\frac{9}{12}=\frac{3}{4}$ and $\frac{BC}{AD}=\frac{6}{8}=\frac{3}{4}$ and $\angle ABC=\angle DAC$ (Given)

$\therefore \Delta ABC ||| \Delta CAD$ (two pairs of corresponding sides are proportional and their included angles are equal.)

(ii) Since the two triangles are similar

$\therefore \frac{AB}{AC}=\frac{AC}{CD}$ (corresponding sides are proportional)

$\therefore \frac{9}{12}=\frac{12}{CD}$ $\therefore CD=16$ cm.

(b) (i) $y=ax^2$ and $y=12x+3$

$ax^2=12x+3 \therefore ax^2-12x-3=0$ (*)

(ii) Since the line is a tangent to the parabola (one point of contact) ∴ the roots are equal.

$\therefore \Delta=0 \therefore (-12)^2-4\times a\times(-3)=0 \therefore a=-12$

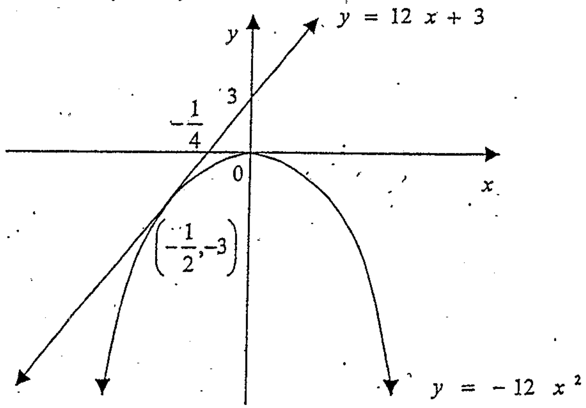
(iii) To find the point of contact, substitute $a = -12$ into equation (*)

$-12x^2-12x-3=0 \therefore 4x^2+4x+1=0 \therefore (2x+1)^2=0$

$x=-\frac{1}{2} \therefore y=12\times\left(-\frac{1}{2}\right)+3 \therefore y=-3$

∴ the point of contact is $\left(-\frac{1}{2},-3\right)$

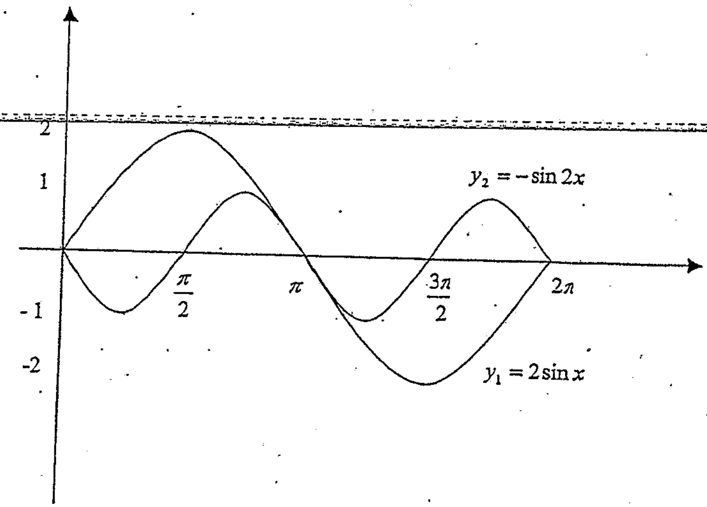
(iv)



Question 5

(a) Base angle is 30° ∴ angles are: $30^\circ, 180^\circ-30^\circ=150^\circ, -210^\circ, -360^\circ+30^\circ=-330^\circ$

(b) (i)



(ii) 0, π and 2π.

