CATHOLIC SECONDARY SCHOOLS' ASSOCIATION OF NEW SOUTH WALES

YEAR TWELVE FINAL TESTS 1999

MATHEMATICS
2/3 UNIT

Morning session

Wednesday 11 August 1999

Time allowed – three hours
(Plus five minutes reading time)

EXAMINERS

P Rockett
B. Cosgrove
R Pantua
J Mann
A Kollias
E Rainert
M Donaghy
A McGahan
C Reichel
K Breen

DIRECTIONS TO CANDIDATES:

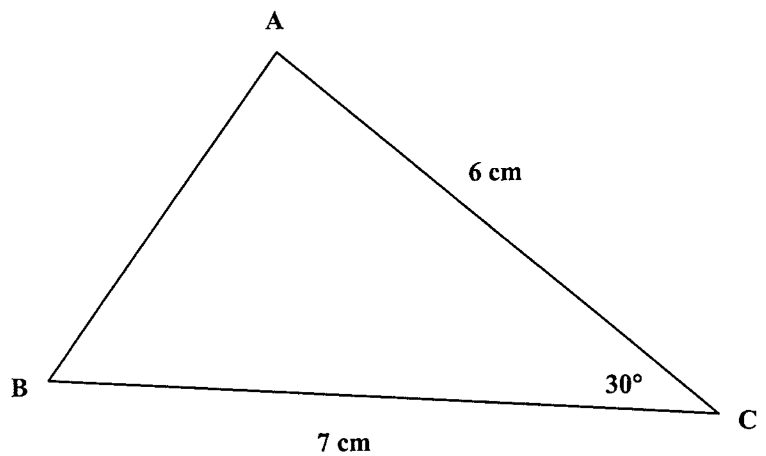
- **ALL** questions may be attempted.
- **ALL** questions are of equal value.
- All necessary working should be shown in every question.
- Full marks may not be awarded for careless or badly arranged work.
- Approved slide rules or calculators may be used.
- Table of standard integrals is printed at the end of the paper.
- The answers to the ten questions in this paper are to be returned in separate writing sheets clearly marked **QUESTION 1, QUESTION 2** etc. on the top of the sheet.
- If required, additional writing sheets may be obtained from the examinations supervisor upon request.

QUESTION 1 *(Begin a new sheet)* **Marks**

(a) Express $\frac{2}{4+\sqrt{3}}$. with a rational denominator. **1**

(b) Find the values of x for which $6x^2 - x - 2 = 0$. **2**

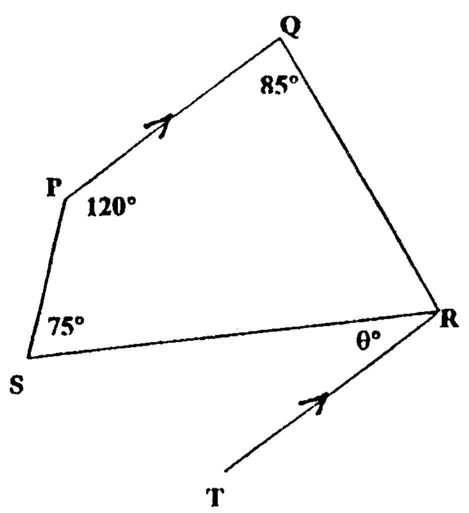
(c) **2**



NOT TO SCALE

In the diagram, AC = 6 cm, BC = 7 cm and $\angle$ ACB = 30°. Use the cosine rule to find the length of AB correct to the nearest cm.

(d) **2**



NOT TO SCALE

In the diagram, PQ||TR, $\angle$PQR = 85°, $\angle$QPS = 120°, $\angle$PSR = 75° and $\angle$SRT = θ°.

Copy the diagram onto your answer sheet.

Find the value of θ.

