CATHOLIC SECONDARY SCHOOLS' ASSOCIATION OF NEW SOUTH WALES

YEAR TWELVE FINAL TESTS 1999

MATHEMATICS

3/4 UNIT

(i.e. 3 UNIT COURSE – ADDITIONAL PAPER:
4 UNIT COURSE – FIRST PAPER)

Afternoon session

Friday 13 August 1999

Time allowed – two hours

EXAMINERS
Graham Arnold, Terra Sancta College, Nirimba
Sandra Hayes, Aquinas College, Menai
Frank Reid, St Ursula's College, Kingsgrove

DIRECTIONS TO CANDIDATES:

- **ALL** questions may be attempted.
- **ALL** questions are of equal value.
- All necessary working should be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- Standard integrals are printed at the end of the exam paper.

Marks

Question 1 Begin a new page

(a) Find the number of ways in which 6 books can be arranged in a row so that the shortest book and the longest book are at the ends. **1**

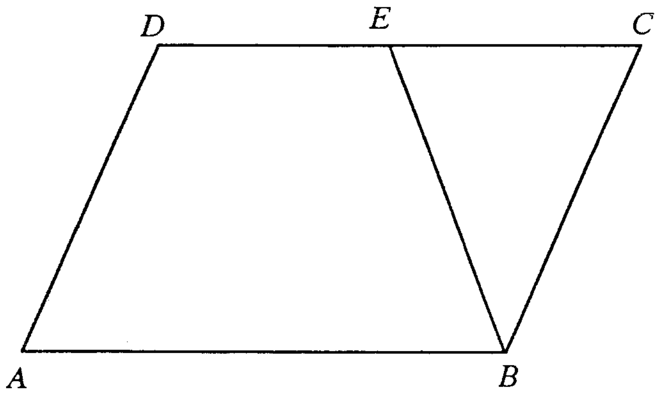
(b) The acute angle between the line $x-2y+3=0$ and the line $y=mx$ is $45°$. **3**

(i) Show that $\left|\dfrac{2m-1}{m+2}\right|=1$

(ii) Find the possible values of m.

(c) Solve the equation $\ln\left(x^3+19\right)=3\ln(x+1)$. **3**

(d) **5**



$ABCD$ is a parallelogram. E is the point on CD such that $BE=BC$.

(i) Copy the diagram showing the above information.

(ii) Show that $ABED$ is a cyclic quadrilateral.

Marks

Question 2 Begin a new page

(a) Find $\lim_{x \to 0} \frac{\sin 2x}{x}$ 1

(b) Solve the inequality $\frac{x^2+9}{x} \le 6$ 3

(c) (i) Factorise $3x^3+3x^2-x-1$ 3

(ii) Solve the equation $3\tan^3\theta+3\tan^2\theta-\tan\theta-1=0$ for $0 \le \theta \le \pi$

(d) $P(2t, t^2)$ is a point on the parabola $x^2=4y$ with focus F. The point M divides the interval FP externally in the ratio $3:1$. 5

(i) Show that as P moves on the parabola $x^2=4y$, then M moves on the parabola $x^2=6y+3$.

(ii) Find the coordinates of the focus and the equation of the directrix of the locus of M.

Question 3 Begin a new page

(a) Find the gradient of the tangent to the curve $y=\tan^{-1}\frac{1}{x}$ at the point on the curve where $x=1$. 2

(b) A function is given by the rule $f(x)=\frac{x+1}{x+2}$. Find the rule for the inverse function $f^{-1}(x)$. 2

(c) At any point on the curve $y=f(x)$ the gradient function is given by $\frac{dy}{dx} = 2\cos^2 x+1$. If $y=\pi$ when $x=\pi$, find the value of y when $x=2\pi$. 4

(d) Use the substitution $x=u^2$, $u>0$, to express the value of $\int_1^{100} \frac{1}{x+2\sqrt{x}}\,dx$ in the form $\ln a$ for some constant $a>0$. 4

Marks

Question 4 Begin a new page

(a) Find the exact value of $\displaystyle\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$. **2**

(b) A particle is moving in a straight line. At time t seconds its displacement x metres from a fixed point O on the line is such that $t = x^2 - 3x + 2$. **2**

(i) Find an expression for its velocity v in terms of x.

