CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NEW SOUTH WALES

YEAR TWELVE FINAL TESTS 2000

MATHEMATICS
4 UNIT COURSE

Morning session

Wednesday 9 August 2000

Time allowed – three hours

EXAMINERS
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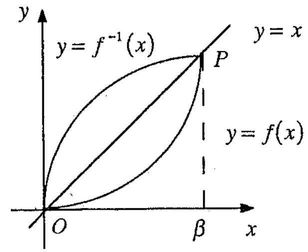
DIRECTIONS TO CANDIDATES:

- **ALL** questions may be attempted.
- **ALL** questions are of equal value.
- All necessary working should be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- Standard integrals are printed at the end of the exam paper.

Question 1 **Begin a new page** **Marks**

(a) **4**

This diagram shows the graphs of $y = f(x)$ and its inverse $y = f^{-1}(x)$, and the line $y = x$. The graphs intersect in $(0, 0)$ and in the point P with x coordinate β.



Use the substitution $u = f^{-1}(x)$ to show that $\int_0^\beta f^{-1}(x)\,dx = \int_0^\beta u\,f'(u)\,du$

and hence show that the area bounded by $y = f(x)$ and $y = f^{-1}(x)$ is given by

$$A = \int_0^\beta \{x\,f'(x) - f(x)\}\,dx.$$

(b) $y = x\sin^{-1}x$ **11**

(i) Show that $\dfrac{dy}{dx} = \sin^{-1}x + \tan\left(\sin^{-1}x\right)$

(ii) By considering the graph of $y = \tan\theta$, deduce that the graph of $y = x\sin^{-1}x$ has exactly one stationary point. Show this stationary point is a minimum turning point at $(0, 0)$.

(iii) Sketch the graph of $y = x\sin^{-1}x$. Show the nature of the curve near the endpoints of its domain.

(iv) If $f(x) = x\sin^{-1}x$, $x \ge 0$, show on a new diagram the graphs of $y = f(x)$, its inverse $y = f^{-1}(x)$, and the line $y = x$. Give the coordinates of any points of intersection and of the endpoints of the curves.

(v) Use the result in (a) to show that the area bounded by the curves $y = f(x)$ and $y = f^{-1}(x)$ between their points of intersection is given by $\int_0^{\sin 1} \dfrac{x^2}{\sqrt{1-x^2}}\,dx$.

Use the substitution $x = \sin\theta$ to evaluate this area.

Question 2 **Begin a new page** **Marks**

(a) (i) Find the two square roots of $2i$. **3**

(ii) Solve $x^2+2x+\left(1-\frac{1}{2}i\right)=0$.

(b) Find **4**

(i) $\int\left(1+\tan^2 x\right)e^{\tan x}\,dx$

(ii) $\int t\,e^{-t}\,dt$

(iii) $\int \cos^3 x\,dx$

(c) Evaluate in simplest exact form $\int_e^{e^2}\frac{1}{x\ln x}\,dx$ **2**

(d) Use the substitution $t=\tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{3}}\frac{1}{1-\sin x}\,dx$, giving your answer in simplest exact form. **3**

(e) If $I=\int_0^{\ln 2}\frac{e^x}{e^x+e^{-x}}\,dx$ and $J=\int_0^{\ln 2}\frac{e^{-x}}{e^x+e^{-x}}\,dx$, find the exact values of $I+J$ and $I-J$ and hence find the exact values of I and J. **3**

Question 3 **Begin a new page** **Marks**

(a) $z_1=1+2i$ and $z_2=3-i$. Find the value of $z_1^{\,2}\div\bar{z}_2$. **2**

(b) $z=\sqrt{3}+i$ **4**

(i) Write z in modulus / argument form.

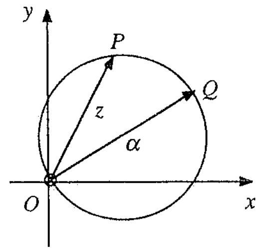
(ii) What can you say about integers n such that $z^n+(\bar{z})^n$ is rational ?

(iii) Find the smallest positive integer n such that $z^n+(\bar{z})^n$ is a negative rational number, and for this value of n, state the value of $z^n+(\bar{z})^n$.

(c) $\alpha=p+iq$ where p and q are real. **9**

(i) If z satisfies $\text{Re}(\alpha z)=1$, show that the locus of the point P representing z in the Argand diagram is the line $px-qy=1$.

(ii) The vector $\overrightarrow{OQ}$ represents α in the Argand diagram. If $z\neq 0$ is represented by the vector $\overrightarrow{OP}$ where P lies on the circle with diameter OQ, copy the diagram and show the vector representing $z-\alpha$.



Show that for such a complex number z, $\frac{z-\alpha}{z}$ is imaginary and hence $\text{Re}\left(\alpha\,\frac{1}{z}\right)=1$.

(iii) Deduce that if z is a non-zero complex number such that the point P representing z in the Argand diagram lies on the circle with diameter OQ, where Q has coordinates $(p\,,\,q)$, then the point representing $\frac{1}{z}$ in the same Argand diagram lies on the line $px-qy=1$.

(iv) $z\neq 0$ satisfies the condition $\left|z-(1+i)\right|=\sqrt{2}$. Sketch the locus of the points representing z and $\frac{1}{z}$ in the same Argand diagram, and label each locus with its equation. Considering only values between $-\pi$ and π , what are the possible values of $\arg z$?

Question 4 **Begin a new page** **Marks**

Hyperbola $\mathcal{H}$ has equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and eccentricity e, while ellipse $\mathcal{E}$ has equation $\frac{x^2}{a^2+b^2} + \frac{y^2}{b^2} = 1$. **15**

(i) Show that $\mathcal{E}$ has eccentricity $\frac{1}{e}$.

(ii) Show that $\mathcal{E}$ passes through one focus of $\mathcal{H}$, and $\mathcal{H}$ passes through one focus of $\mathcal{E}$.

(iii) Sketch $\mathcal{H}$ and $\mathcal{E}$ on the same diagram, showing the foci S, S' of $\mathcal{H}$ and T, T' of $\mathcal{E}$, and the directrices of $\mathcal{H}$ and $\mathcal{E}$. Give the coordinates of the foci and the equations of the directrices in terms of a and e.

(iv) If $\mathcal{H}$ and $\mathcal{E}$ intersect at P in the first quadrant, show that the acute angle α between the tangents to the curves at P satisfies $\tan\alpha = \sqrt{2}\left(e+\frac{1}{e}\right)$.