(e) Find the values of y for which $|9y - 11| > 7$. **2**

(f) Find the primitive function of $\sec^2 4x$. **1**

(g) A country property increased in value by 12½ % to a new value of \$36 000. What was the value of the property before the increase? **2**

QUESTION 2 *(Begin a new sheet)* **Marks**

(a) Differentiate the following functions: **6**

(i) $\sqrt{3x^2 + 2}$

(ii) $(x + 1)\ ln\ x$

(iii) $\dfrac{x}{\sin 2x}$

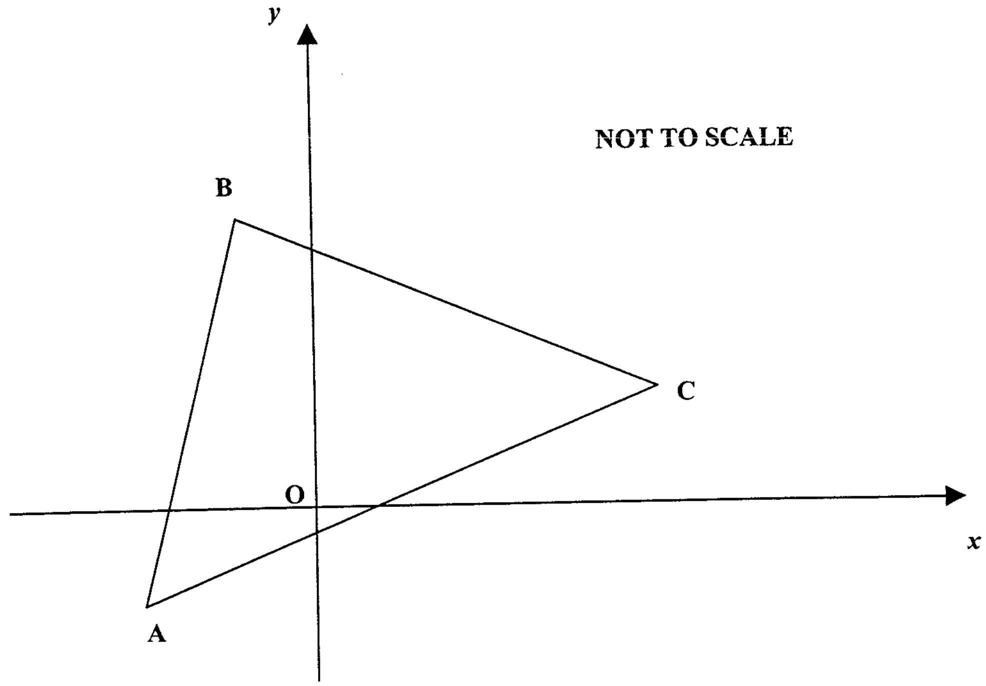
(b) Find: **3**

(i) $\int (x - \frac{2}{x^3})\ dx$

(ii) $\int e^{3x+2}\ dx$

(c) Find the exact value of $\int_0^{\pi/2} \cos\frac{x}{2}\ dx$ **3**

QUESTION 3 *(Begin a new sheet)* **Marks**



The diagram shows points A(-3,-2), B(-1,4) and C(5,2) in the Cartesian plane.

Copy this diagram onto your answer sheet.

(a) Find the gradient of AC. **1**

(b) Point P is the midpoint of AC. Show that the coordinates of P are (1,0). Mark point P on your diagram. **1**

(c) Show that the equation of the line perpendicular to AC and passing through the point P is $2x + y - 2 = 0$. **2**

(d) Show that B lies on the line $2x + y - 2 = 0$. **1**

(e) Show that the length of BP is $2\sqrt{5}$ units. **1**

(f) Point P is the midpoint of the interval BD. **2**
  (i) On your diagram show the position of point D.
  (ii) Find the coordinates of D.

(g) Explain why the quadrilateral ABCD is a rhombus. **1**

(h) Find the area of ΔBPC. **2**

(i) Hence find the area of the rhombus ABCD. **1**

QUESTION 4 *(Begin a new sheet)* **Marks**

(a) The graph of $y = f(x)$ passes through the point (-1, 4) and $f'(x) = 5 - 3x^2$. Find $f(x)$. **2**

(b) The following table gives five values of the function $y = f(x)$. **2**

x	0	1	2	3	4
$f(x)$	1	0·5	0·41	0·37	0·33

Use the five function values and Simpson's rule to approximate $\int_0^4 f(x)\,dx$.