(ii) Find an expression for its acceleration a in terms of x.

(c) Consider the function $y = 2\cos^{-1}(1-x)$. **4**

(i) Find the domain and range of the function.

(ii) Sketch the graph of the function.

(d) The radius r kilometres of a circular oil spill at time t hours after it was first observed is given by $r = \dfrac{1+3t}{1+t}$. Find the exact rate of increase of the area of the oil spill when the radius is 2 kilometres. **4**

Question 5 Begin a new page

(a) Consider the function $f(x) = \dfrac{\ln x}{x}$. **6**

(i) Find the coordinates and the nature of the stationary point on the curve $y = f(x)$.

(ii) Explain why $f(\pi) < f(e)$ and hence show that $\pi^e < e^\pi$.

(iii) $P(X, -2)$ is a point on the curve $y = f(x)$. Starting with an initial approximation of $X = 0\cdot5$, use one application of Newton's method to find an improved approximation to the value of X, giving the answer correct to 2 decimal places.

Marks

Question 5 (Cont)

(b) A machine which initially costs \$49 000 loses value at a rate proportional to the difference between its current value \$$M$ and its final scrap value \$1000. After 2 years the value of the machine is \$25 000. **6**

(i) Explain why $\dfrac{dM}{dt} = -k(M-1000)$ for some constant $k>0$, and verify that $M = 1000 + Ae^{-kt}$, A constant, is a solution of this equation.

(ii) Find the exact values of A and k.

(iii) Find the value of the machine, and the time that has elapsed, when the machine is losing value at a rate equal to one quarter of the initial rate at which it loses value.

Question 6 Begin a new page

(a) Show that the coefficients of x^4, x^5, x^6 in the expansion of $(1+x)^{14}$ are consecutive terms in an arithmetic sequence. **2**

(b) A coin is biased so that in any one throw there is a constant probability p (where $p \neq 0\cdot5$) that the coin shows heads. In 6 throws of the coin the probability of 3 heads is twice the probability of 2 heads. Find the value of p. **4**

(c) A particle moving in a straight line is performing Simple Harmonic Motion. At time t seconds its displacement x metres from a fixed point O on the line is given by $x = 2\sin 3t - 2\sqrt{3}\cos 3t$. **6**

(i) Express x in the form $x = R\sin(3t-\alpha)$ for some constants $R>0$ and $0<\alpha<\frac{\pi}{2}$.

(ii) Describe the initial motion of the particle in terms of its initial position, velocity and acceleration.

(iii) Find the exact value of the first time that the particle is 2 metres to the left of O and moving towards O.

Marks

Question 7   Begin a new page

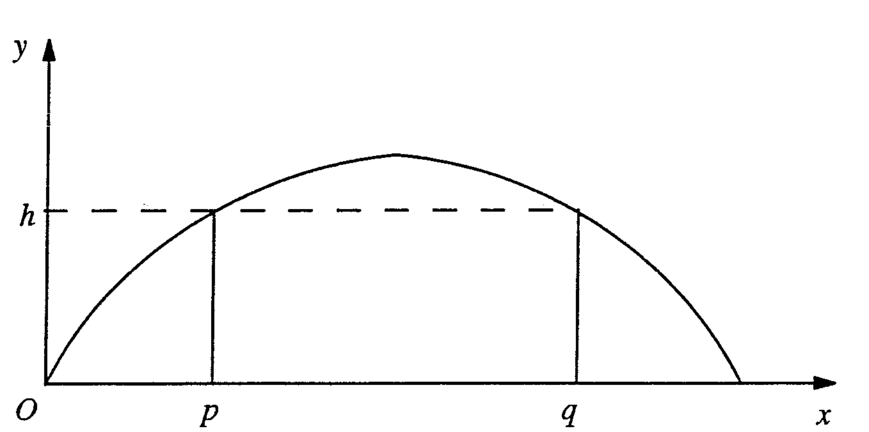
(a) $S_n = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \ldots + \frac{n}{(n+1)!}$, $n = 1, 2, 3, \ldots$ **6**

(i) Use the method of mathematical induction to show that $S_n = 1 - \frac{1}{(n+1)!}$ for all positive integers $n \geq 1$.

