CATHOLIC SECONDARY SCHOOLS' ASSOCIATION OF NEW SOUTH WALES

YEAR TWELVE FINAL TESTS 1997

# MATHEMATICS

## 3/4 UNIT COMMON PAPER

(i.e. 3 UNIT COURSE – ADDITIONAL PAPER:
4 UNIT COURSE – FIRST PAPER)

Afternoon session

Friday 8th August 1997.

*Time Allowed — Two Hours*


**EXAMINERS**

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**DIRECTIONS TO CANDIDATES :**

**ALL** questions may be attempted.

**ALL** questions are of equal value.

All necessary working should be shown in every question.

Full marks may not be awarded for careless or badly arranged work.

Approved calculators may be used.

Standard integrals are printed at the end of the exam paper.

USE RULER TO TEAR OFF HERE

## STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

Marks

## Question 1

(a) Show that $\dfrac{d}{dx}\dfrac{\tan x}{e^{2x}} = \dfrac{(1-\tan x)^2}{e^{2x}}$ 3

(b) Solve the equation $\sin 2x = \tan x$ for $0 \le x \le \pi$ 4

(c) 5

(i) Find the domain and range of the function $y = 4\cos^{-1}\left(\dfrac{x}{3}\right)$

(ii) Sketch the graph of the function $y = 4\cos^{-1}\left(\dfrac{x}{3}\right)$ showing clearly the intercepts on the coordinate axes and the coordinates of any endpoints.

(iii) Find the area of the region in the first quadrant bounded by the curve $y = 4\cos^{-1}\left(\dfrac{x}{3}\right)$ and the coordinate axes.

## Question 2

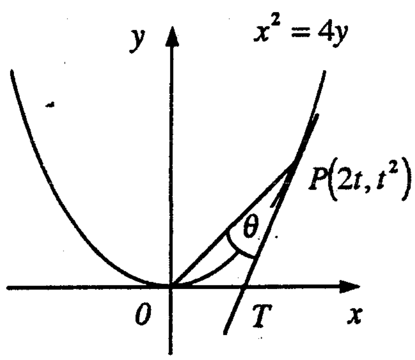
(a) Solve the inequality $\dfrac{1}{|x-3|} \ge \dfrac{1}{2}$ 3

(b)

$P(2t, t^2)$, where $t > 0$, is a point on the parabola $x^2 = 4y$. 4

The tangent to the parabola at $P$ cuts the $x$ axis at $T$.

$\angle OPT = \theta$ .




(i) Find the gradients of $OP$ and $TP$.

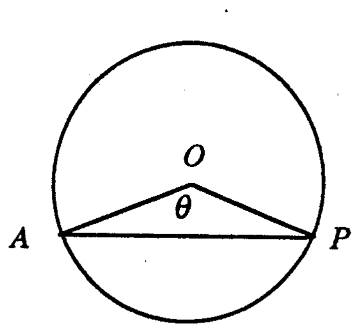
(ii) Show that $\tan\theta = \dfrac{t}{t^2+2}$

(c)

$A$ is a fixed point on a circle centre $O$, radius 1cm. 5

$P$ is a variable point which moves around the circle with a constant speed of one revolution per second.

$\angle AOP = \theta$ .




(i) Show that $\dfrac{d\theta}{dt} = 2\pi$ radians per second.

(ii) Find the rate at which the area of $\Delta AOP$ is changing when $\theta = \frac{2\pi}{3}$

## Question 3

**Marks**

(a) Find $\int \frac{1}{\sqrt{x}\left(1+\sqrt{x}\right)}\,dx$ using the substitution $x=u^2,\ u>0$. **3**

(b) The region bounded by the curve $y=\sin x$, the $x$ axis and the lines $x=\frac{\pi}{12}$ and $x=\frac{\pi}{4}$ is rotated through one complete revolution about the $x$ axis. Find the volume of the solid so formed. **4**

(c) A particle moving in Simple Harmonic Motion starts from rest at a distance 10 metres to the right of its centre of oscillation $O$. The period of the motion is 2 seconds. **5**

(i) Find the speed of the particle when it is 4 metres from its starting point.

(ii) Find the time taken by the particle to first reach the point 4 metres from its starting point, in seconds correct to 2 decimal places.