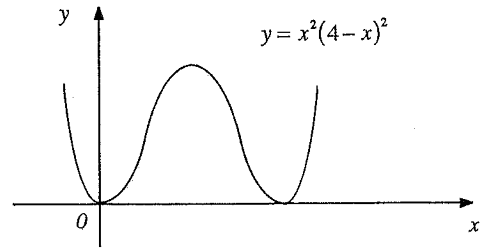
(v) What is the smallest possible acute angle between the tangents to the curves $\mathcal{H}$ and $\mathcal{E}$ at their point of intersection P ?

(vi) Find the acute angle between the tangents to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and the ellipse $\frac{x^2}{25} + \frac{y^2}{9} = 1$ at their points of intersection. Give your answer to the nearest degree.

Question 5 **Begin a new page** **Marks**

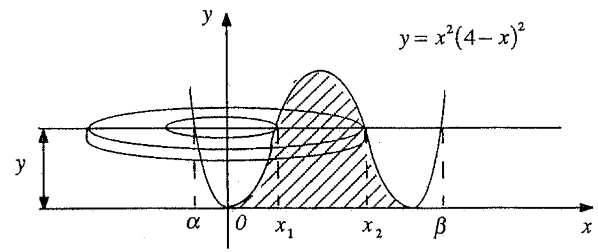
(a) Show that the stationary points of $y = \{f(x)\}^2$ are exactly those points on the curve that have x coordinates which are zeros of either $f(x)$ or $f'(x)$. **2**

(b) **4**



Use the graph of $y = x(4-x)$ to justify the features shown on the graph above.
Copy the graph of $y = x^2(4-x)^2$ and mark on the coordinate axes the values of x and y at the stationary points.

(c) **9**



The shaded region is rotated through one revolution about the y axis. The volume of the solid formed is found by taking slices perpendicular to the y axis. The typical slice shown in the diagram is at a height y above the x axis.

(i) Deduce that α, x_1, x_2, β , as shown in the diagram, are roots of $x^4 - 8x^3 + 16x^2 - y = 0$.

(ii) Use the symmetry in the graph to explain why $\frac{x_1+x_2}{2} = 2$ and $\frac{\alpha+\beta}{2} = 2$.
Hence, by considering the coefficients of the equation in (i), show that $\alpha\beta = -x_1x_2$ and deduce that $x_1x_2 = \sqrt{y}$ and $x_2 - x_1 = 2\sqrt{4-\sqrt{y}}$.

(iii) Show that the volume of the solid of revolution is given by $V = 8\pi\int_0^{16}\sqrt{4-\sqrt{y}}\,dy$.

Use the substitution $y = (4-u)^2$ to evaluate this integral and find the exact volume.

Marks

Question 6 **Begin a new page**

A toy of mass m kg has a parachute device attached. It is released from rest at the top of a vertical cliff 40 m high. During its fall, the forces acting are gravity and, owing to the parachute, a resistance force of magnitude $\frac{1}{10}mv^2$ when the speed of the toy is v ms^{-1}. After $2\ln 2$ seconds, the parachute disintegrates, and then the only force acting on the toy is gravity. The acceleration due to gravity is taken as $g = 10$ ms^{-2}. At time t seconds, the toy has fallen a distance x metres from the top of the cliff, and its speed is v ms^{-1}. **15**

(i) Show that while the parachute is operating, $10\ddot{x} = 100 - v^2$. Hence show that
$v = 10\left(\frac{e^{2t}-1}{e^{2t}+1}\right)$ and $x = -5\ln\left\{1-\left(\frac{v}{10}\right)^2\right\}$.

(ii) Find the exact speed of the toy and the exact distance fallen just before the parachute disintegrates.

(iii) After the parachute disintegrates, find an expression for $\ddot{x}$ and use integration to find the speed of the toy just before it reaches the base of the cliff. Give your answer correct to 2 significant figures.

Question 7 **Begin a new page**

(a) The equation $x^3 + px - 1 = 0$ has three real, non-zero roots α, β, γ. **7**

(i) Find the values of $\alpha^2+\beta^2+\gamma^2$ and $\alpha^4+\beta^4+\gamma^4$ in terms of p, and hence show that p must be strictly negative.

(ii) Find the monic equation, with coefficients in terms of p, whose roots are
$\frac{\alpha}{\beta\gamma}, \frac{\beta}{\gamma\alpha}, \frac{\gamma}{\alpha\beta}$.

(b) (i) If $I_n = \int_0^1 (x^2-1)^n\,dx$, $n = 0, 1, 2, \ldots$, show that **8**

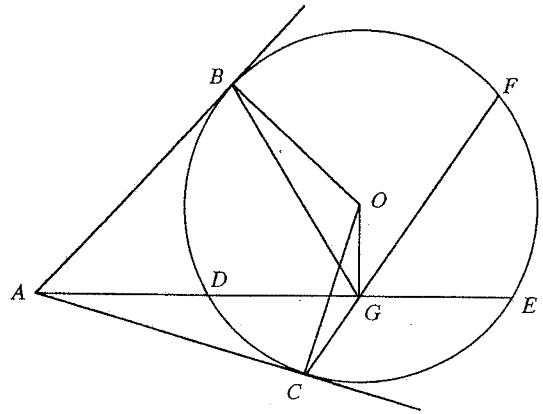
$I_n = \frac{-2n}{2n+1} I_{n-1}$, $n = 1, 2, 3, \ldots$

(ii) Hence use the method of Mathematical Induction to show that $I_n = \frac{(-1)^n 2^{2n} (n!)^2}{(2n+1)!}$ for all positive integers n.

Marks

Question 8 **Begin a new page**

(a) **8**



In the diagram, AB and AC are tangents from A to the circle with centre O, meeting the circle at B and C. ADE is a secant of the circle. G is the midpoint of DE. CG produced meets the circle at F.

(i) Copy the diagram.

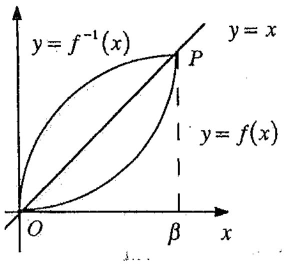
(ii) Show that $ABOC$ and $AOGC$ are cyclic quadrilaterals.

(iii) Show that $BF \parallel ADE$.