(ii) Find the value of $\lim\limits_{n \to \infty} S_n$.

(iii) Find the smallest positive integer n such that $|S_n - 1| < 10^{-6}$.

(b) **6**



A particle is projected with velocity V ms^{-1} from a point O at an angle of elevation α . Axes Ox and Oy are taken horizontally and vertically through O. The particle just clears two vertical chimneys of height h metres at horizontal distances of p metres and q metres from O. The acceleration due to gravity is taken as 10 ms^{-2} and air resistance is ignored.

(i) Write down expressions for the horizontal displacement x and the vertical displacement y of the particle after time t seconds.

(ii) Show that $V^2 = \dfrac{5p^2\left(1 + \tan^2\alpha\right)}{p\tan\alpha - h}$.

(iii) Show that $\tan\alpha = \dfrac{h(p+q)}{pq}$.

QUESTION 1

(a) $2!$ (shortest and longest) $\times\, 4!$ (others)

$= 2 \times 24 = \underline{\underline{48 \text{ ways}}}$.

(b) $x - 2y + 3 = 0$, $2y = x + 3$, $y = \frac{x}{2} + \frac{3}{2}$ gradient $\frac{1}{2}$

$y = mx$ gradient m

(i) $\left| \dfrac{m - (\frac{1}{2})}{1 + m(\frac{1}{2})} \right| = \tan 45^\circ \quad \therefore \left| \dfrac{(2m-1)/2}{(2+m)/2} \right| = 1$

$\therefore \left| \dfrac{2m - 1}{m + 2} \right| = 1$.

(ii) $\dfrac{2m-1}{m+2} = -1$, $2m - 1 = -m - 2$, $3m = -1$, $\underline{\underline{m = -\frac{1}{3}}}$

or $\dfrac{2m-1}{m+2} = 1$, $2m - 1 = m + 2$, $\underline{\underline{m = 3}}$

(c) $\ln(x^3 + 19) = 3\ln(x+1)$

$\therefore \ln(x^3 + 19) = \ln(x+1)^3$

$\therefore \ln(x^3 + 19) = \ln(x^3 + 3x^2 + 3x + 1)$

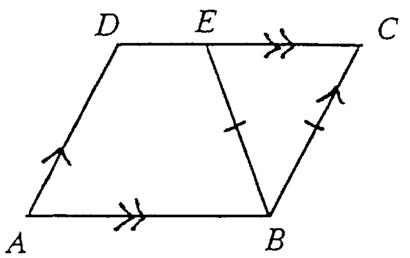
$\therefore x^3 + 19 = x^3 + 3x^2 + 3x + 1$

$\therefore 3x^2 + 3x - 18 = 0 \quad \therefore x^2 + x - 6 = 0$

$\therefore (x+3)(x-2) = 0 \quad \therefore \cancel{x = -3}$ or $\underline{\underline{x = 2}}$

($x \neq -3$ since $\ln((-3)^3 + 19)$, $\ln((-3)+1)$ are not defined)

(d) (i)



(ii) $\angle BEC = \angle BCE$ (in ΔBEC equal angles are opposite equal sides BC and BE)

$\angle BCE = \angle BAD$ (opposite angles are equal in parallelogram $ABCD$)

$\therefore \angle BEC = \angle BAD$

$\therefore ABED$ is a cyclic quadrilateral (exterior angle $\angle BEC$ is equal to interior opposite angle $\angle BAD$)

QUESTION 2

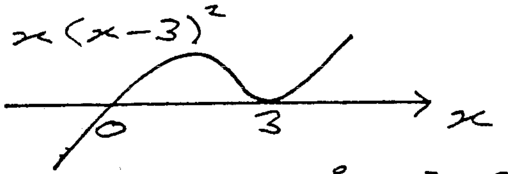
(a) $\lim_{x\to 0} \dfrac{\sin 2x}{x} = 2 \cdot \lim_{x\to 0} \dfrac{\sin 2x}{2x} = 2 \times 1 = \underline{\underline{2}}$.

