CATHOLIC SECONDARY SCHOOLS' ASSOCIATION OF NEW SOUTH WALES

YEAR TWELVE FINAL TESTS 1996

MATHEMATICS

3/4 UNIT COMMON PAPER

(i.e. 3 UNIT COURSE – ADDITIONAL PAPER: 4 UNIT COURSE – FIRST PAPER)

Afternoon session

Friday 9th August 1996.

Time Allowed — Two Hours

EXAMINERS

Graham Arnold, John Paul II Senior High, Marayong
Sandra Hayes, All Saints Catholic Senior High, Casula.
Frank Reid, School of Mathematics, University of NSW.

DIRECTIONS TO CANDIDATES :

ALL questions may be attempted.

ALL questions are of equal value.

All necessary working should be shown in every question.

Full marks may not be awarded for careless or badly arranged work.

Approved calculators may be used.

Standard integrals are printed on a separate page.

Question 1

(a) (3 marks)

Find the acute angle between the lines $2x - y = 0$ and $x + 3y = 0$, giving the answer correctto the nearest minute.

(b) (4 marks)

Consider the function $y = x\ln x - x$.

(i) Solve the equation $y = 0$.

(ii) Find $\dfrac{d^2y}{dx^2}$ and hence show that the function is concave up for all values of x in its domain.

(c) (5 marks)

Consider the polynomial $P(x) = 6x^3 - 5x^2 - 2x + 1$

(i) Show that 1 is a zero of $P(x)$.

(ii) Express $P(x)$ as a product of 3 linear factors.

(iii) Solve the inequality $P(x) \le 0$.

Question 2

(a) (3 marks)

(i) Find $\dfrac{d}{dx}\left(e^{\tan x}\right)$ (ii) Hence find $\displaystyle\int \frac{e^{\tan x}}{\cos^2 x}\,dx$

(b) Use the substitution $u = 1 - x$ to evaluate $\displaystyle\int_{-3}^{0} \frac{x}{\sqrt{1-x}}\,dx$. (4 marks)

(c) . (5 marks)

(i) Find the value of x such that $\sin^{-1} x = \cos^{-1} x$.

(ii) On the same axes sketch the graphs of $y = \sin^{-1} x$ and $y = \cos^{-1} x$.

(iii) On the same diagram as the graphs in (ii), draw the graph of $y = \sin^{-1} x + \cos^{-1} x$.

Question 3

(a) (8 marks)

$$f(x) = \frac{8}{4 + x^2}$$

(i) Show that f is an even function, and the x axis is a horizontal asymptote to the curve $y = f(x)$.

(ii) Find the coordinates and nature of the stationary point on the curve $y = f(x)$.

(iii) Sketch the graph of the curve showing the above features.

(iv) Find the exact area of the region in the first quadrant bounded by the curve $y = f(x)$ and the line $x = 2$.

(b) (4 marks)

A vertical tower of height h metres stands on horizontal ground. From a point P on the ground due east of the tower the angle of elevation of the top of the tower is $45°$. From a point Q on the ground due south of the tower the angle of elevation of the top of the tower is $30°$. If the distance PQ is 40 metres, find the exact height of the tower.

Question 4

(a) (8 marks)

N is the number of animals in a certain population at time t years. The population size N satisfies the equation $\frac{dN}{dt} = -k(N - 1000)$, for some constant k.

(i) Verify by differentiation that $N = 1000 + Ae^{-kt}$, A constant, is a solution of the equation.

(ii) Initially there are 2500 animals but after 2 years there are only 2200 left. Find the values of A and k.

(iii) Find when the number of animals has fallen to 1300.

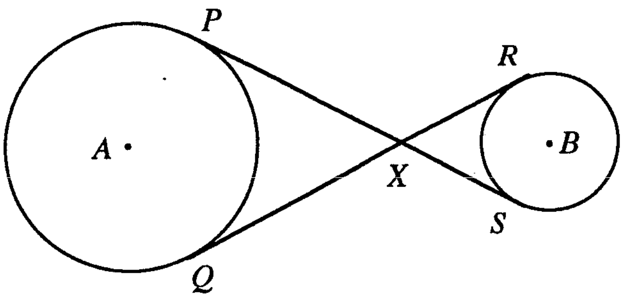
(iv) Sketch the graph of the population size against time.