(b) (i) If $y = x^k + (c-x)^k$, where $c > 0$, $k > 0$, $k \neq 1$, show that y has a single stationary value between $x = 0$ and $x = c$, and show that this stationary value is a maximum if $k < 1$ and a minimum if $k > 1$. **7**

(ii) Hence show that if $a > 0$, $b > 0$, $a \neq b$, then
$\frac{a^k+b^k}{2} < \left(\frac{a+b}{2}\right)^k$ if $0 < k < 1$, and $\frac{a^k+b^k}{2} > \left(\frac{a+b}{2}\right)^k$ if $k > 1$.

estion 1



At P,

$y=x \Rightarrow f(\beta)=f^{-1}(\beta)=\beta$

By *substitution*,

$$u=f^{-1}(x)$$
$$f(u)=x$$
$$f'(u)du=dx$$
$$x=0 \Rightarrow u=0$$
$$x=\beta \Rightarrow u=\beta$$

Hence

$$\int_0^{\beta} f^{-1}(x)\ dx = \int_0^{\beta} u\, f'(u)\ du$$

By *replacement* of the variable of integration u by x in this definite integral,

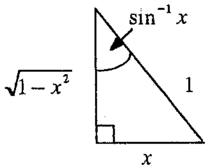
$$\int_0^{\beta} f^{-1}(x)\ dx = \int_0^{\beta} x\, f'(x)\ dx$$

Now the area between the curves is given by

$$\int_0^{\beta} f^{-1}(x)\ dx - \int_0^{\beta} f(x)\ dx$$
$$= \int_0^{\beta} x\, f'(x)\ dx - \int_0^{\beta} f(x)\ dx$$
$$\therefore A = \int_0^{\beta} \{ x\, f'(x) - f(x) \}\ dx$$

(i) $y = x\sin^{-1}x$ has domain $-1 \le x \le 1$

$$\frac{dy}{dx} = \sin^{-1}x + x \, . \, \frac{1}{\sqrt{1-x^2}} , \quad -1<x<1$$



For $0<x<1$

$$\tan\left(\sin^{-1}x\right) = \frac{x}{\sqrt{1-x^2}}$$

For $-1<x<0$, $0<-x<1 \Rightarrow$

$$\frac{(-x)}{\sqrt{1-(-x)^2}} = \tan\left(\sin^{-1}(-x)\right) = -\tan\left(\sin^{-1}x\right)$$

$$\therefore \frac{x}{\sqrt{1-x^2}} = \tan\left(\sin^{-1}x\right)$$

Also $\tan\left(\sin^{-1}0\right) = \tan 0 = 0$

Hence $\dfrac{x}{\sqrt{1-x^2}} = \tan\left(\sin^{-1}x\right)$, $-1<x<1$

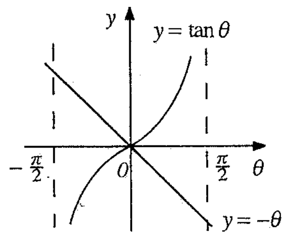
$$\therefore \frac{dy}{dx} = \sin^{-1}x + \tan\left(\sin^{-1}x\right)$$

$$\frac{dy}{dx} = 0 \Rightarrow \tan\left(\sin^{-1}x\right) = -\sin^{-1}x$$

$-1<x<1 \Rightarrow -\frac{\pi}{2} < \sin^{-1}x < \frac{\pi}{2}$

Hence for stationary points, $\sin^{-1}x = \theta$ where $\tan\theta = -\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

(b) (ii) cont.



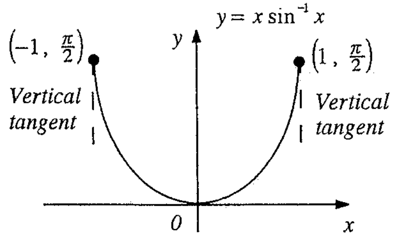
Since solutions are θ values at points of intersection of the graphs shown above, only solution is $\theta=0$.

But $\theta = 0 \Rightarrow \sin^{-1}x = 0 \Rightarrow x = 0$, hence $y = x\sin^{-1}x$ has exactly one stationary point, and this point is $(0,0)$.

$x<0 \Rightarrow \sin^{-1}x<0$ and $\tan\left(\sin^{-1}x\right)<0 \Rightarrow \dfrac{dy}{dx}<0$

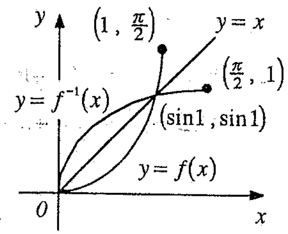
$x>0 \Rightarrow \sin^{-1}x>0$ and $\tan\left(\sin^{-1}x\right)>0 \Rightarrow \dfrac{dy}{dx}>0$

Hence $(0,0)$ is a minimum turning point.



Question 1 (cont)

(b)(iv)



The curves $y=f(x)$, $y=f^{-1}(x)$ are reflections in the line $y=x$, and hence their points of intersection lie on the line $y=x$.

The curves intersect where

$$x\sin^{-1}x = x$$
$$x\left(\sin^{-1}x - 1\right) = 0$$
$$x=0 \text{ or } \sin^{-1}x = 1$$
$$x=0 \text{ or } x=\sin 1$$

1(b)(v) Using (a), and

$$x\,f'(x) = x\left\{\sin^{-1}x + \frac{x}{\sqrt{1-x^2}}\right\}$$
$$= x\sin^{-1}x + \frac{x^2}{\sqrt{1-x^2}}$$

$$A = \int_0^{\sin 1} \{x\,f'(x) - f(x)\}\ dx = \int_0^{\sin 1} \frac{x^2}{\sqrt{1-x^2}}\,dx$$

$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

$x = \sin\theta$

$dx = \cos\theta\ d\theta$

$x=0 \Rightarrow \theta = 0$

$x=\sin 1 \Rightarrow \theta = 1$

$$\frac{x^2}{\sqrt{1-x^2}} = \frac{\sin^2\theta}{\sqrt{\cos^2\theta}} = \frac{\sin^2\theta}{\cos\theta}$$

$$\int_0^{\sin 1} \frac{x^2}{\sqrt{1-x^2}}\,dx$$
$$= \int_0^1 \frac{\sin^2\theta}{\cos\theta}\cos\theta\ d\theta$$
$$= \tfrac{1}{2}\int_0^1 (1-\cos 2\theta)\ d\theta$$
$$= \tfrac{1}{2}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^1$$
$$= \tfrac{1}{4}(2-\sin 2)$$

Hence area is $\frac{1}{4}(2-\sin 2)$ sq. units.