(b) $\dfrac{x^2+9}{x} \le 6 \quad \therefore \dfrac{x^2+9}{x} \times x^2 \le 6 \times x^2$

$\therefore x^3 + 9x \le 6x^2 \quad \therefore x^3 - 6x^2 + 9x \le 0$

$\therefore x(x^2 - 6x + 9) \le 0 \quad \therefore x(x-3)^2 \le 0$

$y = x(x-3)^2$



$\therefore \underline{\underline{x < 0 \text{ or } x = 3}}$

($x \ne 0$ since $\{(0)^2+9\}/(0)$ is not defined)

(c) (i) $3x^3 + 3x^2 - x - 1 = (x+1)(3x^2-1)$

(ii) $3\tan^3\theta + 3\tan^2\theta - \tan\theta - 1 = 0 \quad (0 \le \theta \le \pi)$

$\therefore (\tan\theta + 1)(3\tan^2\theta - 1) = 0$

$\therefore \tan\theta + 1 = 0,\ \tan\theta = -1,\ \underline{\underline{\theta = 3\pi/4}}$

or $3\tan^2\theta - 1 = 0,\ 3\tan^2\theta = 1,\ \tan^2\theta = 1/3,$

$\tan\theta = -1/\sqrt{3},\ \underline{\underline{\theta = 5\pi/6}}$ or $\tan\theta = 1/\sqrt{3},\ \underline{\underline{\theta = \pi/6}}$.

(d) $x^2 = 4y$. $F(0,1)$, $P(2t, t^2)$

(i) $M(x,y)$ divides FP externally in the ratio $3:1$.

$\therefore x = \dfrac{3(2t) - 1(0)}{3-1} = 3t,\ y = \dfrac{3(t^2) - 1(1)}{3-1} = \dfrac{3t^2-1}{2}$

$x = 3t \quad \therefore t = x/3$

$y = (3t^2-1)/2 \quad \therefore 2y = 3t^2 - 1$

$\therefore 2y = 3(x/3)^2 - 1 \quad \therefore 2y = \cancel{3}(x^2/\cancel{9}_3) - 1$

$\therefore 6y = x^2 - 3 \quad \therefore x^2 = 6y + 3$.

$\therefore M$ moves on the parabola $x^2 = 6y + 3$.

(ii) $x^2 = 6y + 3 \quad \therefore x^2 = 6(y + 1/2)$

$\therefore x^2 = 4(3/2)(y + 1/2) \quad \therefore (x-0)^2 = 4(3/2)(y - (-1/2))$

$\therefore$ vertex $V(0, -1/2)$, focal length $a = 3/2$

$\therefore$ focus $(0, -1/2 + 3/2)$ ie. $\underline{(0,1)}$

$\therefore$ directrix $y = -1/2 - 3/2$ ie. $\underline{\underline{y = -2}}$

QUESTION 3

(a) $y = \tan^{-1} 1/x \quad \therefore \dfrac{dy}{dx} = \dfrac{1}{1+(1/x)^2} \cdot (-1/x^2)$

$\therefore$ when $x = 1$, $dy/dx = 1/(1+1) \cdot (-1) = -1/2$

$\therefore$ the gradient of the tangent is $\underline{\underline{-1/2}}$.

(b) $f(x) = \dfrac{x+1}{x+2} \quad \therefore y = \dfrac{x+1}{x+2}$

interchange x and y $\quad \therefore x = \dfrac{y+1}{y+2}$.

$\therefore x(y+2) = y+1 \quad \therefore xy + 2x = y+1$

$\therefore xy - y = 1 - 2x \quad \therefore y(x-1) = 1-2x$

$\therefore y = \dfrac{1-2x}{x-1} \quad \therefore \underline{\underline{f^{-1}(x) = \dfrac{1-2x}{x-1}}}$.

(c) $\dfrac{dy}{dx} = 2\cos^2 x + 1 \quad \therefore y = \int (2\cos^2 x + 1)\,dx$

$\therefore y = \int (\cos 2x + 1 + 1)\,dx \quad \therefore y = \int (2 + \cos 2x)\,dx$

$\therefore y = 2x + 1/2 \sin 2x + c$.