(b) (4 marks)

Use Mathematical Induction to show that $\cos(x + n\pi) = (-1)^n \cos x$ for all positive integers $n \geq 1$.

Question 5

(a) (7 marks)



In the diagram PS and QR are tangents to each of the circles with centres A and B. The tangents intersect at X and A, X, B are collinear.

(i) Copy the diagram and show that $\Delta APX \;|||\; \Delta BSX$.

(ii) Suppose that the diagram represents two circles of radii 5 cm and 3 cm that are placed in the same plane with their centres 16 cm apart. A taut string surrounds the circles and crosses itself between them. Find the exact length of the string.

(b) (5 marks)

The interior of a circle is divided into two segments with areas in the ratio $3:1$ by a chord which subtends an angle θ radians at the centre of the circle.

(i) Show that $\theta - \sin\theta = \frac{\pi}{2}$.

(ii) Taking $\theta = 2\cdot 5$ as a first approximation, use Newton's method twice to find a better approximation to θ , giving the answer correct to 2 decimal places.

Question 6

(a) (7 marks)

A group consisting of 3 men and 6 women attends a prizegiving ceremony.

(i) If the members of the group sit down at random in a straight line, find the probability that the 3 men sit next to each other.

(ii) If 5 prizes are awarded at random to members of the group, find the probability that exactly 3 of the prizes are awarded to women if

(α) there is a restriction of at most one prize per person.

(β) there is no restriction on the number of prizes per person.

Question 6 (cont.)

(b) (5 marks)

A particle moving in a straight line is performing Simple Harmonic Motion about a fixed point O on the line. At time t seconds the displacement x metres of the particle from O is given by

$$x = a\cos nt \quad , \qquad \text{where } a > 0 \text{ and } 0 < n < \pi \ .$$

After 1 second the particle is 1 metre to the right of O, and after 2 seconds the particle is 1 metre to the left of O.

(i) Find the values of n and a .

(ii) Find the amplitude and period of the motion.

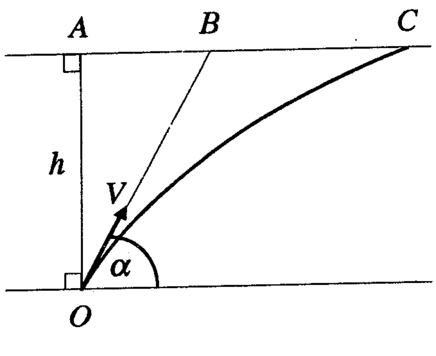
Question 7

(a) (4 marks)

(i) Write down the Binomial expansion of $(1+x)^n$ in ascending powers of x.
Hence show that $\quad {}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n = 2^n$.

(ii) Find how many groups of 1 or more digits can be formed from the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 where repetition is not allowed.

(b) (8 marks)



In the diagram an aircraft is flying with constant velocity U at a constant height h above horizontal ground. When the plane is at A it is directly over a gun at O. When the plane is at B a shell is fired from the gun at the aircraft along OB. The shell is fired with initial velocity V at an angle of elevation α .

(i) If x and y are the horizontal and vertical displacements of the shell from O at time t seconds, show that if g is the acceleration due to gravity,

$$x = Vt\cos\alpha \quad \text{and} \quad y = Vt\sin\alpha - \tfrac{1}{2}gt^2 \ .$$

(ii) Show that if the shell hits the aircraft at time T at point C, then

$$VT\cos\alpha = \frac{h}{\tan\alpha} + UT \ .$$

(iii) Show that if the shell hits the aircraft then $2U\,(V\cos\alpha - U)\tan^2\alpha = gh$.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

CSSA MATHEMATICS 3 UNIT SOLUTIONS 1996

QUESTION 1

(a) $2x - y = 0$ has gradient $m_1 = 2$.
$x + 3y = 0$ has gradient $m_2 = -1/3$.
Let θ be the acute angle between the lines.
$\therefore \tan\theta = \left|\dfrac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\dfrac{2 - (-1/3)}{1 + 2(-1/3)}\right| = 7$
$\therefore \theta = 81°52'$ to the nearest minute.