Question 2

(a) (i) $2i = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

Hence the two square roots of $2i$ are

$\pm\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \pm(1+i)$

(ii) $x^2 + 2x + \left(1-\frac{1}{2}i\right) = 0$

$\Delta = 4 - 4\left(1-\frac{1}{2}i\right) = 2i$

Using the quadratic formula $x = \dfrac{-b \pm \sqrt{\Delta}}{2a}$,

$$x = \frac{-2 \pm (1+i)}{2}$$

$x = -\frac{1}{2}(3+i)$, or $x = \frac{1}{2}(-1+i)$

(b)(i) $\displaystyle\int \left(1+\tan^2 x\right) e^{\tan x} dx = \int \sec^2 x\ e^{\tan x} dx$

$= e^{\tan x} + c$

(ii) $\displaystyle\int t\,e^{-t}\ dt = -t\,e^{-t} + \int e^{-t}\ dt$

$= -t\,e^{-t} - e^{-t} + c$

(iii) $\displaystyle\int \cos^3 x\ dx = \int \cos x\left(1-\sin^2 x\right)\ dx$

$= \sin x - \frac{1}{3}\sin^3 x + c$

(c) $\displaystyle\int_e^{e^2} \frac{1}{x\ln x}\,dx = \left[\ln(\ln x)\right]_e^{e^2} = \ln 2 - \ln 1 = \ln 2$

(d)

$t = \tan\frac{x}{2}$

$dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx$

$dt = \frac{1}{2}\left(1+t^2\right)dx$

$\dfrac{2}{1+t^2}\,dt = dx$

$x=0 \Rightarrow t=0$

$x=\frac{\pi}{3} \Rightarrow t = \frac{1}{\sqrt{3}}$

$$1-\sin x = 1 - \frac{2t}{1+t^2}$$
$$= \frac{1+t^2-2t}{1+t^2}$$
$$= \frac{(1-t)^2}{1+t^2}$$
$$\frac{1}{1-\sin x} = \frac{1+t^2}{(1-t)^2}$$

$$\int_0^{\frac{\pi}{3}} \frac{1}{1-\sin x}\,dx$$
$$= \int_0^{\frac{1}{\sqrt{3}}} \frac{1+t^2}{(1-t)^2}\,\frac{2}{1+t^2}\,dt$$
$$= 2\int_0^{\frac{1}{\sqrt{3}}} (1-t)^{-2}\ dt$$
$$= 2\left[(1-t)^{-1}\right]_0^{\frac{1}{\sqrt{3}}}$$

$$2\left\{\frac{1}{1-\frac{1}{\sqrt{3}}} - 1\right\}$$
$$= 2\left\{\frac{\sqrt{3}\left(\sqrt{3}+1\right)}{3-1} - 1\right\}$$
$$= 3+\sqrt{3} - 2$$
$$= 1+\sqrt{3}$$

Question 2 (cont)

(d)

$$I+J=\int_0^{\ln 2}\frac{e^{x}+e^{-x}}{e^{x}+e^{-x}}\,dx=\int_0^{\ln 2}1\,dx=\ln 2$$

$$I-J=\int_0^{\ln 2}\frac{e^{x}-e^{-x}}{e^{x}+e^{-x}}\,dx=\left[\ln\left(e^{x}+e^{-x}\right)\right]_0^{\ln 2}=\ln\left(2+\tfrac{1}{2}\right)-\ln 2$$

$$I-J=\ln\left(\tfrac{5}{2}\div 2\right)=\ln\tfrac{5}{4}$$

$I+J=\ln 2$

$I-J=\ln\frac{5}{4}$

$2I=\ln\left(2\times\frac{5}{4}\right)\ \Rightarrow\ I=\frac{1}{2}\ln\frac{5}{2}$

$2J=\ln\left(2\div\frac{5}{4}\right)\ \Rightarrow\ J=\frac{1}{2}\ln\frac{8}{5}$

Question 3

(a)

$z_1^{\,2}=(1+2i)^2=-3+4i$

$$z_1^{\,2}\div\bar z_2=\frac{-3+4i}{3+i}=\frac{(-3+4i)(3-i)}{9-1}=\frac{-5+15i}{8}$$

(b) (i)

$\sqrt{3}+i=2\left(\frac{\sqrt{3}}{2}+\frac{1}{2}i\right)$

$z=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

(b) (ii)

$z^{n}=2^{n}\left(\cos\frac{n\pi}{6}+i\sin\frac{n\pi}{6}\right)$

$\bar z^{n}=2^{n}\left(\cos\frac{n\pi}{6}-i\sin\frac{n\pi}{6}\right)$

$z^{n}+\bar z^{n}=2^{n}\,.\,2\cos\frac{n\pi}{6}$

$\cos\frac{n\pi}{6}$ is rational when $\frac{n\pi}{6}=k\frac{\pi}{2}$, k integral

or $\frac{n\pi}{6}=k\frac{\pi}{3}$, k integral

Hence $z^{n}+\bar z^{n}$ is rational when n is even or a multiple of 3.

(b)(iii) $n=4$ is the smallest positive integer for which $\cos\frac{n\pi}{6}$ is rational and negative. Hence $n=4$ and $z^{4}+\bar z^{4}=2^{5}\cos\frac{4\pi}{6}=-16$

(c) (i) Let $z=x+iy$

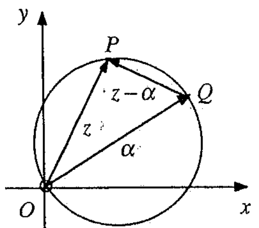
$\alpha z=(p+iq)(x+iy)$

$=(px-qy)+i(qx+py)$

$\mathrm{Re}(\alpha z)=1\Rightarrow px-qy=1$

Hence locus of P is the line $px-qy=1$.

(c)(ii)



$O\hat PQ=90^\circ$ ($\angle$ in semi – circle is rt angle.)