## Question 4

(a) One of the roots of the equation $x^3+ax^2+1=0$ is equal to the sum of the other two roots. **4**

(i) Show that $x=-\frac{a}{2}$ is a root of the equation.

(ii) Find the value of $a$.

(b) **8**

(i) Sketch the graph of the function $f(x)=e^x-4$ showing clearly the coordinates of any points of intersection with the axes and the equations of any asymptotes.

(ii) On the same diagram sketch the graph of the inverse function $f^{-1}(x)$ showing clearly the coordinates of any points of intersection with the axes and the equations of any asymptotes.

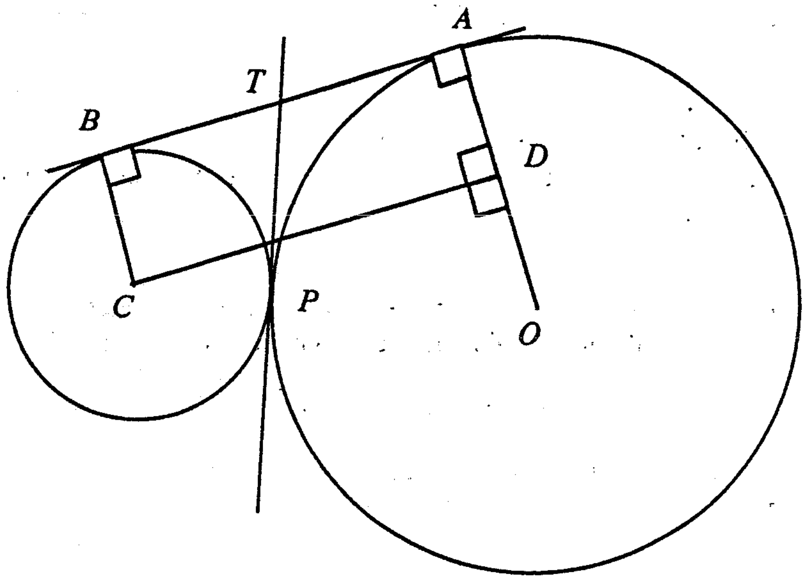
(iii) Explain why the $x$ coordinate of any point of intersection of the graphs $y=f(x)$ and $y=f^{-1}(x)$ satisfies the equation $e^x-x-4=0$.

(iv) Show the equation $e^x-x-4=0$ has a root between $x=0$ and $x=2$, and use the 'halving the interval' method to find this root correct to the nearest whole number.

## Question 5

Marks

(a) 6




Two circles, one with centre $C$ and radius 1cm and the other with centre $O$ and radius 3cm, touch each other externally at $P$. $AB$ is a common tangent to the two circles. $CD$ is drawn perpendicular to $OA$ to complete the rectangle $ABCD$. The common tangent to the two circles at $P$ meets $AB$ at $T$.

(i) Copy the diagram.

(ii) Find the exact length of $AB$ giving reasons for your answer.

(iii) Find the exact length of $PT$ giving reasons for your answer.

(b) A particle is moving on a straight line. At time t seconds it has displacement $x$ metres from a fixed point $O$ on the line, velocity $v$ ms$^{-1}$ and acceleration $a$ ms$^{-2}$. The particle starts from $O$ and at time $t$ seconds $v=(1-x)^2$. 6

(i) Find an expression for $a$ in terms of $x$.

(ii) Find an expression for $x$ in terms of $t$.

(iii) Find the time taken for the particle to slow down to a speed of 1% of its initial speed.

## Question 6 Marks

(a) 4

(i) Write down the Binomial expansion of $(1+x)^n$ in ascending powers of $x$.

(ii) Find the value of n such that the coefficient of $x^4$ is twice the coefficient of $x^3$.

(b) A motorway pay station has five toll gates, three of which are automatically operated and two of which are manually operated. Drivers with the exact money can use any one of the five gates, but drivers requiring change have to use one of the two manually operated gates. A Ford driver, a Holden driver and a Toyota driver use the motorway every day. 4

(i) On one day the Ford driver requires change but the other two drivers have the exact money. Find the number of ways in which the three drivers can go through the pay station so that they all use different gates.

(ii) On another day, all three drivers have the exact money. Find the number of ways in which the three drivers can go through the pay station so that exactly one goes through a manually operated gate.

(c) It is known that 5% of men are colourblind. A random sample of 20 men is chosen. 4

(i) Find the probability, correct to two decimal places, that the sample contains at most one colourblind man.