(b) $y = x\ln x - x$ has domain $x > 0$.

(i) $x\ln x - x = 0$
$\therefore x(\ln x - 1) = 0$
$\therefore$ ~~$x = 0$~~ (not in domain) or $\ln x = 1$
$\therefore x = e$.

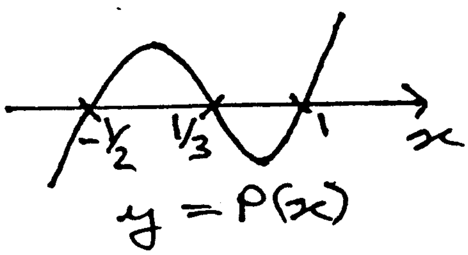
(ii) $dy/dx = (x)(1/x) + (\ln x)(1) - (1)$
$= 1 + \ln x - 1$
$= \ln x$
$d^2y/dx^2 = 1/x$.
$\therefore$ for $x > 0$, $d^2y/dx^2 > 0$ and the function is concave up for all x in its domain.

(c) $P(x) = 6x^3 - 5x^2 - 2x + 1$

(i) $P(1) = 6 - 5 - 2 + 1 = 0$
$\therefore 1$ is a zero of $P(x)$.

(ii) $(x - 1)$ is a factor of $P(x)$
$\therefore P(x) = 6x^3 - 5x^2 - 2x + 1$
$= (x-1)(6x^2 + x - 1)$
$= (x-1)(3x-1)(2x+1)$

(iii)



$P(x) \le 0$ when the graph lies on or below the x axis
$\therefore x \le -1/2$ or $1/3 \le x \le 1$.

QUESTION 2

(a) (i) $\frac{d}{dx} e^{\tan x} = e^{\tan x} \cdot \sec^2 x$
$= \sec^2 x \, e^{\tan x}$

(ii) $\int \frac{e^{\tan x}}{\cos^2 x} dx = \int \sec^2 x \, e^{\tan x} dx$
$= e^{\tan x} + c$

(b) $\int_{-3}^{0} \frac{x}{\sqrt{1-x}} dx$

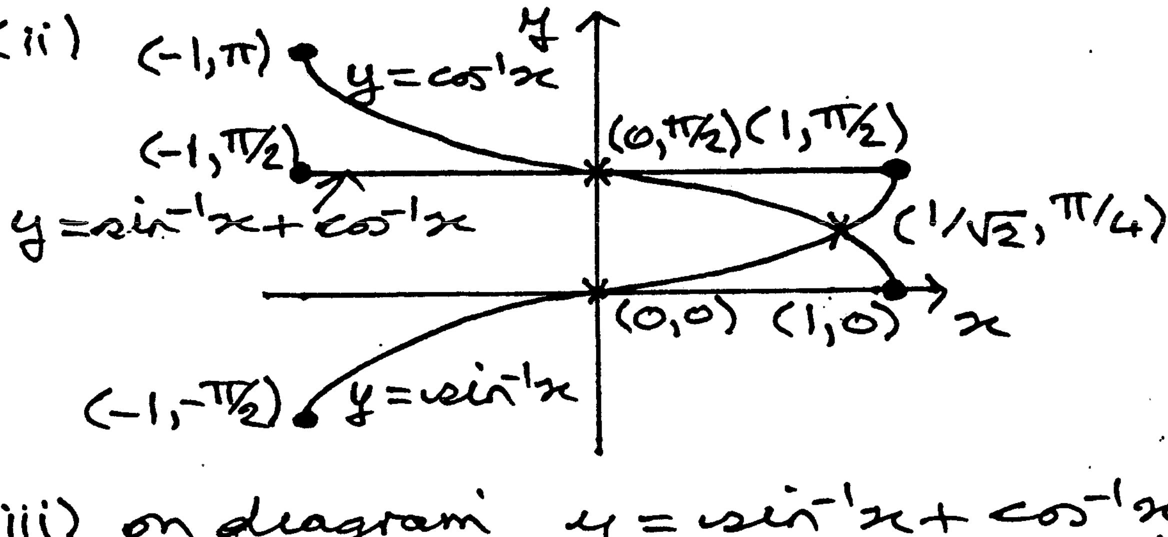
$u = 1 - x$
$x = 1 - u$
$dx/du = -1$
$dx = -du$
when $x = -3, u =$
when $x = 0, u =$