$\arg(z-\alpha)-\arg z=\pm\frac{\pi}{2}$

$\therefore\ \dfrac{z-\alpha}{z}=ki$, k integral

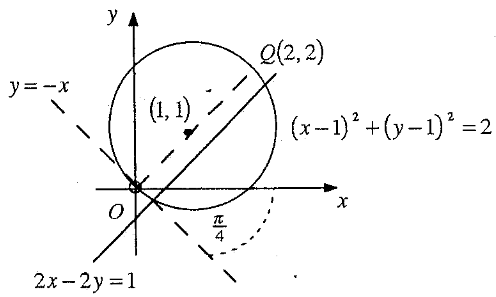
Hence $\dfrac{z-\alpha}{z}$ is imaginary and $\mathrm{Re}\left(\dfrac{z-\alpha}{z}\right)=0$.

But $\mathrm{Re}\left(\dfrac{z-\alpha}{z}\right)=\mathrm{Re}\left(1-\dfrac{\alpha}{z}\right)=1-\mathrm{Re}\left(\dfrac{\alpha}{z}\right)$

$\therefore\ \mathrm{Re}\left(\dfrac{z-\alpha}{z}\right)=0\Rightarrow\mathrm{Re}\left(\dfrac{\alpha}{z}\right)=\mathrm{Re}\left(\alpha\,\dfrac{1}{z}\right)=1$

(c)(iii) If P lies on circle with diameter OQ, where $Q(p,q)$ represents $\alpha=p+iq$, then from (ii) $\mathrm{Re}\left(\alpha\,\dfrac{1}{z}\right)=1$ and hence from (i), the point representing $\dfrac{1}{z}$ lies on the line $px-qy=1$.

(c)(iv)



The locus of z is the circle with centre $(1,1)$ and radius $\sqrt{2}$. Its diameter is OQ as shown.

Since Q has coordinates $(2,2)$, the locus of $\dfrac{1}{z}$ is the line $2x-2y=1$, using (iii).

The line $y=-x$ is perpendicular to OQ and hence is tangent to the circle at O.

$\therefore\ -\frac{\pi}{4}<\arg z<\frac{3\pi}{4}$

Question 4

(i) For the hyperbola $\mathcal{H}$,

$b^{2}=a^{2}\left(e^{2}-1\right)$

$$e^{2}=\frac{b^{2}}{a^{2}}+1=\frac{b^{2}+a^{2}}{a^{2}}$$

If the ellipse $\mathcal{E}$ has eccentricity ε,

$b^{2}=\left(a^{2}+b^{2}\right)\left(1-\varepsilon^{2}\right)$

$$\varepsilon^{2}=1-\frac{b^{2}}{a^{2}+b^{2}}$$

$$\therefore\ \varepsilon^{2}=\frac{a^{2}}{a^{2}+b^{2}}=\frac{1}{e^{2}}$$

Hence the ellipse $\mathcal{E}$ has eccentricity $\frac{1}{e}$.

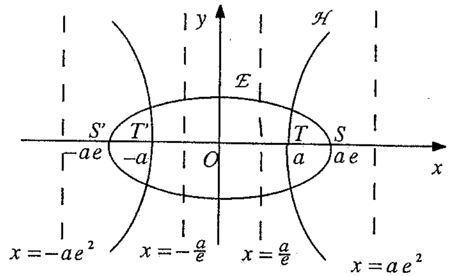
(ii) Since $a^{2}+b^{2}=a^{2}e^{2}$, the equation of the ellipse can be rewritten as $\dfrac{x^{2}}{a^{2}e^{2}}+\dfrac{y^{2}}{b^{2}}=1$.

One focus of $\mathcal{H}$ is $S(ae,0)$, and this point clearly lies on the ellipse $\mathcal{E}$.

One focus of the ellipse is $T\left(ae\,.\,\frac{1}{e},\ 0\right)\equiv T(a,0)$ and this point is clearly on the hyperbola $\mathcal{H}$.

(iii) Hyperbola $\mathcal{H}$ has foci $S(ae,0)$, $S'(-ae,0)$ and directrices $x=\dfrac{a}{e}$, $x=-\dfrac{a}{e}$.

Ellipse $\mathcal{E}$ has foci $T(a,0)$, $T'(-a,0)$ and directrices $x=\dfrac{ae}{\left(\frac{1}{e}\right)}=ae^{2}$, $x=-ae^{2}$.



(vi) Hyperbola $\mathcal{H}$: $\dfrac{x^{2}}{16}-\dfrac{y^{2}}{9}=1$, with eccentricity e given by $9=16\left(e^{2}-1\right)\Rightarrow e=\frac{5}{4}$,

and ellipse $\mathcal{E}$: $\dfrac{x^{2}}{25}+\dfrac{y^{2}}{9}=1$ are two such conics.

Using the symmetry in their graphs, at all of their points of intersection, the acute angle α between the tangents to the curves is given by $\tan\alpha=\sqrt{2}\left(\frac{5}{4}+\frac{4}{5}\right)$

Hence $\alpha\approx 71^\circ$ (to the nearest degree)

(iv) Where the curves intersect,

$$\frac{x^{2}}{a^{2}}-\frac{y^{2}}{b^{2}}=1\quad(1)$$

$$\frac{x^{2}}{a^{2}e^{2}}+\frac{y^{2}}{b^{2}}=1\quad(2)$$

$(1)+(2)\ \Rightarrow\ \dfrac{x^{2}}{a^{2}e^{2}}\left(e^{2}+1\right)=2$

$e^{2}\times(2)-(1)\ \Rightarrow\ \dfrac{y^{2}}{b^{2}}\left(e^{2}+1\right)=e^{2}-1$

$b^{2}=a^{2}\left(e^{2}-1\right)\ \Rightarrow\ \dfrac{y^{2}}{a^{2}\left(e^{2}-1\right)}\left(e^{2}+1\right)=e^{2}-1$

$\therefore$ at P, $x=ae\sqrt{\dfrac{2}{e^{2}+1}}$, $y=\dfrac{a\left(e^{2}-1\right)}{\sqrt{e^{2}+1}}$

For the hyperbola, at P

$\dfrac{x^{2}}{a^{2}}-\dfrac{y^{2}}{b^{2}}=1$

$\dfrac{2x}{a^{2}}-\dfrac{2y}{b^{2}}\dfrac{dy}{dx}=0$

$\therefore\ \dfrac{dy}{dx}=\dfrac{b^{2}}{a^{2}}\dfrac{x}{y}=\left(e^{2}-1\right)\dfrac{x}{y}=\sqrt{2}\,e$