(c) (i) $M=M_0e^{-kt}$ $\therefore \frac{dM}{dt}=-kM_0e^{-kt}$ $\therefore \frac{dM}{dt}=-kM$

(ii) (α) $80=100\,e^{-20t}$ $\therefore 0.8=e^{-20k}$ $\therefore k=\frac{\ln 0.8}{-20}=0.011157$

(β) $M=100\,e^{-30\times0.011157}=72$ grams (to the nearest gram)

(iii) $50=100e^{-0.011157t}$ $\therefore t=\frac{\ln 0.5}{-0.011157}=62$ hours (to the nearest hour)

Question 6

(a) (i) $y=x^3+ax^2+7x-5 \therefore \frac{dy}{dx}=3x^2+2ax+7$

At $x=1$, $\frac{dy}{dx}=0 \therefore 3+2a+7=0 \therefore a=-5$

(ii) $y=x^3-5x^2+7x-5 \therefore \frac{dy}{dx}=3x^2-10x+7$

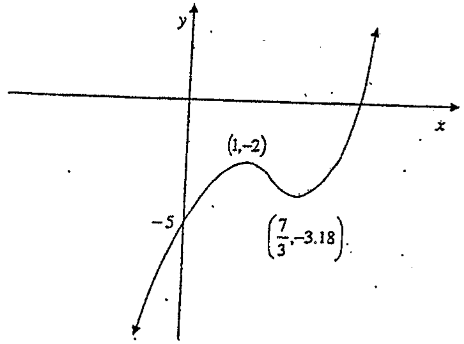
$\therefore \frac{dy}{dx}=0 \therefore (x-1)(3x-7)=0 \therefore x=1$ or $x=\frac{7}{3}$

$\therefore$ stationary points are: $(1,-2)$ and $\left(\frac{7}{3},-3.18\right)$

(iii) $\therefore \frac{d^2y}{dx^2}=6x-10$, for $x=1 \therefore \frac{d^2y}{dx^2}=-4<0 \therefore (1,-2)$ is a local max.

for $x=\frac{7}{3} \therefore \frac{d^2y}{dx^2}=4>0 \therefore \left(\frac{7}{3},-3.18\right)$ is a local min.

(iv)



y = f(x) is increasing for $x<1$ or $x>\frac{7}{3}$.

(b) (i) Since $T_n=a+(n-1)d \therefore T_K=a+(K-1)d \therefore L=a+(K-1)d$ (1)

(ii) Similarly $T_L=a+(L-1)d \therefore K=a+(L-1)d$ (2)

(iii) (1) − (2) $\therefore L-K=(K-L)d \therefore d=-1$

(iv) Substitute $d=-1$ into equation (1)

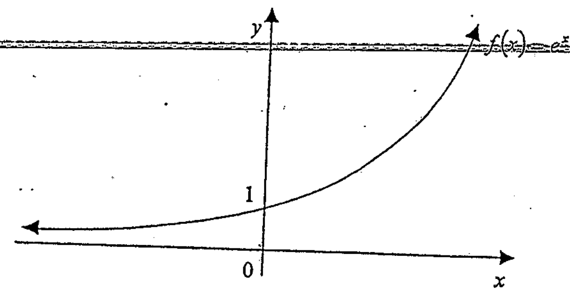
$\therefore L=a+(k-1)(-1) \therefore L=a-K+1 \therefore a=L+K-1$

Question 7

(a) (i) $A=\int_1^3 \frac{1}{x^2}dx=\left[-\frac{1}{x}\right]_1^3=-\frac{1}{3}+1=\frac{2}{3}$ square units

(ii) $V_x=\pi\int_1^3 y^2dx=\pi\int_1^3 \frac{1}{x^4}dx=\pi\int_1^3 x^{-4}dx=\frac{\pi}{-3}\left[x^{-3}\right]_1^3=\frac{26}{81}\pi$ cubic units

(b) (i)



Range: $\{y: y>0\}$

(ii) The volume of the solid obtained by the rotation of the curve y=f(x) about y – axis between $y=3$ and $y=5$ is given by:

$V_y=\pi\int_3^5 x^2dy$, and making x the subject from $y=e^x \therefore x=\ln y$

$\therefore V_y=\pi\int_3^5 (\ln y)^2dy$, as required.