$= \int_4^1 \frac{(1-u)}{\sqrt{u}} \cdot (-du)$
$= \int_1^4 (u^{-1/2} - u^{1/2}) du$
$= \left[2\sqrt{u} - 2/3 \, u\sqrt{u} \right]_1^4$
$= \{4 - 16/3\} - \{2 - 2/3\}$
$= -8/3.$

(c) (i) Let $\sin^{-1} x = a$ $(-1 \le x \le 1, -\pi/2 \le a \le \pi/2)$
Let $\cos^{-1} x = a$ $(-1 \le x \le 1, \ 0 \le a \le \pi)$

$\therefore \left.\begin{array}{l} \sin a = x \\ \cos a = x \end{array}\right\}$ $\therefore \tan a = 1$, $\therefore a = \pi/4$
$\therefore x = 1/\sqrt{2}.$

(ii)



(iii) on diagram $y = \sin^{-1} x + \cos^{-1} x$
$\therefore y = \pi/2 \quad (-1 \le x \le 1)$

QUESTION 3

(a) (i) $f(-x) = \dfrac{8}{4+(-x)^2} = \dfrac{8}{4+x^2} = f(x)$

for all values of x. $\therefore$ function is even.

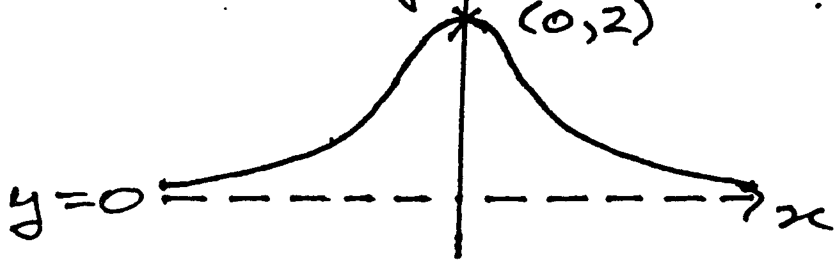
$\displaystyle\lim_{x \to \pm\infty} \frac{8}{4+x^2} = 0$ $\therefore y = 0$ is a horizontal asymptote.

(ii) $\dfrac{dy}{dx} = \dfrac{-16x}{(4+x^2)^2} = 0$ at stationary point $(0,2)$.

x	:	0^-	0	0^+
dy/dx	:	+ve	0	−ve

$\therefore (0,2)$ is a maximum turning point.

(iii)

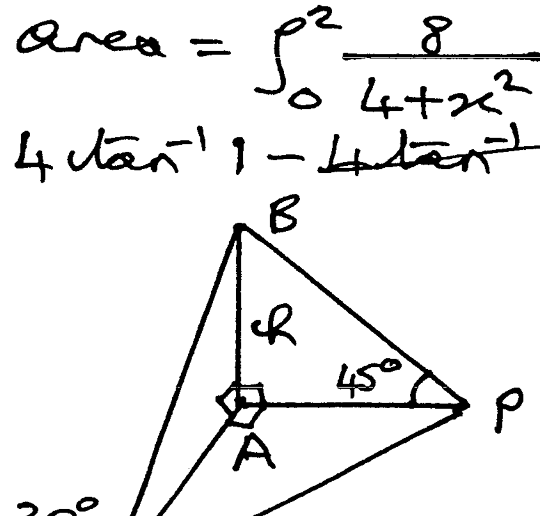


$y = \dfrac{8}{4+x^2}$

(iv) area $= \displaystyle\int_0^2 \frac{8}{4+x^2}\,dx = \left[8 \cdot \tfrac{1}{2}\tan^{-1}\tfrac{x}{2}\right]_0^2$