When $x = \pi$, $y = \pi \quad \therefore \pi = 2\pi + 1/2 \sin 2\pi + c$

$\therefore \pi = 2\pi + 0 + c \quad \therefore c = -\pi$

$\therefore y = 2x + 1/2 \sin 2x - \pi$

$\therefore$ when $x = 2\pi$, $y = 4\pi + 1/2 \sin 4\pi - \pi$

$\therefore y = 4\pi + 0 - \pi \quad \therefore \underline{\underline{y = 3\pi}}$.

(d) $\displaystyle\int_1^{100} \frac{1}{x + 2\sqrt{x}}\,dx$ $\qquad x = u^2 \ (u > 0)$

$\therefore dx/du = 2u, \ dx = 2u\,du$

$\displaystyle = \int_1^{10} \frac{1}{u^2 + 2u} \cdot 2u\,du$ $\qquad$ when $x = 1$, $u = 1$; when $x = 100$, $u = 10$

$\displaystyle = \int_1^{10} \frac{1}{\cancel{u}(u+2)} \cdot 2\cancel{u}\,du = 2\int_1^{10} \frac{1}{u+2}\,du$

$= 2\left[\ln(u+2)\right]_1^{10} = 2(\ln 12 - \ln 3)$

$= 2 \ln 12/3 = 2 \ln 4$.

$= \ln 4^2 = \underline{\underline{\ln 16}}$

QUESTION 4

(a) $\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx = \int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{2^2-x^2}}\,dx$

$= \left[\sin^{-1} x/2\right]_{\sqrt{2}}^{\sqrt{3}} = \sin^{-1}\sqrt{3}/2 - \sin^{-1}\sqrt{2}/2$

$= \pi/3 - \pi/4 = \underline{\underline{\pi/12}}$

(b) (i) $t = x^2 - 3x + 2 \quad \therefore dt/dx = 2x - 3$

$v = dx/dt = 1/dt/dx \quad \therefore \underline{\underline{v = 1/(2x-3)}}$

(ii) $a = d/dx\,(\tfrac{1}{2}v^2) = d/dx\,(\tfrac{1}{2}\cdot 1/(2x-3)^2)$

$\therefore a = \tfrac{1}{2}\cdot -2/(2x-3)^3 \cdot 2 \quad \therefore \underline{\underline{a = -2/(2x-3)^3}}$

(c) $y = 2\cos^{-1}(1-x)$

(i) domain : $-1 \le 1-x \le 1$

$\therefore -2 \le -x \le 0 \quad \therefore \underline{\underline{0 \le x \le 2}}$

range : $0 \le \cos^{-1}(1-x) \le \pi$

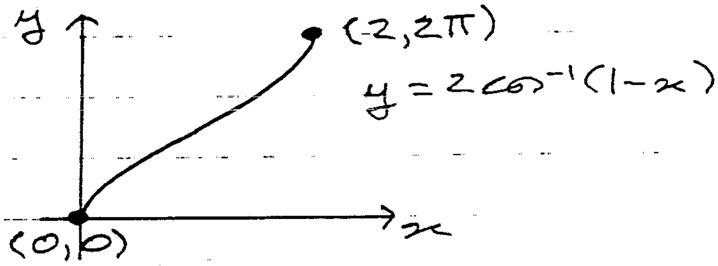
$\therefore 0 \le 2\cos^{-1}(1-x) \le 2\pi \quad \therefore \underline{\underline{0 \le y \le 2\pi}}$

(ii) when $x = 0$

$y = 2\cos^{-1}(1) = 0$

when $x = 2$

$y = 2\cos^{-1}(-1) = 2\pi$



(d) $r = \frac{1+3t}{1+t} \quad \therefore \frac{dr}{dt} = \frac{(1+t)(3)-(1+3t)(1)}{(1+t)^2} = \frac{2}{(1+t)^2}$

when $r = 2$, $\frac{1+3t}{1+t} = 2$

$\therefore 1+3t = 2+2t \quad \therefore t = 1$

$A = \pi r^2 \quad \therefore \frac{dA}{dt} = \frac{dA}{dr}\cdot\frac{dr}{dt} = 2\pi r \cdot \frac{2}{(1+t)^2}$

when $r = 2$, $t = 1$

$\frac{dA}{dt} = 2\pi(2)\cdot\frac{2}{(1+(1))^2} = 2\pi$

$\therefore$ the area of the oil spill is increasing at a rate of $\underline{\underline{2\pi \text{ kilometres}^2/\text{hour}}}$.