(iii) let $f(y)=(\ln y)^2$

y	3	3.5	4	4.5	5
f(y)	1.2069…	1.5694…	1.9218…	2.2622…	2.5902…

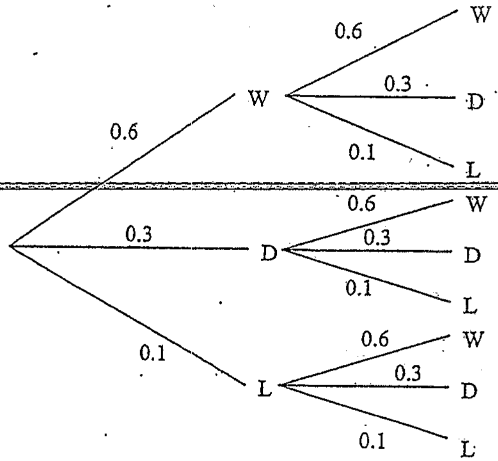
$h=\frac{5-3}{4}=0.5$

$\therefore V_y=\pi\frac{0.5}{3}[1.2069...+4\times(1.5694...+2.2622)+2\times1.9218...+2.5902...]$

$\therefore V_y=12$ cubic units (to 2 sign. fig.)

Question 8

(a) (i)



(ii) P(winning at least one match) = [P(WW) + P(WD) + P(WL)] + P(DW) + P(LW) = [0.36 + 0.18 + 0.06] + 0.18 + 0.06 = 0.6 + 0.18 + 0.06 = 0.84

(iii) P(not win either match) = 1 − P(winning at least one match)
= 1 − 0.84 = 0.16

(b)

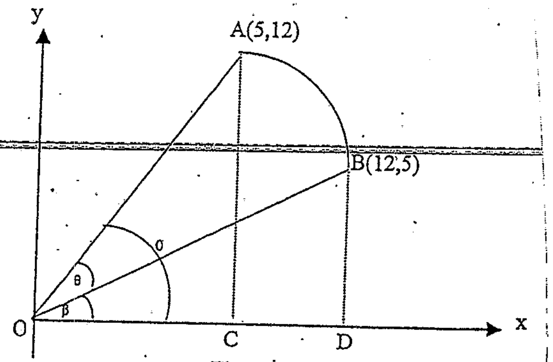


Fig.1

(i) $1 \text{ rad} = \frac{180^\circ}{\pi} = 57^\circ 18'$

(ii) From Δ OAC, $\tan \alpha = \frac{12}{5} \therefore \alpha = 1.12$ radians.

and from ΔOBD, $\tan \beta = \frac{5}{\ }$ $\therefore \beta = 0.34$ ...

$\therefore \theta = 1.116 - 0.335 \doteq 0.78$ rad

$\therefore \angle AOB = 0.78$ rad.

Question 8 (continu

(iii) $OA^2 = OC^2 + AC^2 = 25 + 144 = 169$
$\therefore OA = 13 = OB$
The length of the arc $AB = r\,\theta = 13 \times 0.78 = 10.14$
$\therefore$ The perimeter of sector OAB = 13 + 13 + 10.14 = 36.14

Question 9

(a) (i) $V = (50-2x)(20-2x)x = 4x^3 - 140x^2 + 1000x$ (cm^3)

(ii) $\frac{dV}{dx} = 12x^2 - 280x^2 + 1000 \therefore \frac{dV}{dx} = 0 \therefore x = 4.4$ (cm), correct to one decimal place.

$\frac{d^2V}{dx^2} = 24x - 560x$, and for $x = 4.4$ $\frac{d^2V}{dx^2} = -2359.39 < 0 \therefore$ Volume is maximum.