$= 4\tan^{-1}1 - \cancel{4\tan^{-1}0} = 4 \cdot \pi/4 = \pi$ units^2.

(b)



In ΔPAB, $\tan 45° = h/PA$

$\therefore PA = \dfrac{h}{\tan 45°} = h$

In ΔQAB, $\tan 30° = h/QA$

$\therefore QA = \dfrac{h}{\tan 30°} = h\sqrt{3}$.

In ΔPAQ, $PA^2 + QA^2 = PQ^2$

$\therefore (h)^2 + (h\sqrt{3})^2 = 40^2$, $\therefore h^2 + 3h^2 = 1600$

$\therefore 4h^2 = 1600$, $\therefore h^2 = 400$, $\therefore h = 20$ $(h > 0)$

$\therefore$ the height of the tower is 20 metres.

QUESTION 4

(a) (i) $LHS = dN/dt = d/dt\,(1000 + Ae^{-kt})$

$= -kAe^{-kt} = -k(N - 1000) = RHS.$

(ii) When $t = 0$, $N = 2500$, $\therefore 2500 = 1000 + A$

$\therefore A = 1500$, $\therefore N = 1000 + 1500e^{-kt}$.

When $t = 2$, $N = 2200$, $\therefore 2200 = 1000 + 1500e^{-2k}$

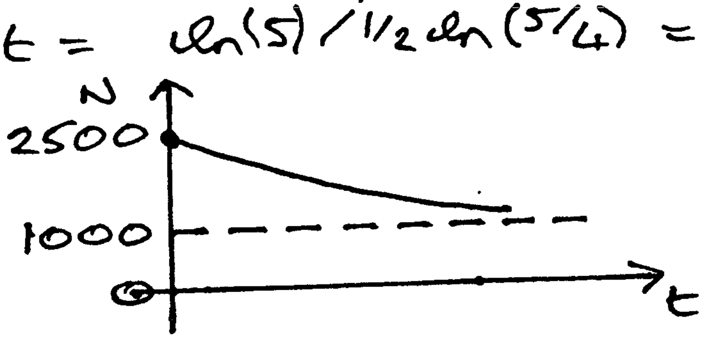
$\therefore e^{2k} = 5/4$, $\therefore k = \frac{1}{2}\ln(5/4)$.

(iii) When $N = 1300$, $1300 = 1000 + 1500e^{-kt}$

$\therefore e^{kt} = 5$, $\therefore t = \frac{1}{k}\ln(5)$

$\therefore t = \ln(5) / \frac{1}{2}\ln(5/4) = 14 \cdot 4$ years (to 1 d.p.)

(iv)



as $t \to \infty$, $N \to 1000$

$N = 1000$ is a horizontal asymptote

(b) $S(n)$: $\cos(x + n\pi) = (-1)^n \cos x$ $(n \geq 1)$

When $n = 1$, $LHS = \cos(x + \pi) = -\cos x$

$RHS = (-1)^1 \cos x = -\cos x$

$\therefore S(1)$ is true.

If $S(k)$ is true $(k \geq 1)$

i.e. if $\cos(x + k\pi) = (-1)^k \cos x$

then $\cos(x + (k+1)\pi)$

$= \cos((x + k\pi) + \pi)$

$= \cos(x + k\pi)\cos\pi - \sin(x + k\pi)\sin\pi$

$= (-1)^k \cos x \,.\, (-1) - \sin(x + k\pi) \,.\, 0$

$= (-1)^{k+1} \cos x$

and $S(k+1)$ is also true.