(ii) Find the probability, correct to two decimal places, that the sample contains at least two colourblind men.

## Question 7

(a) When a particle is fired in the open from a point $O$ at a speed of $40 \text{ ms}^{-1}$ and at an angle $\theta$ above the horizontal, where $0 < \theta < \frac{\pi}{2}$, *you may assume without proof* that the horizontal displacement ($x$ metres) and the vertical displacement ($y$ metres) of the particle from $O$ at time $t$ seconds after firing are given by 6

$$x = 40\,t\cos\theta \qquad \text{and} \qquad y = 40\,t\sin\theta - 5t^2$$

If the particle is fired with the same speed from a point $O$ on the floor of a horizontal tunnel of height 20 metres, find the maximum horizontal range of the particle along the tunnel.

(b) 6

(i) Use the method of mathematical induction to show that if $x$ is a positive integer then $(1+x)^n - 1$ is divisible by $x$ for all positive integers $n \geq 1$.

(ii) Factorise $12^n - 4^n - 3^n + 1$. Without using the method of mathematical again, use the result of part (i) to deduce that $12^n - 4^n - 3^n + 1$ is divisible by 6 for all positive integers $n \geq 1$.

# CSSA MATHEMATICS 3 UNIT SOLUTIONS 1997

## QUESTION 1

(a) $\dfrac{d}{dx}\dfrac{\tan x}{e^{2x}} = \dfrac{e^{2x}.\sec^2 x - \tan x.2e^{2x}}{(e^{2x})^2}$

$= \dfrac{\cancel{e^{2x}}(\sec^2 x - 2\tan x)}{(e^{2x})^{\cancel{2}}}$

$= \dfrac{1+\tan^2 x - 2\tan x}{e^{2x}} = \dfrac{(1-\tan x)^2}{e^{2x}}$

(b) $\sin 2x = \tan x \quad \therefore 2\sin x\cos x = \dfrac{\sin x}{\cos x}$

$\therefore 2\sin x\cos^2 x = \sin x \quad (\cos x \neq 0)$

$\therefore 2\sin x\cos^2 x - \sin x = 0$

$\therefore \sin x\,(2\cos^2 x - 1) = 0 \quad \therefore \sin x\cos 2x = 0$

$\therefore \sin x = 0$ or $\cos 2x = 0$

$\therefore x = 0$ or $\pi$ or $2x = \pi/2$ or $3\pi/2 \quad (0 \le x \le \pi)$

$x = \pi/4$ or $3\pi/4$

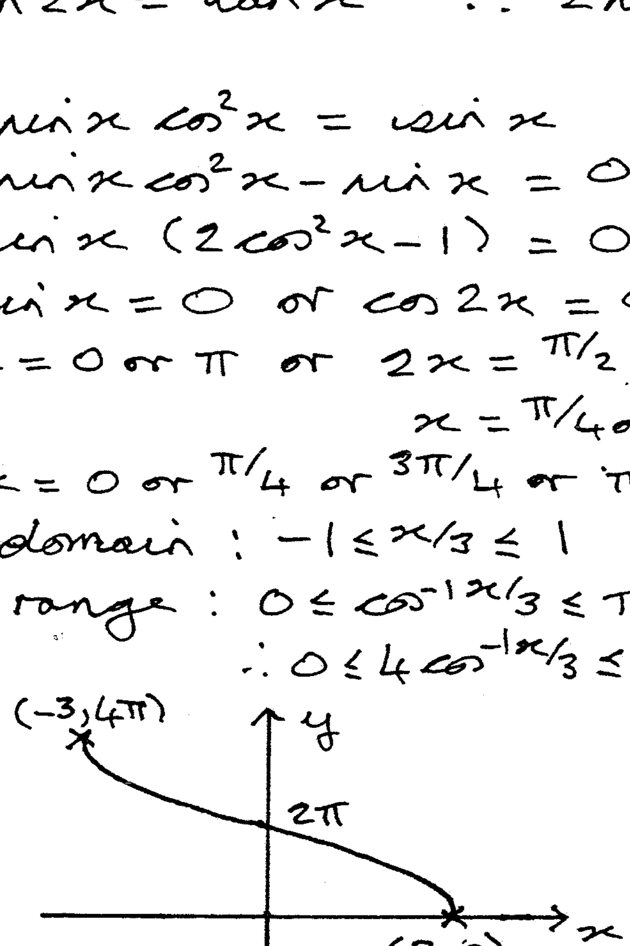
$\therefore x = 0$ or $\pi/4$ or $3\pi/4$ or $\pi \quad (0 \le x \le \pi)$