QUESTION 5

(a) $f(x) = \ln x / x \quad (x > 0)$

(i) $f'(x) = \{(x)(1/x) - (\ln x)(1)\}/x^2 = (1-\ln x)/x^2$

When $f'(x) = 0$; $(1-\ln x)/x^2 = 0$, $1-\ln x = 0$, $\ln x = 1$, $x = e$

When $x = e$, $y = \ln e / e = 1/e$

When $0 < x < e$, $\ln x < 1$ and $f'(x) = (1-\ln x)/x^2 > 0$

When $x > e$, $\ln x > 1$ and $f'(x) = (1-\ln x)/x^2 < 0$

$\therefore$ $(e, 1/e)$ is a maximum turning point.

(ii) $f(e)$ is a local maximum and $f'(x) < 0$ for $x > e$ so that $f(x)$ is a decreasing function for $x > e$. $\therefore f(\pi) < f(e)$

$\therefore \ln\pi / \pi < \ln e / e \quad \therefore e\ln\pi < \pi \ln e$

$\therefore \ln\pi^e < \ln e^\pi \quad \therefore \pi^e < e^\pi$.

(iii) $\ln x / x = -2 \quad \therefore \ln x = -2x \quad \therefore \ln x + 2x = 0$

$P(x) = \ln x + 2x$, $P'(x) = 1/x + 2$

$\therefore$ with initial approximation of $X = 0.5$, improved approximati

$= 0.5 - \dfrac{P(0.5)}{P'(0.5)} = 0.5 - \dfrac{\ln 0.5 + 2(0.5)}{1/(0.5) + 2} = 0.42$ (to 2d.p

(b) (i) $dM/dt < 0$, $dM/dt \propto (M - 1000)$

$\therefore dM/dt = k(M-1000)$ $(k<0)$ or $dM/dt = -k(M-1000)$ $(k>0$

If $M = 1000 + Ae^{-kt}$

$dM/dt = 0 + A(-ke^{-kt}) = -k(Ae^{-kt}) = -k(M-1000)$

$\therefore M = 1000 + Ae^{-kt}$ is a solution of $dM/dt = -k(M-1000)$

(ii) When $t = 0$, $M = 49000$. $\therefore 49000 = 1000 + Ae^{-k(0)}$

$\therefore 49000 = 1000 + A \quad \therefore A = 48000$

When $t = 2$, $M = 25000 \quad \therefore 25000 = 1000 + 48000e^{-k(2)}$

$\therefore 24000 = 48000e^{-2k} \quad \therefore e^{2k} = 2 \quad \therefore k = \frac{1}{2}\ln 2$

(iii) $dM/dt = -k(M-1000)$; when $t = 0$, $M = 49000$

$\therefore$ initial rate of losing value $= -k(49000-1000) = -48000k$

When $dM/dt = \frac{1}{4}(-48000k) = -12000k$,

$-k(M-1000) = -12000k$, $M - 1000 = 12000$, $M = \$13000$

When $M = 13000$, $13000 = 1000 + 48000e^{-kt}$

$\therefore 12000 = 48000e^{-kt} \quad \therefore e^{kt} = 4 \quad \therefore kt = \ln 4$

$\therefore t = \dfrac{\ln 4}{k} = \dfrac{\ln 4}{\frac{1}{2}\ln 2} = 4$ years

QUESTION 6

(a) In the expansion of $(1+x)^{14}$ the coefficients of x^4, x^5, x^6 are ${}^{14}C_4 = 1001$, ${}^{14}C_5 = 2002$, ${}^{14}C_6 = 3003$ which are consecutive terms in an arithmetic sequence with common difference 1001.