For the ellipse, at P

$\dfrac{x^{2}}{a^{2}e^{2}}+\dfrac{y^{2}}{b^{2}}=1$

$\dfrac{2x}{a^{2}e^{2}}+\dfrac{2y}{b^{2}}\dfrac{dy}{dx}=0$

$\therefore\ \dfrac{dy}{dx}=-\dfrac{b^{2}}{a^{2}e^{2}}\dfrac{x}{y}=-\dfrac{\left(e^{2}-1\right)}{e^{2}}\dfrac{x}{y}=-\sqrt{2}\,\dfrac{1}{e}$

Hence the gradients of the tangents to $\mathcal{H}$ and $\mathcal{E}$ at P are $\sqrt{2}\,e$ and $-\sqrt{2}\,\frac{1}{e}$ respectively.

$$\tan\alpha=\left|\frac{\sqrt{2}\,e-\left(-\sqrt{2}\,\frac{1}{e}\right)}{1+\sqrt{2}\,e\left(-\sqrt{2}\,\frac{1}{e}\right)}\right|=\sqrt{2}\left|\frac{e+\frac{1}{e}}{1-2}\right|$$

$\therefore\ \tan\alpha=\sqrt{2}\left(e+\frac{1}{e}\right)$

(v) For the hyperbola $\mathcal{H}$, $e>1$

$\left(e+\frac{1}{e}\right)^{2}=\left(e-\frac{1}{e}\right)^{2}+4\Rightarrow\left(e+\frac{1}{e}\right)^{2}>4$

and $\left(e+\frac{1}{e}\right)^{2}\to 4$ as $e\to 1^{+}$.

$\therefore\ \left(e+\frac{1}{e}\right)>2$ and $\left(e+\frac{1}{e}\right)\to 2$ as $e\to 1^{+}$

Hence $\tan\alpha>2\sqrt{2}\ \Rightarrow\ \alpha>\tan^{-1}\left(2\sqrt{2}\right)$,

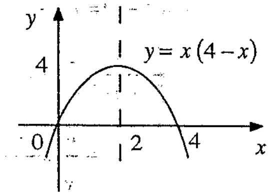
and $\alpha\to\tan^{-1}\left(2\sqrt{2}\right)$ as $e\to 1^{+}$.

Question 5

(a) $\frac{d}{dx}\{f(x)\}^2 = 2f(x).f'(x)$

Stationary points on $y=\{f(x)\}^2$ occur when $\frac{dy}{dx}=0$, that is when $f(x)=0$ or $f'(x)=0$.

(b)



Parabola, with axis of symmetry $x=2$ and vertex $(2,4)$

$y=\{f(x)\}^2$ where $f(x)=x(4-x)$

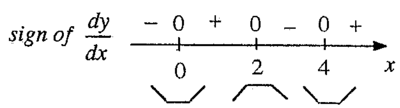
$f(x)=0 \Rightarrow x=0, x=4$

$f'(x)=0 \Rightarrow x=2$

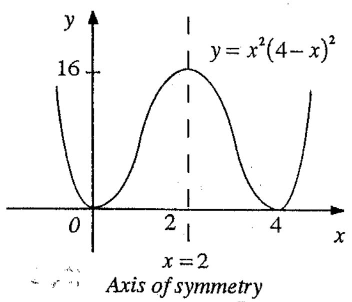
Hence $y=x^2(4-x)^2$ has stationary points at $(0,0)$, $(4,0)$, $(2,16)$, from (a).

$\frac{d}{dx}\{f(x)\}^2>0$ when $f(x)$, $f'(x)$ have same sign

$\frac{d}{dx}\{f(x)\}^2<0$ when $f(x)$, $f'(x)$ have opposite sign



Hence stationary points have nature shown in the diagram.



(c) (i) α, x_1, x_2, β satisfy $y=x^2(4-x)^2$

$y=x^2(4-x)^2 \Rightarrow x^2(x^2-8x+16)-y=0$

Hence α, x_1, x_2, β are roots of $x^4-8x^3+16x^2-y=0$.

(ii) Since $x=2$ is an axis of symmetry for the parabola and hence for $y=x^2(4-x)^2$, 2 is the midpt of the interval between x_1 and x_2, and of the interval between α and β.

$$\therefore \frac{x_1+x_2}{2} = \frac{\alpha+\beta}{2} = 2$$

Hence $x_1+x_2=\alpha+\beta=4$, and

$$0=\alpha\beta x_1+\alpha\beta x_2+x_1x_2\alpha+x_1x_2\beta$$
$$0=(x_1+x_2)\alpha\beta+(\alpha+\beta)x_1x_2$$
$$0=4\alpha\beta+4x_1x_2 \Rightarrow \alpha\beta=-x_1x_2$$

Then

$$-y=\alpha\beta x_1x_2=-(x_1x_2)^2 \Rightarrow x_1x_2=\sqrt{y}$$

and $(x_2-x_1)^2=(x_2+x_1)^2-4x_1x_2=16-4\sqrt{y}$

$$\therefore x_2>x_1 \Rightarrow x_2-x_1=\sqrt{16-4\sqrt{y}}=2\sqrt{4-\sqrt{y}}$$

(iii) The slice has volume

$$\delta V=\pi(x_2^2-x_1^2)\delta y$$
$$\delta V=\pi(x_2+x_1)(x_2-x_1)\delta y$$
$$=8\pi\sqrt{4-\sqrt{y}}\ \delta y$$

Hence

$$V=\lim_{\delta y\to 0}\sum_{y=0}^{16} 8\pi\sqrt{4-\sqrt{y}}\ \delta y$$
$$=8\pi\int_0^{16}\sqrt{4-\sqrt{y}}\ dy$$

$y=(4-u)^2,\ u\le 4$

$dy=-2(4-u)\,du$

$y=0 \Rightarrow u=4$

$y=16 \Rightarrow u=0$

$4-\sqrt{y}=4-(4-u)$

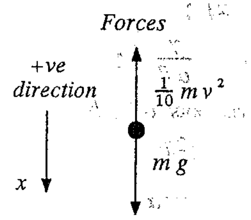
$\sqrt{4-\sqrt{y}}=\sqrt{u}$

$$V=8\pi\int_4^0 u^{\frac{1}{2}}\ .-2(4-u)du$$
$$=16\pi\int_0^4\left(4u^{\frac{1}{2}}-u^{\frac{3}{2}}\right)du$$
$$=16\pi\left[\tfrac{8}{3}u^{\frac{3}{2}}-\tfrac{2}{5}u^{\frac{5}{2}}\right]_0^4$$
$$=16\pi\left(\tfrac{64}{3}-\tfrac{64}{5}\right)$$

Hence volume is $\frac{2048\pi}{15}$ cubic units.