(c) (i) domain : $-1 \le x/3 \le 1 \quad \therefore -3 \le x \le 3$.

range : $0 \le \cos^{-1} x/3 \le \pi$

$\therefore 0 \le 4\cos^{-1} x/3 \le 4\pi \quad \therefore 0 \le y \le 4\pi$.

(ii)




$y = 4\cos^{-1} x/3$.

(iii) Area $= \int_0^{2\pi} x\,dy$

$= \int_0^{2\pi} 3\cos y/4\,dy$

$= \left[12\sin y/4\right]_0^{2\pi}$

$= 12(1) - 12(0)$

$= 12$ units$^2$.

QUESTION 2

(a) $\dfrac{1}{|x-3|} \geq \dfrac{1}{2}$ $\quad (\therefore x \neq 3)$

$\therefore |x-3| \leq 2 \quad (x \neq 3) \quad \therefore -2 \leq x-3 \leq 2 \ (x \neq 3)$

$\therefore 1 \leq x \leq 5 \quad (x \neq 3)$

$\therefore 1 \leq x < 3$ or $3 < x \leq 5$.

(b) (i) gradient OP : $m_{OP} = \dfrac{t^2-0}{2t-0} = \dfrac{t^2}{2t} = \dfrac{t}{2}$.

$x^2 = 4y \quad \therefore y = x^2/4$

$\therefore dy/dx = 2x/4 = x/2$

$\therefore$ at P, $dy/dx = 2t/2 = t$

$\therefore$ gradient of tangent at $P = t$

$\therefore$ gradient TP : $m_{TP} = t$.

(ii) $\tan\theta = \left| \dfrac{m_{TP} - m_{OP}}{1 + m_{TP}\, m_{OP}} \right|$

$= \left| \dfrac{t - t/2}{1 + t \cdot t/2} \right| = \left| \dfrac{(2t-t)/2}{(2+t^2)/2} \right|$

$= \left| \dfrac{t}{t^2+2} \right| = \dfrac{t}{t^2+2} \quad (t > 0)$

(c) (i) P moves around the circle at a constant speed of one revolution per second.

$\therefore \theta$ changes at a constant rate of $2\pi$ radians per second

$\therefore d\theta/dt = 2\pi$ radians per second.

(ii) area $\triangle AOP$ : $A = \frac{1}{2}(1)(1)\sin\theta \quad \therefore A = \frac{1}{2}\sin\theta$

$\therefore \dfrac{dA}{dt} = \dfrac{dA}{d\theta} \cdot \dfrac{d\theta}{dt} = \frac{1}{2}\cos\theta \cdot 2\pi = \pi\cos\theta$

$\therefore$ when $\theta = 2\pi/3$, $\dfrac{dA}{dt} = \pi\cos 2\pi/3 = \pi(-\frac{1}{2}) = -\pi/2$

$\therefore$ the area is decreasing at a rate of $\pi/2\ cm^2 s^{-1}$.

QUESTION 3

(a) $\int \frac{1}{\sqrt{x}(1+\sqrt{x})}\,dx$ $\qquad x = u^2 \quad (u = \sqrt{x})$

$\therefore \frac{dx}{du} = 2u$

$= \int \frac{1}{u(1+u)}\,2u\,du$ $\qquad \therefore dx = 2u\,du$

$= 2\ln(1+u) + c$

$= 2\ln(1+\sqrt{x}) + c.$

(b) Volume $= \pi \int_{\pi/12}^{\pi/4} y^2\,dx$

$= \pi \int_{\pi/12}^{\pi/4} \sin^2 x\,dx$

$= \pi \int_{\pi/12}^{\pi/4} \frac{1}{2}(1-\cos 2x)\,dx$

$= \pi \left[ x/2 - \frac{1}{4}\sin 2x \right]_{\pi/12}^{\pi/4}$

$= \pi \{ (\pi/8 - \frac{1}{4}(1)) - (\pi/24 - \frac{1}{4}(\frac{1}{2})) \}$

$= \pi/24\,(3\pi - 6 - \pi + 3)$

$= \pi/24\,(2\pi - 3)$ units$^3$.