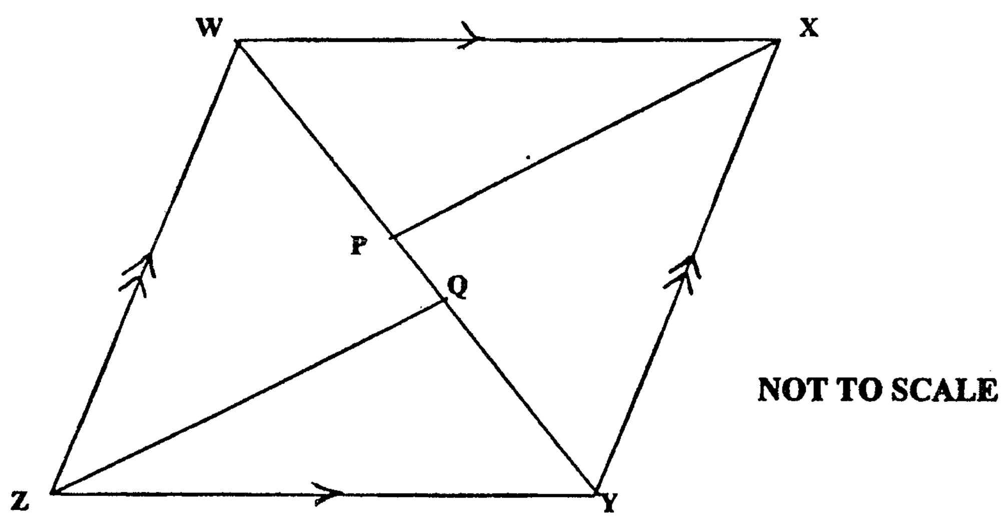
(Give your answer correct to 2 decimal places.)

(c) The equation of a parabola is $(x-3)^2 = -12(y-1)$. Find the: **3**

(i) coordinates of its vertex.

(ii) equation of its directrix.

(d) **5**



WXYZ is a parallelogram. XP bisects ∠WXY and ZQ bisects ∠WZY.

Copy the diagram onto your answer sheet.

(i) Explain why ∠WXY = ∠WZY.

(ii) Prove Δ WXP is congruent to Δ YZQ.

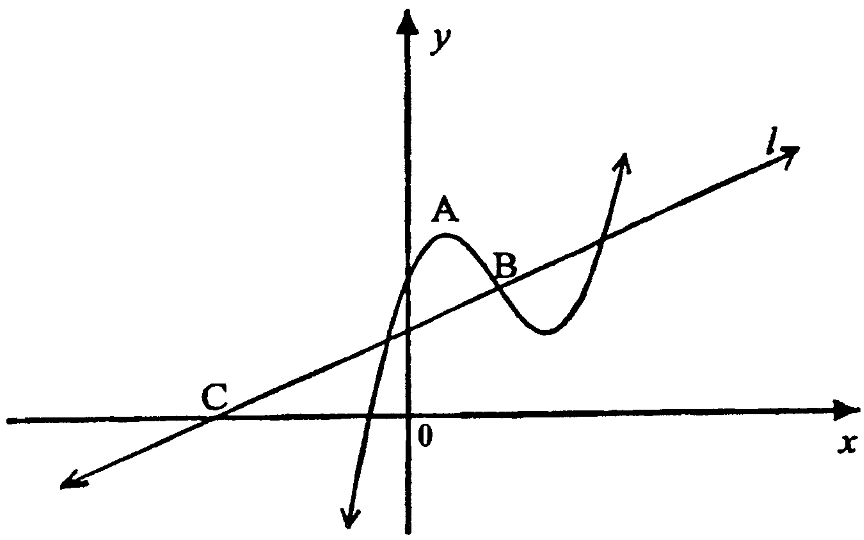
(iii) Hence find the length of PQ given WY = 20 cm and QY = 8 cm.

QUESTION 5 *(Begin a new sheet)* **Marks**

(a) (i) Write down the discriminant of $x^2 - 2kx + 6k$. **3**

(ii) For what values of k is $x^2 - 2kx + 6k$ always positive?

(b) **9**



NOT TO SCALE

The diagram shows a sketch of the curve $y = x^3 - 6x^2 + 9x + 4$. The curve has a local maximum point at A and a point of inflexion at B. The line l is a normal to the curve at point B and meets the x axis at point C.

(i) Find the coordinates of point A.