Since $S(1)$ is true and if $S(k)$ $(k \geq 1)$ is true then $S(k+1)$ is also true it follows that $S(n)$ is true for all positive integers $n \geq 1$.

$\therefore \cos(x + n\pi) = (-1)^n \cos x$ for all $n \geq 1$.

QUESTION 5

(a) (i) In ΔAPX and ΔBSX

$\angle APX = \angle BSX$ ($=90°$) (tangent $\perp$ radius)

$\angle PXA = \angle SXB$ (vertically opposite angles)

$\angle XAP = \angle XBS$ (alternate angles $AP \parallel SB$)

$\therefore \Delta APX \,|||\, \Delta BSX$ (equiangular)

(ii) $\frac{AX}{BX} = \frac{AP}{BS} \left(= \frac{5}{3}\right)$ (corresponding sides are proportional)

But $AB = 16$ cm $\therefore AX = 10$ cm, $BX = 6$ cm.

Now $\cos P\hat{A}X = \cos S\hat{B}X = 1/2$

$\therefore P\hat{A}X = S\hat{B}X = \pi/3$, $P\hat{A}Q = S\hat{B}R = 2\pi/3$

$\therefore$ Length of string

= length of major arc $PQ + PX + QX$

+ length of major arc $SR + SX + RX$

$= 5(4\pi/3) + 5\sqrt{3} + 5\sqrt{3} + 3(4\pi/3) + 3\sqrt{3} + 3\sqrt{3}$

$= (32\pi/3 + 16\sqrt{3})$ cm.

(b) (i) Area minor segment = 1/4 area circle

$\therefore \frac{1}{2}\cancel{r^2}(\theta - \sin\theta) = \frac{1}{4}\pi \cancel{r^2}$

$\therefore \theta - \sin\theta = \pi/2$.

(ii) $P(\theta) = \theta - \sin\theta - \pi/2$

$\therefore P'(\theta) = 1 - \cos\theta$

If $\theta = 2.5$ is a first approximation then a second approximation is

$$2.5 - \frac{P(2.5)}{P'(2.5)} = 2.5 - \frac{2.5 - \sin 2.5 - \pi/2}{1 - \cos 2.5}$$

$$= 2.32 \text{ (to 2 d.p.)}$$

and a third approximation is

$$2.32 - \frac{P(2.32)}{P'(2.32)} = 2.32 - \frac{2.32 - \sin 2.32 - \pi/2}{1 - \cos 2.32}$$

$$= 2.31 \text{ (to 2 d.p.)}$$

QUESTION 6

(a) (i) $(M_1, M_2, M_3)\ W_1\ W_2\ W_3\ W_4\ W_5\ W_6$

$\therefore$ P(the 3 men sit next to each other)

$= \dfrac{7! \times 3!}{9!} = \dfrac{7! \times 3 \times 2 \times 1}{9 \times 8 \times 7!} = \dfrac{1}{12}.$

(ii) (α) With repetition of people not allowed

P(exactly 3 of the prizes are awarded to women)

$= \dfrac{{}^6C_3 \times {}^3C_2}{{}^9C_5} = \dfrac{20 \times 3}{126} = \dfrac{10}{21}$

(β) With repetition of people allowed (binomial probability distribution)

P(exactly 3 of the prizes are awarded to women)

$= {}^5C_3 (6/9)^3 (3/9)^2$

$= 10 (8/27)(1/9) = 80/243.$

(b)(i) $x = a \cos nt \quad (a > 0,\ 0 < n < \pi)$

When $t = 1, x = 1 \quad \therefore a \cos n = 1$

When $t = 2, x = -1 \quad \therefore a \cos 2n = -1$

$\therefore a \cos n + a \cos 2n = 0$

$\therefore \cos n + \cos 2n = 0$

$\therefore \cos n + 2\cos^2 n - 1 = 0, \quad \therefore 2\cos^2 n + \cos n - 1 = 0$

$\therefore (2\cos n - 1)(\cos n + 1) = 0, \quad \therefore \cos n = 1/2$ or $\cos n = -1$

$\therefore n = \pi/3, 5\pi/3$ or $n = \pi$

$\therefore n = \pi/3 \quad (0 < n < \pi)$

$\therefore a \cos \pi/3 = 1, \quad \therefore a(1/2) = 1.$

$\therefore a = 2$

(ii) Amplitude $= a = 2$ metres.