(c) $a = 10$

$T = 2\pi/n \quad \therefore 2 = 2\pi/n \quad \therefore n = 2\pi/2 \quad \therefore n = \pi$

(i) $v^2 = n^2(a^2 - x^2) \quad \therefore v^2 = \pi^2(100 - x^2)$

$\therefore$ when $x = 6$, $v^2 = \pi^2(100 - 36)$

$\therefore v^2 = 64\pi^2 \quad \therefore |v| = 8\pi$

$\therefore$ speed $= 8\pi$ ms$^{-1}$.

(ii) $x = a\cos(nt + \alpha) \quad \therefore x = 10\cos(\pi t + \alpha)$

when $t = 0$, $x = 10 \quad \therefore \alpha = 0$.

$\therefore x = 10\cos \pi t$

$\therefore$ when $x = 6$, $6 = 10\cos \pi t$

$\therefore \cos \pi t = 0.6 \quad \therefore t = 0.30$ (to 2 d.p.)

$\therefore$ time taken $= 0.30$ seconds (to 2 d.p.)

QUESTION 4

(a) (i) Let the roots of $x^3 + ax^2 + 1 = 0$ be $\alpha, \beta$ and $\gamma$ where $\gamma = \alpha + \beta$

sum of roots : $\alpha + \beta + \gamma = -{(a)}/{(1)}$

$\therefore 2\gamma = -a \quad \therefore \gamma = -a/2$

$\therefore x = -a/2$ is a root of the equation.

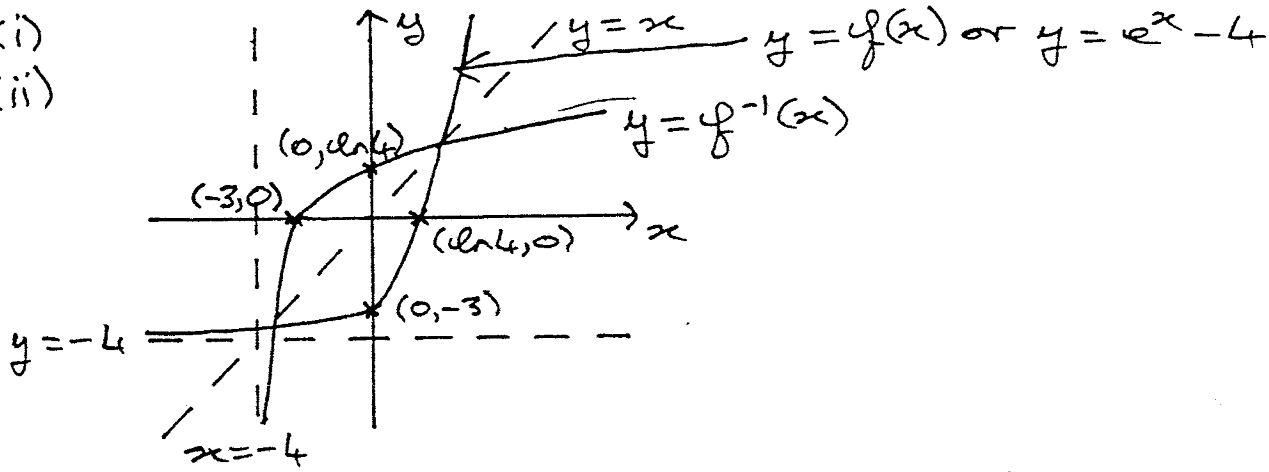
(ii) $(-a/2)^3 + a(-a/2)^2 + 1 = 0$

$\therefore -a^3/8 + a^3/4 + 1 = 0 \quad \therefore -a^3 + 2a^3 + 8 = 0$

$\therefore a^3 = -8 \quad \therefore a = -2$.

(b) (i)

(ii)




(iii) The x coordinate of any point of intersection of the graphs $y = f(x)$ and $y = f^{-1}(x)$ lies on the line $y = x$, and satisfies the equation $f(x) = x$, ie. $e^x - 4 = x$, ie. $e^x - x - 4 = 0$.