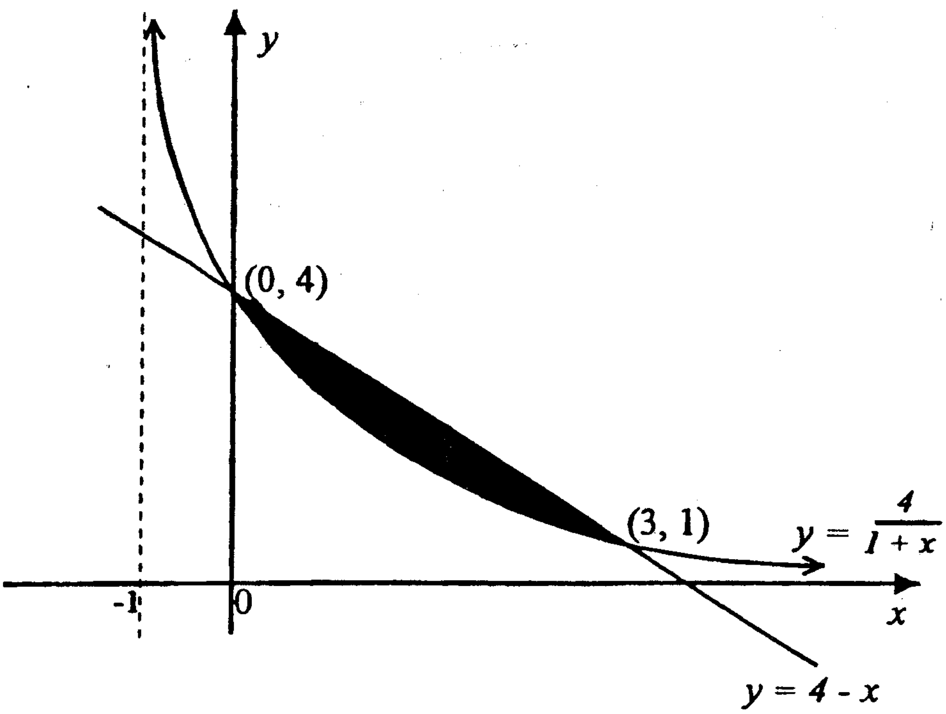
(ii) Show that the coordinates of point B is (2,6).

(iii) Show that the equation of the line l is $x - 3y + 16 = 0$.

(iv) Find the coordinates of point C.

QUESTION 6 *(Begin a new sheet)* **Marks**

(a) 3



NOT TO SCALE

The diagram shows part of the hyperbola $y = \frac{4}{1+x}$ and the line $y = 4 - x$.
The hyperbola and line intersect at the points (0,4) and (3,1).
Calculate the exact area of the shaded region.

(b) In an opinion poll conducted at a school, it was found that 70% of the students favoured an Olympic parade through the city before the opening of the Olympic Games. Three students were selected at random from the school and interviewed separately. 4

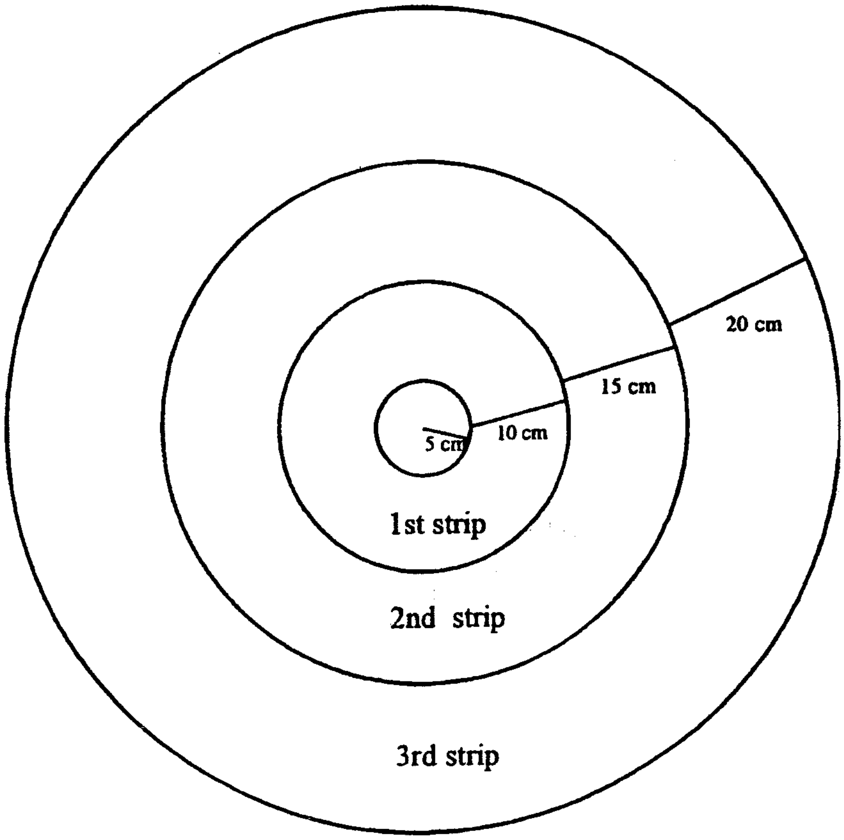
By drawing a tree diagram, or otherwise, find the probability that:

(i) all three students favoured the parade.

(ii) at least one of the students favoured the parade.

QUESTION 6 *(Continued)* **Marks**

(c) 5



NOT TO SCALE

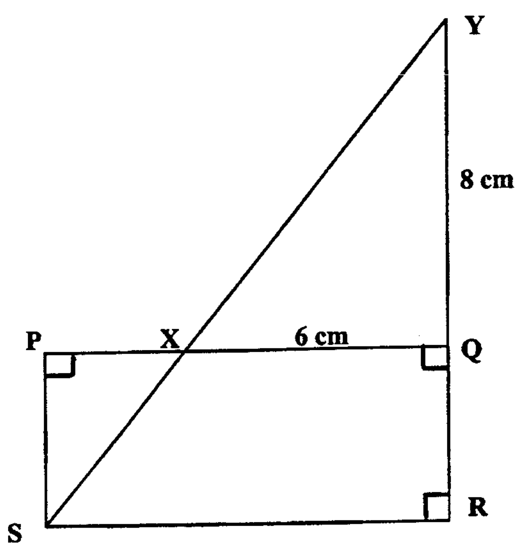
Beginning with a circular piece of fabric of radius 5 cm, Le sewed together circular strips of different coloured fabrics which increased in width to make a circular tablecloth. The finished width of the first strip was 10cm, the second was 15 cm, the third was 20 cm and so on.

(i) Show that the width of the tenth strip was 55 cm.

(ii) The radius of the table cloth was 455 cm. How many strips were sewn to the edge of the first circular piece?

QUESTION 7 *(Begin a new sheet)* **Marks**

(a) 3



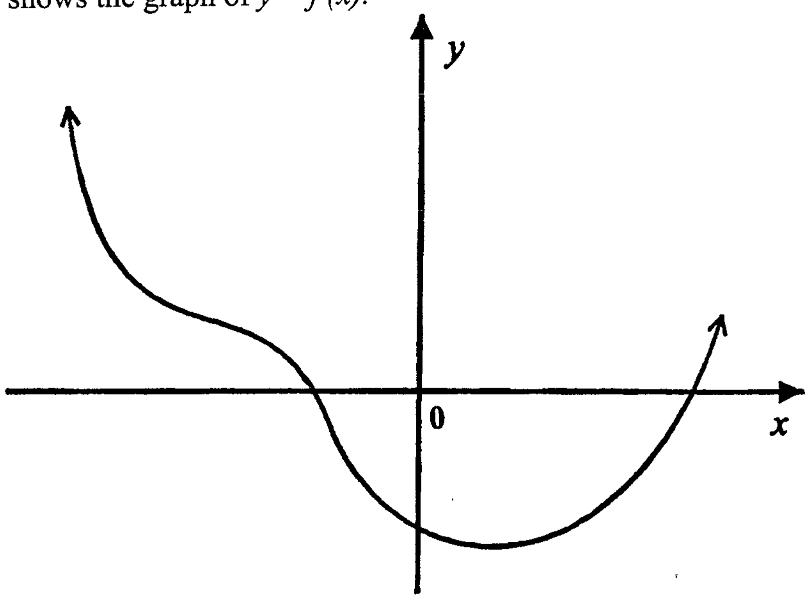
NOT TO SCALE

In the diagram, PQRS is a rectangle and SR = 3 PS. R, Q and Y are collinear points. XQ = 6 cm and YQ = 8 cm.