(iii) For $x = 4.4$... $\therefore V = 2030.34$ cm^3

(b) (i) $M_n = \left(1 + \frac{r}{100}\right) M_{n-1}$ When n = 2

$M_2 = \left(1 + \frac{r}{100}\right) M_1$
$M_2 = 500(1.12)$

$M_2 = \left(1 + \frac{12}{100}\right) 500$
$\therefore M_2 = \$560$

(ii) $M_3 = 1.12 M_2$ $\therefore M_3 = 500(1.12)^2$.
$\therefore M_4 = 500(1.12)^3$.
$\therefore M_5 = 500(1.12)^4$.
....
....
$\therefore M_{20} = 500(1.12)^{19} = \$\ 4306.38$

(iii) The total value is given by:
$500 + 500(1.12) + 500(1.12)^2 + 500(1.12)^3 + \ldots + 500(1.12)^{19}$
$\therefore S_n = \frac{a(r^n - 1)}{r - 1}$ $\therefore S_{20} = \frac{500(1.12^{20} - 1)}{1.12 - 1} = \$36026{,}22$

Question 10

(a) (i) $\frac{dv}{dt} = k \therefore v = \int k.dt \therefore v = kt + c_1$ (1)

(ii) $\frac{dx}{dt} = kt + c_1 \therefore x = \int (kt + c_1)dt \therefore x = \frac{kt^2}{2} + c_1 t + c_2$ (2)
When t = 0, x = 1 ∴ $1 = 0 + 0 + c_2 \therefore c_2 = 1$.
When t = 1, x = 2 ∴ $2 = \frac{1}{2}k + c_1 + 1 \therefore k + 2c_1 = 2$ (3)
When t = 2, x = 9 ∴ $9 = 2k + 2c_1 + 1 \therefore k + c_1 = 4$ (4)
(3) − (4) $\therefore c_1 = -2$ sub. into (3) ∴ k = 6
∴ sub. into eq. (2) $\therefore x = 3t^2 - 2t + 1$

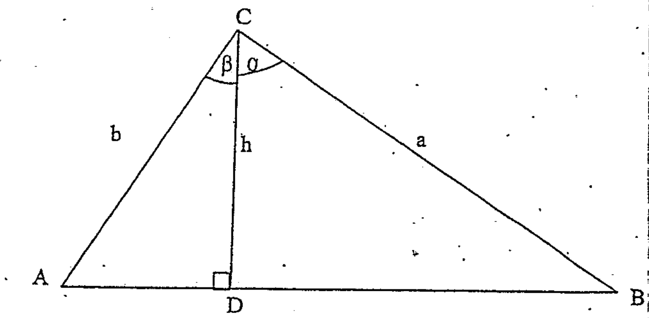
Question 10 (continued)

(iii) The particle at rest when $v=0 \therefore$ from (1)

$v = 6t - 2 \therefore 0 = 6t - 2 \therefore t = \frac{1}{3}$

$\therefore$ the particle come to the rest at $t = \frac{1}{3}$ sec.

(b)



(i) In Δ ADC, $\cos\beta = \frac{h}{b}$ $\qquad \therefore h = b\cos\beta$

In Δ BCD, $\cos\alpha = \frac{h}{a}$ $\qquad \therefore h = a\cos\alpha$

$\therefore h = b\cos\beta = b\cos\alpha$

(ii) Area of Δ ACD $= \frac{1}{2} \times AC \times CD \sin\beta = \frac{1}{2} \times b \times h \sin\beta$

$= \frac{1}{2} \times b \times a\cos\alpha \times \sin\beta$

$= \frac{1}{2} a\, b \sin\beta \cos\alpha$

(iii) Area of Δ BCD $= \frac{1}{2} \times a \times h \sin\alpha$

$= \frac{1}{2} \times a \times b\cos\beta \sin\alpha$

$= \frac{1}{2} a\, b \cos\beta \sin\alpha$

(iv) Area of Δ ACB $= \frac{1}{2} \times AC \times BC \sin(\alpha + \beta)$

$= \frac{1}{2} a\, b \sin(\alpha + \beta)$

(v) Area of Δ ACB = Area of Δ ACD + Area of Δ BCD

~~$\frac{1}{2} a\, b \sin(\alpha + \beta) = \frac{1}{2} a\, b \sin\beta \cos\alpha + \frac{1}{2} a\, b \cos\beta \sin\alpha$~~

$\therefore \sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta.$