(iv) Let $P(x) = e^x - x - 4$

$\therefore P(0) = -3 < 0$

$P(2) = e^2 - 6 = 1{\cdot}39$ (to 2d.p.) $> 0$

$\therefore$ $P(x)$ is continuous and $P(0)$ and $P(2)$ have opposite signs. $\therefore P(x) = 0$ or $e^x - x - 4 = 0$ has a root between $x = 0$ and $x = 2$.

$P(1) = e - 5 = -2{\cdot}28$ (to 2d.p.) $< 0$

$\therefore$ root lies between $x = 1$ and $x = 2$.

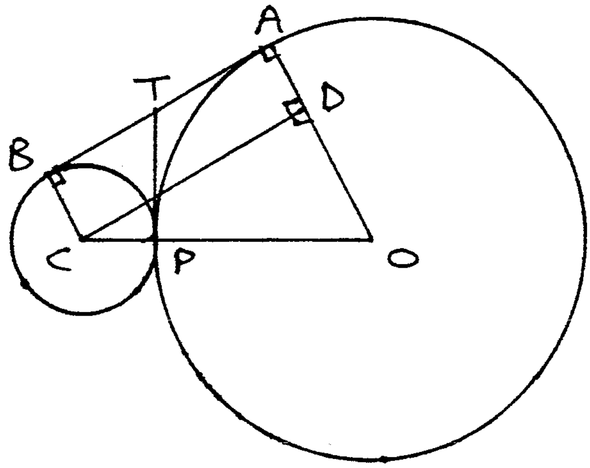
$P(1{\cdot}5) = e^{1{\cdot}5} - 5{\cdot}5 = -1{\cdot}02$ (to 2d.p.) $< 0$

$\therefore$ root lies between $x = 1{\cdot}5$ and $x = 2$

$\therefore$ root is $x = 2$ (correct to nearest whole number)

QUESTION 5

(a) (i)




(ii) Draw the line CPO (line of centres passes through point of contact)

$\therefore CO = CP + PO = 1 + 3 = 4$

$DO = AO - AD = 3 - 1 = 2$

$CO^2 = CD^2 + DO^2$

$\therefore CD^2 = 4^2 - 2^2 = 12$

$\therefore CD = \sqrt{12} = 2\sqrt{3}$

$\therefore AB = CD = 2\sqrt{3}$ cm.

(iii) $TB = TP$ (tangents from T to circle centre C are equal)

$TA = TP$ (tangents from T to circle centre O are equal)

$\therefore TB = TA = \frac{1}{2}AB = \frac{1}{2}(2\sqrt{3}) = \sqrt{3}$

$\therefore PT = TB = TA = \sqrt{3}$ cm.

(b) (i) $v = (1-x)^2$

$\therefore a = \frac{d}{dx}(\frac{1}{2}v^2) = \frac{d}{dx}(\frac{1}{2}(1-x)^4) = -2(1-x)^3$.

(ii) $v = (1-x)^2$

$\therefore dx/dt = (1-x)^2 \quad \therefore dt/dx = 1/(1-x)^2$

$\therefore t = \int \frac{1}{(1-x)^2}\,dx \quad \therefore t = \frac{1}{1-x} + c$

When $t = 0, x = 0 \quad \therefore c = -1$

$\therefore t = \frac{1}{1-x} - 1 \quad \therefore \frac{1}{1-x} = t + 1$

$\therefore 1 - x = \frac{1}{t+1} \quad \therefore x = 1 - \frac{1}{t+1}$

$\therefore x = \frac{t+1-1}{t+1} \quad \therefore x = \frac{t}{t+1}$.

(iii) When $t = 0, x = 0, v = 1 \quad \therefore$ initial speed $v = 1$

$\therefore$ when $v = 1/100, \quad (1-x)^2 = 1/100$

$\therefore 1 - x = 1/10 \quad \therefore x = 1 - 1/10 \quad \therefore x = 9/10$

$\therefore$ when $x = 9/10, \quad t/(t+1) = 9/10$

$\therefore 10t = 9(t+1) \quad \therefore 10t = 9t + 9 \quad \therefore t = 9$

$\therefore$ the particle takes 9 seconds to slow down to a speed of 1% of its initial speed.