(i) Prove ΔPXS is similar to ΔQXY.

(ii) Hence find the length of PS.

(b) The graph shows the graph of $y = f(x)$. 2



(i) Copy this graph onto your answer sheet.

(ii) On the same set of axes, sketch the graph of its derivative, $f'(x)$.

(c) Consider the function $y = \sin x + \cos x$ in the domain $0 \le x \le 2\pi$. **7**

(i) Find $\dfrac{dy}{dx}$.

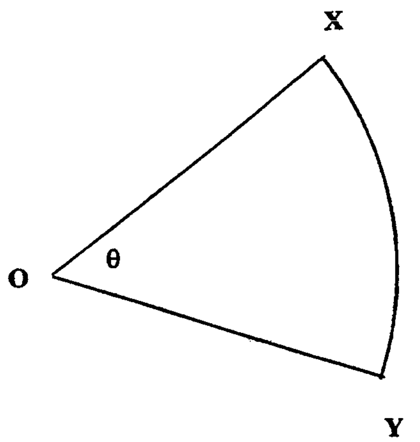
(ii) Find the maximum and minimum values of $\sin x + \cos x$ in the given domain.

(iii) Show that the curve cuts the x axis at $x = \dfrac{3\pi}{4}$ and at $x = \dfrac{7\pi}{4}$.

(iv) Hence sketch the curve of $y = \sin x + \cos x$ in the domain $0 \le x \le 2\pi$.

QUESTION 8 *(Begin a new sheet)* **Marks**

(a) **2**



NOT TO SCALE

In the diagram, XY is an arc of a circle with centre O and radius 12 cm. The length of the arc XY is 4π cm. Find the:

(i) exact size of θ in radians.

(ii) area of the sector OXY.

(b) The region bounded by the curve $y = e^x + e^{-x}$, the x axis and the lines $x = 0$ and $x = 2$ is rotated about the x axis. Find the volume of the solid formed. (Leave your answer in terms of e.) **3**

(c) A particle moves along a straight line about a fixed point O so that its acceleration, a ms^{-2}, at time t seconds is given by $a = 8\cos\left(2t + \frac{\pi}{6}\right)$. Initially the particle is moving to the right with a velocity of 2 ms^{-1} from a position $\sqrt{3}$ metres to the left of O. **7**

(i) Find expressions for the velocity and position of the particle at any time t.

(ii) Show that the particle changes directions when $t = \frac{5\pi}{12}$ seconds.

(iii) At what time does the particle return to its initial position for the first time?

QUESTION 9 *(Begin a new sheet)* **Marks**

(a) A die numbered 1 to 6 is rolled twice. The sum S of the numbers which appear uppermost on the die is calculated. **4**

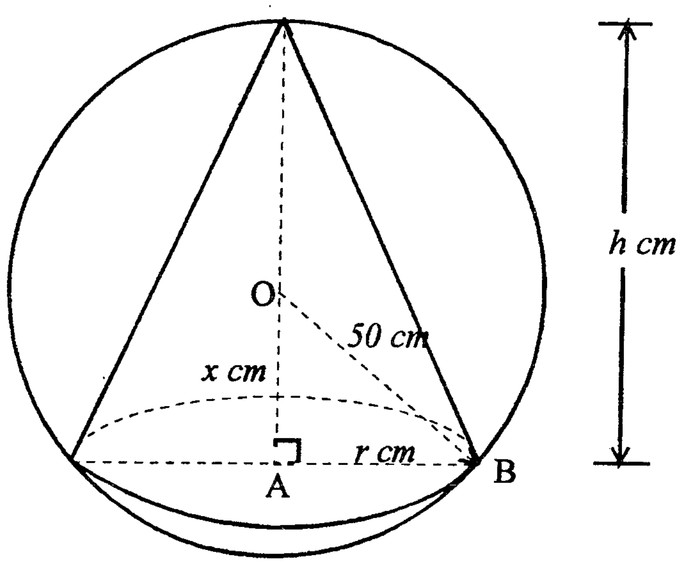
(i) Find the probability that S is greater than 9.