Question 6

) While parachute is operating



Initial conditions

$$t=0 \Rightarrow \begin{cases}x=0\\ v=0\end{cases}$$

Equation of motion

$$m\ddot{x}=mg-\tfrac{1}{10}mv^2$$
$$\ddot{x}=10-\tfrac{1}{10}v^2$$
$$10\ddot{x}=100-v^2$$

o find x in terms of v:

$$5\frac{dv^2}{dx}=100-v^2$$
$$\frac{1}{5}\frac{dx}{d(v^2)}=\frac{1}{100-v^2}$$
$$-\tfrac{1}{5}x=\ln\{(100-v^2)B\},\quad B \text{ constant}$$
$$\left.\begin{matrix}x=0\\ v=0\end{matrix}\right\} \Rightarrow \ln(100B)=0 \Rightarrow B=\tfrac{1}{100}$$
$$-\tfrac{1}{5}x=\ln\{\tfrac{1}{100}(100-v^2)\}$$
$$x=-5\ln\{1-(\tfrac{v}{10})^2\}$$

To find v in terms of t:

$$10\frac{dv}{dt}=100-v^2$$
$$\frac{1}{10}\frac{dt}{dv}=\frac{1}{(10+v)(10-v)}$$
$$2\frac{dt}{dv}=\frac{1}{(10+v)}+\frac{1}{(10-v)}$$
$$2t=\ln\left\{\frac{10+v}{10-v}A\right\},\quad A \text{ constant.}$$
$$\left.\begin{matrix}t=0\\ v=0\end{matrix}\right\} \Rightarrow \ln A=0 \Rightarrow A=1$$
$$2t=\ln\left\{\frac{10+v}{10-v}\right\}$$
$$e^{2t}=\frac{10+v}{10-v}$$
$$e^{2t}(10-v)=10+v$$
$$10(e^{2t}-1)=v(e^{2t}+1)$$
$$v=10\left(\frac{e^{2t}-1}{e^{2t}+1}\right)$$

) Parachute disintegrates when $t=2\ln 2$

$$v=10\left(\frac{e^{2t}-1}{e^{2t}+1}\right)$$
$$t=2\ln 2 \Rightarrow v=10\left(\frac{2^4-1}{2^4+1}\right)=\tfrac{150}{17}$$
$$x=-5\ln\{1-(\tfrac{v}{10})^2\}$$
$$t=2\ln 2 \Rightarrow x=-5\ln\{1-(\tfrac{15}{17})^2\}=10\ln\tfrac{17}{8}$$

Just before parachute disintegrates, speed is $\frac{150}{17}$ ms^{-1}, and distance fallen is $10\ln\frac{17}{8}$ m.

(iii) After parachute disintegrates, $\ddot{x}=10$.

$$\frac{1}{2}\frac{dv^2}{dx}=10$$
$$v^2=20x+c,\quad c \text{ constant}$$
$$\left.\begin{matrix}v=\tfrac{150}{17}\\ x=10\ln\tfrac{17}{8}\end{matrix}\right\} \Rightarrow (\tfrac{150}{17})^2=200\ln\tfrac{17}{8}+c$$
$$v^2-(\tfrac{150}{17})^2=20x-200\ln\tfrac{17}{8}$$
$$x=40 \Rightarrow v^2=(\tfrac{150}{17})^2+800-200\ln\tfrac{17}{8}$$
$$v\approx 26\cdot 96$$

At base of cliff, speed is 27ms^{-1} (to 2 sig.fig).

Question 7

(a) (i) $x^3+px-1=0$ has roots α, β, γ

$$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$$
$$\alpha^2+\beta^2+\gamma^2=0-2p=-2p$$

Since α, β, γ are non-zero and real, $-2p>0$ and hence $p<0$.

α, β, γ each satisfy $x^3+px-1=0$

$$x^4=-px^2+x$$
$$\alpha^4=-p\alpha^2+\alpha$$
$$\beta^4=-p\beta^2+\beta$$
$$\gamma^4=-p\gamma^2+\gamma$$
$$\alpha^4+\beta^4+\gamma^4=-p\left(\alpha^2+\beta^2+\gamma^2\right)+(\alpha+\beta+\gamma)$$
$$\alpha^4+\beta^4+\gamma^4=2p^2+0=2p^2$$

(a) (ii)

$$\alpha\beta\gamma=1 \Rightarrow \frac{\alpha}{\beta\gamma}=\frac{\alpha^2}{\alpha\beta\gamma}=\alpha^2$$

similarly $\frac{\beta}{\gamma\alpha}=\beta^2$, $\frac{\gamma}{\alpha\beta}=\gamma^2$

Required equation has roots $\alpha^2, \beta^2, \gamma^2$.

$$\alpha^2+\beta^2+\gamma^2=-2p$$

$$2\left(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2\right)$$
$$=\left(\alpha^2+\beta^2+\gamma^2\right)^2-\left(\alpha^4+\beta^4+\gamma^4\right)$$
$$=(-2p)^2-\left(2p^2\right)$$
$$\therefore \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2=p^2$$

$$\alpha^2\beta^2\gamma^2=(\alpha\beta\gamma)^2=1$$

Hence required monic equation is
$x^3+2px^2+p^2x-1=0$.

(b)(i) For $n=1, 2, 3, \ldots$

$$I_n=\int_0^1\left(x^2-1\right)^n\,dx$$
$$=\left[x\left(x^2-1\right)^n\right]_0^1-2n\int_0^1 x^2\left(x^2-1\right)^{n-1}\,dx$$
$$=0-2n\int_0^1\left(x^2-1+1\right)\left(x^2-1\right)^{n-1}\,dx$$
$$=-2n\int_0^1\left(x^2-1\right)^n+\left(x^2-1\right)^{n-1}\,dx$$
$$=-2n\left(I_n+I_{n-1}\right)$$
$$(2n+1)\,I_n=-2n\,I_{n-1}$$
$$I_n=\frac{-2n\,I_{n-1}}{2n+1}$$

(ii) $I_0=\int_0^1 1\,dx=1 \Rightarrow I_1=\frac{-2}{2+1}\,I_0=-\frac{2}{3}$

For $n=1, 2, 3, \ldots$
let $S(n)$ be the statement

$$I_n=\frac{(-1)^n\,2^{2n}\,(n!)^2}{(2n+1)!}$$

$$\frac{(-1)^1\,2^2\,(1!)^2}{(2+1)!}=\frac{-4}{3\times 2}=-\frac{2}{3}=I_1$$

$\therefore S(1)$ is true.