QUESTION 6

(a) (i) $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r$

$= 1 + {}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + {}^nC_4 x^4 + \dots + x^n$

(ii) coefficient of $x^4$ = 2 coefficient of $x^3$

$\therefore {}^nC_4 = 2\,{}^nC_3$

$\therefore \frac{\cancel{n(n-1)(n-2)}(n-3)}{\cancel{1\times2\times3}\times4} = 2\,\frac{\cancel{n(n-1)(n-2)}}{\cancel{1\times2\times3}}$

$\therefore n-3 = 2\times4 \quad \therefore n-3 = 8 \quad \therefore n = 11.$

(b) 2 manual gates $M_1, M_2$. 3 automatic gates $A_1, A_2, A_3$

(i) the Ford driver has 2 choices,
then the Holden driver has 4 choices,
then the Toyota driver has 3 choices.
$\therefore$ number of ways $= 2\times4\times3 = 24$.

(ii) there are ${}^3C_1 = 3$ ways of choosing the one driver to go through a manually operated gate,
this driver has 2 choices,
the other two drivers have $3\times3$ choices.
$\therefore$ number of ways $= 3\times2\times3\times3 = 54$.

(c) (i) P(at most one colourblind man)

$= P(\text{none}) + P(\text{one})$

$= (0{\cdot}95)^{20} + {}^{20}C_1 (0{\cdot}95)^{19}(0{\cdot}05)$

$= 0{\cdot}74$ (to 2 d.p.)

(ii) P(at least two colourblind men)

$= 1 - P(\text{at most one colourblind man})$

$= 1 - 0{\cdot}74$ (to 2 d.p.)

$= 0{\cdot}26$ (to 2 d.p.)

QUESTION 7

(a) $y = 40t\sin\theta - 5t^2 = 5t(8\sin\theta - t)$

$\therefore y = 0$ when $t = 0$ or $t = 8\sin\theta$ (time of flight)

when $t = 8\sin\theta$, $x = 40(8\sin\theta)\cos\theta$

$\therefore x = 320\sin\theta\cos\theta = 160\sin 2\theta$ (horizontal range)

$\therefore v_y = dy/dt = 40\sin\theta - 10t = 10(4\sin\theta - t)$

$\therefore$ at maximum height $v_y = 0$ $\therefore t = 4\sin\theta$

when $t = 4\sin\theta$, $y = 40(4\sin\theta)\sin\theta - 5(4\sin\theta)^2$

$\therefore y = 160\sin^2\theta - 80\sin^2\theta = 80\sin^2\theta$ (maximum height)

now maximum height $\leq 20$ (height of tunnel)

$\therefore 80\sin^2\theta \leq 20$ $\therefore \sin^2\theta \leq 1/4$

$\therefore \sin\theta \leq 1/2$ $\therefore 0 < \theta \leq \pi/6$ $(0 < \theta < \pi/2)$

$\therefore$ when $\theta = \pi/6$, maximum horizontal range

$= 160\sin 2\pi/6 = 160 . \sqrt{3}/2 = 80\sqrt{3}$ metres.

(b)(i) $S(n)$: $(1+x)^n - 1$ is divisible by $x$

When $n=1$, $(1+x)^1 - 1 = 1 + x - 1 = x = x(1)$

$\therefore S(1)$ is true.

If $S(k)$ is true for some $k \geq 1$, so that

$(1+x)^k - 1 = x(M)$ for some integer $M$, then

$$(1+x)^{k+1} - 1 = (1+x)^k(1+x) - 1$$
$$= (Mx+1)(1+x) - 1 = Mx + \cancel{1} + Mx^2 + x - \cancel{1}$$
$$= x(Mx + M + 1)$$
$$= x(N) \text{ for some integer } N,$$

and so $S(k+1)$ is also true.

Since $S(1)$ is true and if $S(k)$ is true for $k \geq 1$ then $S(k+1)$ is also true, it follows that $S(1+1) = S(2)$ is true, $S(2+1) = S(3)$ is true etc.

$\therefore S(n)$ is true for all positive integers $n \geq 1$.

(ii) $12^n - 4^n - 3^n + 1 = (4^n - 1)(3^n - 1)$

From (i), $4^n - 1$ is divisible by 3 and $3^n - 1$ is divisible by 2 for all positive integers $n \geq 1$.

$\therefore (4^n - 1)(3^n - 1)$ is divisible by $3 \times 2 = 6$.

$\therefore 12^n - 4^n - 3^n + 1$ is divisible by 6 for all positive integers $n \geq 1$.