(ii) It is known that 5 appears on the die at least once in the two throws. Find the probability that S is greater than 9.

(b) *The median house price, $P, in a certain suburb is falling at an increasing rate after a recent peak.* **2**

What does this tell you about $\frac{dP}{dt}$ and $\frac{d^2P}{dt^2}$?

(c) **6**



NOT TO SCALE

The diagram shows a cone of base radius r cm and height h cm inscribed in a sphere of radius 50 cm. The centre of the sphere is O and $\angle OAB = 90°$.

Let $OA = x$ cm.

(i) Show that $r = \sqrt{2500 - x^2}$.

(ii) Hence show that the volume, V cm^3, of the cone is given by:

$$V = \frac{\pi}{3}(2500 - x^2)(50 + x)$$

(iii) Find the radius of the largest cone which can be inscribed in the sphere. (Give your answer to the nearest mm.)

QUESTION 10 *(Begin a new sheet)* **Marks**

(a) In a fish hatchery, the fish population, N, satisfies the equation $N = N_0 e^{kt}$ where N_0 and k are constants and t is measured in months. **6**

(i) Initially there were 1 000 fish in the hatchery and at the end of 5 months there were 5000. Find the value of k correct to 3 decimal places.

(ii) Find the number of fish in the hatchery at the end of 8 months. (Give your answer correct to the nearest hundred.)

(iii) At the end of which month will the fish population exceed 50 000 for the first time?

(iv) At what rate is the population increasing at the end of six months? (Give your answer correct to the nearest hundred fish per month.)

(b) Mario and Fei Lin worked out that they would save \$50 000 in five years by depositing all their combined monthly salary of \$S at the beginning of each month into a savings account and withdrawing \$1 600 at the end of each month for living expenses. The savings account paid interest at the rate of 3% p.a. compounding monthly. **6**

(i) Show that at the end of the second month, the balance in their savings account, immediately after making their withdrawal of \$1 600, would be given by $\$[(1.0025^2 + 1.0025)S - 1\,600\,(1.0025 + 1)]$.

(ii) Hence write down an expression for the balance in their account at the end of the sixtieth month.

(iii) Calculate their combined monthly salary.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{(a^2-x^2)}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0,\; -a < x < a$$

$$\int \frac{1}{\sqrt{(x^2-a^2)}}\,dx = \ln\left\{x+\sqrt{(x^2-a^2)}\right\}, \quad |x| > |a|$$

$$\int \frac{1}{\sqrt{(x^2+a^2)}}\,dx = \ln\left\{x+\sqrt{(x^2+a^2)}\right\}$$

NOTE : $\ln x = \log_e x; \quad x > 0$

(1)(a) $\frac{2\left(4-\sqrt{3}\right)}{13}$

(b) $x = \frac{2}{3}, \ -\frac{1}{2}$

(c) 4 cm (to the nearest cm)

(d) 15^0

(e) $y > 2$ or $y < \frac{4}{9}$

(f) $\frac{1}{4}\tan 4x + c$

(g) $32 000

(2) (a) (i) $\frac{3x}{\sqrt{3x^2+2}}$

(ii) $1 + \ln x + \frac{1}{x}$

(ii) $\frac{\sin 2x - 2x\cos 2x}{\sin^2 x}$

(b) (i) $\frac{x^2}{2} + \frac{1}{x^2} + c$

(ii) $\frac{1}{3}e^{3x+2} + c$

(c) $\sqrt{2}$

(3)(a) $\frac{1}{2}$ (b) $(1, 0)$ (c) Proof

(d) Proof (e) Proof

(f) (i) Diagram (ii) $D(3, -4)$

(g) Diagonals bisect each other

(h) 10 sq units (c) 40 sq units

(4) (a) $f(x) = 5x - x^3 + 8$

(b) Area ≈ 1.88 (to 2 d.p.)

(c)(i) $V(3, 1)$ (ii) $y = 4$

(d) (i) Opposite angles of a parallelogram are equal.