(b) In any one throw $P(\text{heads}) = p$.

$P(3 \text{ heads in } 6 \text{ throws}) = 2P(2 \text{ heads in } 6 \text{ throws})$

$\therefore \; {}^6C_3\, p^3(1-p)^3 = 2\, {}^6C_2\, p^2(1-p)^4$

$\therefore \; 20p^3(1-p)^3 = 2.15p^2(1-p)^4$

$\therefore \; 20p^3(1-p)^3 = 30p^2(1-p)^4 \quad \therefore 2p = 3(1-p)$

$\therefore \; 2p = 3 - 3p \quad \therefore 5p = 3 \quad \therefore \underline{p = 3/5}$

(c) $x = 2\sin 3t - 2\sqrt{3}\cos 3t$

(i) $2\sin 3t - 2\sqrt{3}\cos 3t \equiv R\sin(3t - \alpha)$

$\equiv R(\sin 3t\cos\alpha - \cos 3t\sin\alpha)$

$\equiv (R\cos\alpha)\sin 3t - (R\sin\alpha)\cos 3t$

$\therefore R\cos\alpha = 2 \quad (1) \qquad R\sin\alpha = 2\sqrt{3} \quad (2)$

$(1)^2 \quad R^2\cos^2\alpha = 4 \qquad R^2(\cos^2\alpha + \sin^2\alpha) = 16$

$(2)^2 \quad R^2\sin^2\alpha = 12 \quad + \qquad R^2 = 16, \quad R = 4$

$\dfrac{(2)}{(1)} \quad \dfrac{R\sin\alpha}{R\cos\alpha} = \dfrac{2\sqrt{3}}{2} \qquad \tan\alpha = \sqrt{3}, \quad \alpha = \pi/3$

$\therefore \; \underline{\underline{x = 4\sin(3t - \pi/3)}}$

(ii) $x = 4\sin(3t - \pi/3)$; when $t = 0$, $x = -2\sqrt{3}$

$v = \dot{x} = 12\cos(3t - \pi/3)$; when $t = 0$, $v = 6$

$a = \ddot{x} = -36\sin(3t - \pi/3)$; when $t = 0$, $a = 18\sqrt{3}$

$\therefore$ initially the particle is $2\sqrt{3}$ metres to the left of O moving to the right at a speed of $6\,\text{ms}^{-1}$ and speeding up at a rate of $18\sqrt{3}\,\text{ms}^{-2}$.

(iii) When $x = -2$, $4\sin(3t - \pi/3) = -2$

$\therefore \sin(3t - \pi/3) = -1/2 \qquad \therefore 3t - \pi/3 = -\pi/6, \ldots$

$\therefore 3t = -\pi/6 + \pi/3, \ldots \qquad \therefore 3t = \pi/6, \ldots$

$\therefore t = \pi/18, \ldots \qquad$ When $t = \pi/18$, $v = 6\sqrt{3} > 0$

$\therefore$ the first time is after $\underline{\underline{\pi/18 \text{ seconds}}}$

QUESTION 7

(a) $S_n = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{n}{(n+1)!} \quad (n = 1, 2, 3, \dots)$

(i) $S(n) : \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

When $n = 1$, $LHS = 1/2! = 1/2$

$RHS = 1 - 1/(1+1)! = 1 - 1/2! = 1 - 1/2 = 1/2$

$\therefore$ statement is true when $n = 1$.

If the statement is true when $n = k$ for some $k \geq 1$

then $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \dots + \frac{k}{(k+1)!} + \frac{(k+1)}{(k+2)!}$

$= 1 - \frac{1}{(k+1)!} + \frac{(k+1)}{(k+2)!}$

$= 1 - \frac{\{(k+2) - (k+1)\}}{(k+2)!} = 1 - \frac{1}{(k+2)!}$

and the statement is also true when $n = k+1$.

Since $S(1)$ is true and if $S(k)$ is true for some $k \geq 1$ then $S(k+1)$ is also true it follows that $S(1+1) = S(2)$ is true, $S(2+1) = S(3)$ is true, ... and so $S(n)$ is true for all positive integers $n \geq 1$.