(b) (ii) (continued)

If $S(k)$ is true, $I_k=\frac{(-1)^k\,2^{2k}\,(k!)^2}{(2k+1)!}$ **

Consider $S(k+1)$, k some positive integer.

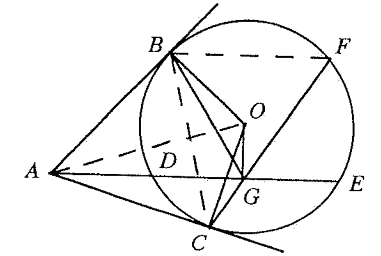
$$I_{k+1}=\frac{-2(k+1)}{2(k+1)+1}\,I_{n-1}$$
$$=\frac{-2(k+1)}{2(k+1)+1}\cdot\frac{(-1)^k 2^{2k}(k!)^2}{(2k+1)!}\quad \text{, if } S(k) \text{ is true using **}$$
$$=\frac{(-1)^{k+1}2^{2k+1}(k+1)\,(k!)^2}{(2k+3)\,(2k+1)!}$$
$$=\frac{(-1)^{k+1}2^{2k+1}(2k+2)(k+1)(k!)^2}{(2k+3)(2k+2)\,(2k+1)!}$$
$$=\frac{(-1)^{k+1}2^{2(k+1)}(k+1)^2(k!)^2}{(2k+3)!}$$
$$=\frac{(-1)^{k+1}2^{2(k+1)}\{(k+1)!\}^2}{\{2(k+1)+1\}!}$$

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(1)$ is true, hence $S(2)$ is true and then $S(3)$ is true, and so on. By Mathematical Induction, $S(n)$ is true for $n=1, 2, 3, \ldots$

Hence $I_n=\frac{(-1)^n\,2^{2n}\,(n!)^2}{(2n+1)!}$ for all positive integers n.

Question 8

(a)(i)



(a)(ii) Construct AO. Then

$A\hat{B}O=90°$ ($\angle$ between tangent AB and radius OB at point of contact B is $90°$.)

Similarly $A\hat{C}O=90°$

$\therefore ABOC$ is cyclic (one pair of opp. $\angle$'s add to $180°$)

$A\hat{G}O=90°$ (line from centre O to midpt G of chord DE is perpendicular to DE)

Then $A\hat{G}O=A\hat{C}O=90°$

$\therefore AOGC$ is cyclic ($\angle$ s subtended by interval AO at C, G on same side are equal)

(a)(iii) Construct BC and BF.

$B\hat{F}C=A\hat{B}C$ ($\angle$ between tangent AB and chord BC is equal to $\angle BFC$ in alternate segment in circle $BFEC$)

$A\hat{B}C=A\hat{O}C$ ($\angle$'s subtended at circumference of circle $ABOC$ by arc AC are equal)

$A\hat{O}C=A\hat{G}C$ ($\angle$'s subtended at circumference of circle $AOGC$ by arc AC are equal)

$\therefore B\hat{F}C=A\hat{G}C$

$\therefore BF \parallel ADE$ (equal corresponding angles on transversal FC)

(b)(i)

$$y=x^k+(c-x)^k$$
$$\frac{dy}{dx}=k\,x^{k-1}-k\,(c-x)^{k-1}$$
$$\frac{dy}{dx}=0 \Rightarrow x^{k-1}=(c-x)^{k-1}\Rightarrow\left(\frac{c-x}{x}\right)^{k-1}=1$$
$$\left(\frac{c-x}{x}\right)^{k-1}=1,\quad k\neq 1 \Rightarrow \frac{c}{x}-1=\pm 1$$

$\therefore$ since $c>0$, $\frac{dy}{dx}=0 \Rightarrow \frac{c}{x}=2$

Hence y has a single stationary value at $x=\frac{1}{2}c$.

$$\frac{dy}{dx}=k\,x^{k-1}-k\,(c-x)^{k-1}$$
$$\frac{d^2y}{dx^2}=k\,(k-1)x^{k-2}+k\,(k-1)(c-x)^{k-2}$$
$$=\tfrac{1}{2}c \Rightarrow \frac{d^2y}{dx^2}=2k\,(k-1)\left(\tfrac{1}{2}c\right)^{k-2}$$
$$<k<1 \Rightarrow \frac{d^2y}{dx^2}<0 \Rightarrow y \text{ has max. at } x=\tfrac{1}{2}c$$
$$k>1 \Rightarrow \frac{d^2y}{dx^2}>0 \Rightarrow y \text{ has min. at } x=\tfrac{1}{2}c$$

(b)(ii)

Let $c=a+b$, $a>0$, $b>0$, $a\neq b$
Consider $y=x^k+(c-x)^k$, $k>0$, $k\neq 1$.
From (i), for $0<x<c$,

$0<k<1 \Rightarrow y$ has a maximum value of $\left(\frac{1}{2}c\right)^k+\left(c-\frac{1}{2}c\right)^k=2\left(\frac{1}{2}c\right)^k$ when $x=\frac{1}{2}c$.

$$0<k<1 \Rightarrow x^k+(c-x)^k<2\left(\tfrac{1}{2}c\right)^k \quad \text{for } x\neq\tfrac{1}{2}c$$
$$a^k+(c-a)^k<2\left(\tfrac{1}{2}c\right)^k \quad \left(a\neq\tfrac{1}{2}c\right)$$
$$\therefore 0<k<1 \Rightarrow \frac{a^k+b^k}{2}<\left(\frac{a+b}{2}\right)^k$$

$k>1 \Rightarrow y$ has a minimum value of $\left(\frac{1}{2}c\right)^k+\left(c-\frac{1}{2}c\right)^k=2\left(\frac{1}{2}c\right)^k$ when $x=\frac{1}{2}c$.

$$k>1 \Rightarrow x^k+(c-x)^k>2\left(\tfrac{1}{2}c\right)^k \quad \text{for } x\neq\tfrac{1}{2}c$$
$$a^k+(c-a)^k>2\left(\tfrac{1}{2}c\right)^k \quad \left(a\neq\tfrac{1}{2}c\right)$$
$$\therefore k>1 \Rightarrow \frac{a^k+b^k}{2}>\left(\frac{a+b}{2}\right)^k$$