(ii) $\Delta WXP \equiv \Delta YZQ$ (A.S.A. test)

(iii) $PQ = 4$ cm

(5) (a) (i) $\Delta = 4k^2 - 24k$

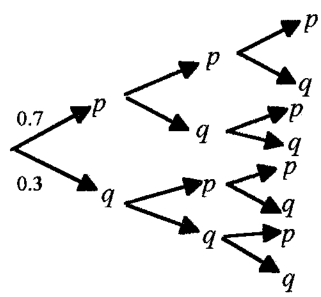
(ii) $0 < k < 6$

(b) (i) $A(1, 8)$ (ii) Proof

(iii) Proof (iv) $C(-16, 0)$

(6) (a) Area $\approx 7\frac{1}{2} - 8\ln 2$

(b)



p =In favour q =Not in favour

(i) $P(\text{All in favour}) = (0.7)^3 = 0.343$

(ii) $P(\text{At least one in favour})$

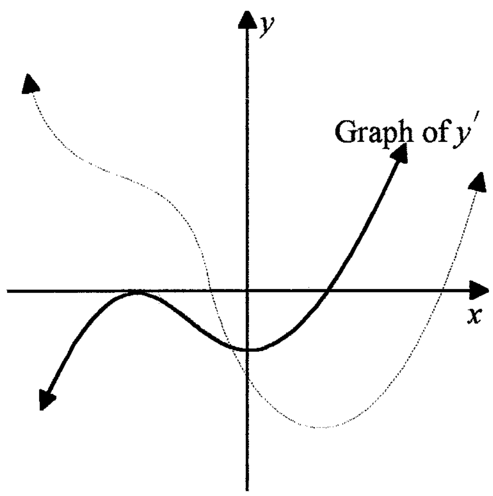
$= 1 - (0.3)^3 = 0.973$

(c) (i) 55 cm (ii) 12 strips

(7)(a)(i) $\Delta PXS \,|||\, \Delta QXY$ (Equiangular)

(ii) $PS = 2\frac{2}{3}$ cm

(7) (b) (i)



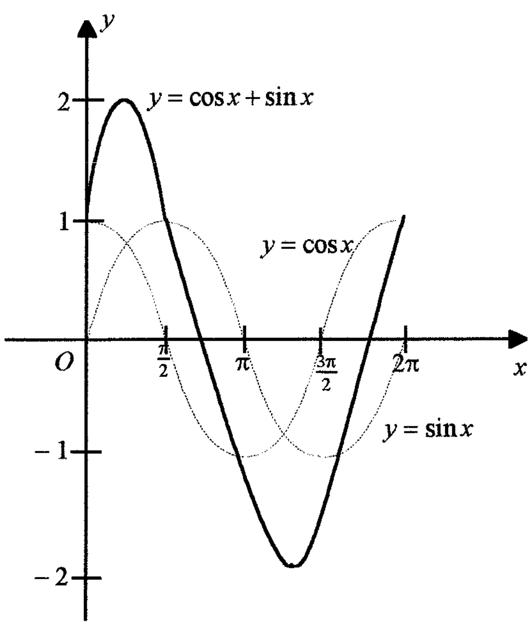
(c) (i) $y' = \cos x - \sin x$

(ii) Max. at $y = \sqrt{2}$

Min. at $y = -\sqrt{2}$

(iii) Proof

(iv)



(8) (a) (i) $\frac{\pi}{3}$ rad (ii) 24π cm^2

(b) $\frac{\pi}{2e^4}(e^8 + 8e^4 - 1)$

(c) (i) $v = 4\sin\left(2t + \frac{\pi}{6}\right)$

$x = -2\cos\left(2t + \frac{\pi}{6}\right)$

(ii) $t = \frac{5\pi}{12}$ sec

(iii) $t = \frac{\pi}{6}$ sec

(9) (a) (i) $\frac{1}{6}$ (ii) $\frac{3}{11}$

(b) $\frac{dP}{dt} < 0$ and $\frac{d^2P}{dt^2} < 0$

(c) (i) Proof (ii) Proof (iii) $r = 471$ mm

(10) (a) (i) $k = \frac{\ln 5}{5} \approx 0.322$ (to 3 d.p)

(ii) 13 100 fish

(iii) $t = 12.15 \approx 13$ months

(iv) $\frac{dN}{dt} \approx 2\,200$ fish/month

(b) (i) Proof

(ii) $A_{60} = \frac{(1.0025S - 1600)(1.0025^{60} - 1)}{0.0025}$

(iii) $S = \$2\,367.52$