(ii) $\lim_{n \to \infty} S_n = \lim_{n \to \infty} 1 - \frac{1}{(n+1)!} = 1 - 0 = \underline{\underline{1}}$

(iii) $|S_n - 1| < 10^{-6}$

$\therefore \left| 1 - \frac{1}{(n+1)!} - 1 \right| < \frac{1}{10^6} \quad \therefore \left| \frac{-1}{(n+1)!} \right| < \frac{1}{10^6}$

$\therefore \frac{1}{(n+1)!} < \frac{1}{10^6} \quad \therefore (n+1)! > 10^6$

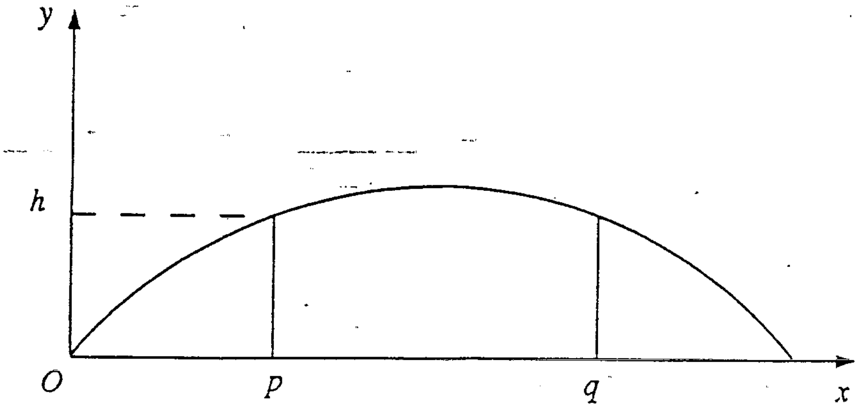
Now $9! = 362880$, $10! = 3628800$

$\therefore n+1 \geq 10 \quad \therefore n \geq 9$

$\therefore$ the smallest positive integer is $\underline{\underline{n = 9}}$.

QUESTION 7

(b)



(i) $x = (V\cos\alpha)t$ (1), $y = (V\sin\alpha)t - \frac{1}{2}gt^2$

$y = (V\sin\alpha)t - 5t^2$ (2)

(ii) When $x = p$, $y = h$

$\therefore$ in (1) $p = (V\cos\alpha)t$ $\therefore t = p/V\cos\alpha$

$\therefore$ in (2) $h = (V\sin\alpha)t - 5t^2$

$\therefore h = (V\sin\alpha)(p/V\cos\alpha) - 5(p/V\cos\alpha)^2$

$\therefore h = p\tan\alpha - 5p^2/V^2\cos^2\alpha$

$\therefore h = p\tan\alpha - 5p^2/V^2 \sec^2\alpha$

$\therefore h = p\tan\alpha - 5p^2/V^2(1+\tan^2\alpha)$

$\therefore 5p^2/V^2(1+\tan^2\alpha) = p\tan\alpha - h$

$$\therefore V^2 = \frac{5p^2(1+\tan^2\alpha)}{p\tan\alpha - h}$$

(iii) When $x = q$, $y = h$

$$\therefore \text{similarly } V^2 = \frac{5q^2(1+\tan^2\alpha)}{q\tan\alpha - h}$$

$$\therefore \frac{\cancel{5}p^2\cancel{(1+\tan^2\alpha)}}{p\tan\alpha - h} = \frac{\cancel{5}q^2\cancel{(1+\tan^2\alpha)}}{q\tan\alpha - h}$$

$\therefore p^2(q\tan\alpha - h) = q^2(p\tan\alpha - h)$

$\therefore p^2q\tan\alpha - p^2h = pq^2\tan\alpha - q^2h$

$\therefore p^2h - q^2h = p^2q\tan\alpha - pq^2\tan\alpha$

$\therefore (p^2 - q^2)h = (p^2q - pq^2)\tan\alpha$

$\therefore (p+q)\cancel{(p-q)}h = pq\cancel{(p-q)}\tan\alpha$

$$\therefore \tan\alpha = \frac{h(p+q)}{pq}$$