Period $= \dfrac{2\pi}{n} = \dfrac{2\pi}{\pi/3} = 6$ seconds.

QUESTION 7

(a)(i) $(1+x)^n$

$= 1 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_r x^r + \dots + x^n$

$= {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_r x^r + \dots + {}^nC_n x^n$

$\therefore$ putting $x = 1$.

${}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n = 2^n$.

(ii) Total number of groups of 1 or more digits

$= {}^{10}C_1$ (groups of 1 digit) $+ {}^{10}C_2$ (groups of 2 digits) $+ \dots + {}^{10}C_{10}$ (groups of 10 digits)

$= {}^{10}C_1 + {}^{10}C_2 + \dots + {}^{10}C_{10}$

$= 2^{10} - {}^{10}C_0$

$= 1024 - 1$

$= 1023$.

(b)(i) Horizontally $a_x = 0$

$\therefore v_x = \int 0 \, dt$

$\therefore v_x = c_1$

When $t = 0$, $v_x = V\cos\alpha$ $\therefore c_1 = V\cos\alpha$

$\therefore v_x = V\cos\alpha$

$\therefore x = \int V\cos\alpha \, dt$

$\therefore x = (V\cos\alpha)t + c_2$

When $t = 0$, $x = 0$ $\therefore c_2 = 0$

$\therefore x = Vt\cos\alpha$.

Vertically $a_y = -g$

$\therefore v_y = \int -g \, dt$

$\therefore v_y = -gt + c_3$

When $t = 0$, $v_y = V\sin\alpha$ $\therefore c_3 = V\sin\alpha$

$\therefore v_y = V\sin\alpha - gt$

$\therefore y = \int (V\sin\alpha - gt) \, dt$

$\therefore y = (V\sin\alpha)t - \frac{1}{2}gt^2 + c_4$

When $t = 0$, $y = 0$ $\therefore c_4 = 0$

$\therefore y = Vt\sin\alpha - \frac{1}{2}gt^2$.

(ii) If the shell hits the aircraft at time T then horizontal distance AC (for shell)
= horizontal distance AB
+ horizontal distance BC (for aircraft).

$$\therefore VT\cos\alpha = \frac{h}{\tan\alpha} + UT \qquad (1)$$

(iii) If the shell hits the aircraft then also vertical distance (for shell)
= height h (for aircraft)

$$\therefore VT\sin\alpha - \tfrac{1}{2}gT^2 = h \qquad (2)$$

From (1), $VT\cos\alpha - UT = \dfrac{h}{\tan\alpha}$

$$\therefore T(V\cos\alpha - U) = \frac{h}{\tan\alpha}, \quad \therefore T = \frac{h}{(V\cos\alpha - U)\tan\alpha}$$

In (2),

$$\frac{V\sin\alpha \cdot h}{(V\cos\alpha - U)\tan\alpha} - \frac{gh^2}{2(V\cos\alpha - U)^2\tan^2\alpha} = h$$

$$\therefore V\cos\alpha \cdot 2(V\cos\alpha - U)\tan^2\alpha - gh$$
$$= 2(V\cos\alpha - U)^2\tan^2\alpha$$

$$\therefore 2V\cos\alpha(V\cos\alpha - U)\tan^2\alpha - 2(V\cos\alpha - U)^2\tan^2\alpha = gh$$

$$\therefore 2(V\cos\alpha - U)\tan^2\alpha[V\cos\alpha - (V\cos\alpha - U)] = gh$$

$$\therefore 2U(V\cos\alpha - U)\tan^2\alpha = gh.$$