FILE ONLY

CATHOLIC SECONDARY SCHOOLS' ASSOCIATION OF NEW SOUTH WALES

YEAR TWELVE FINAL TESTS 1992

MATHEMATICS

3/4 UNIT COMMON PAPER

(i.e. 3 UNIT COURSE – ADDITIONAL PAPER; 4 UNIT COURSE – FIRST PAPER)

Afternoon session

Friday 21st August 1992

Time Allowed — Three Hours

EXAMINERS

Glenn Abrahams, Patrician Brothers' College, Fairfield
Graham Arnold, John Paul II Senior High, Marayong
Margot Cooper, Rosebank College, Five Dock
Sandra Hayes, All Saints Catholic Senior High, Casula
John Hennessy, Marcellin College, Randwick
Frank Reid, St. Ursula's College, Kingsgrove

DIRECTIONS TO CANDIDATES :

ALL questions may be attempted.

ALL questions are of equal value.

All necessary working should be shown in every question.

Full marks may not be awarded for careless or badly arranged work.

Approved calculators may be used.

Standard integrals are printed on a separate page.

QUESTION 1

(a) Solve the equation

$$\cos 2A = \cos A \quad \text{where} \quad 0° \le A \le 360°.$$

(b) Evaluate $\displaystyle\int_3^5 x\sqrt{x^2 - 9}\,dx$

using the substitution $u = x^2 - 9$.

(c) (i) Sketch the graph of the function

$$y = \sin^{-1}\left(\tfrac{x}{2}\right)$$

(ii) State the domain and the range of the function.

(iii) Find the exact equation of the tangent to the curve

$$y = \sin^{-1}\left(\tfrac{x}{2}\right)$$

at the point where $x = 1$.

QUESTION 2

(a) Given that $$\int_0^1 \frac{1}{x^2 + 3}\, dx = k\pi$$

find the value of the constant k.

(b) α, β and γ are the roots of the equation $x^3 + 2x^2 - 3x + 5 = 0$.

(i) State the values of
$\alpha + \beta + \gamma$, $\alpha\beta + \alpha\gamma + \beta\gamma$ and $\alpha\beta\gamma$.

(ii) Find the value of $(\alpha - 1)(\beta - 1)(\gamma - 1)$.

(c) A particle is moving in a straight line with Simple Harmonic Motion. If the amplitude of the motion is 4 cm and the period of the motion is 3 seconds, calculate

(i) the maximum velocity of the particle,

(ii) the maximum acceleration of the particle,

(iii) the speed of the particle when it is 2cm from the centre of the motion.

Cont...

QUESTION 3

(a) Solve the inequality $\frac{2x + 3}{x - 4} > 1$.

(b) (i) Show that the equation

$$5x^4 - 4x^5 - 0 \cdot 9 = 0$$

has a root near $x = 1$.

(ii) Starting with the approximation $x_0 = 1$ attempt to find an improved value for this root using Newton's Method. Explain why this attempt fails.

(c) (i) At any time t the rate of cooling of the temperature T of a body, when the surrounding temperature is P, is given by the equation

$\frac{dT}{dt} = -k(T - P)$, for some constant k.

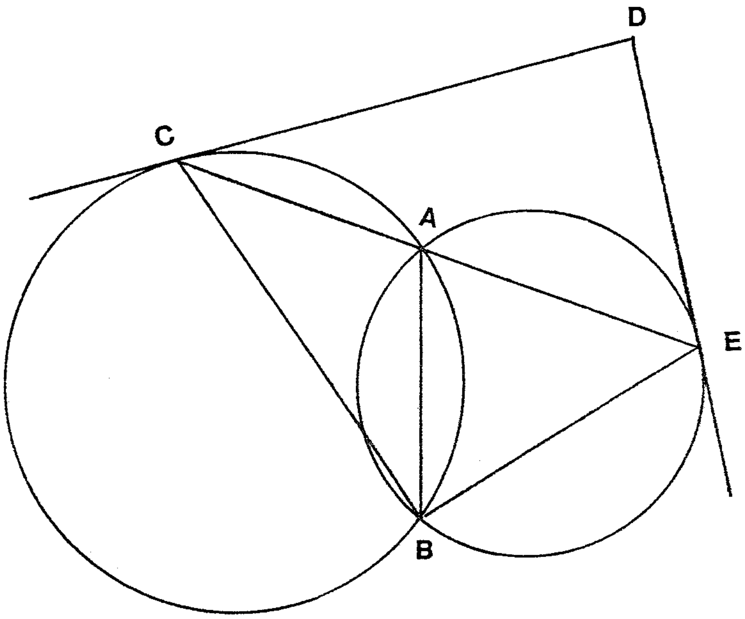
Show that the solution

$T = P + Ae^{-kt}$, for some constant A, satisfies this equation.

(ii) A metal bar has a temperature of 1340° C and cools to 1010° C in 12 minutes, when the surrounding temperature is 25° C. Find how much longer it will take the bar to cool to 60° C, giving your answer correct to the nearest minute.

QUESTION 4

(a)



Two circles intersect at A and B. CAE is a straight line where C is a point on the first circle and E is a point on the second circle. The tangent at C to the first circle and the tangent at E to the second circle meet at D.

(i) Copy the diagram.

(ii) Prove that BCDE is a cyclic quadrilateral.

(b) (i) Solve the equation

$$\sqrt{3}\cos x - \sin x = 1 \text{ for } 0 \leq x \leq 2\pi.$$

(ii) What is the general solution of the equation?

QUESTION 5

(a) A is the point $(-4,1)$ and B is the point $(2,4)$. Q is the point which divides AB internally in the ratio 2:1 and R is the point which divides AB externally in the ratio 2:1. $P(x,y)$ is a variable point which moves so that $PA = 2PB$.

(i) Find the coordinates of Q and R.

(ii) Show that the locus of P is a circle on QR as diameter.

(b) Twelve people are going to the local swimming pool. Five are to go by car and the rest are to walk to the pool.

(i) How many different groups of five of the people can be found to fill the car?

(ii) In one of these groups it is found that only one person can drive. In how many ways can the seats be filled in this group under this condition?

Cont...

QUESTION 6

(a) (i) Write down the first three terms in the expansion of $(1 + ax)^n$ where a is a constant and n is a positive integer.

(ii) If the first three terms of the expansion are $1 - 12x + 63x^2$

(α) find the values of a and n ,

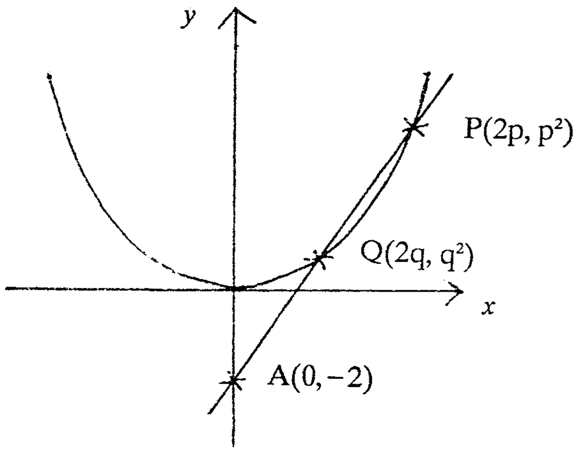
(β) find the next term in the expansion.

(b) Use the Principle of Mathematical Induction to show that $9^{n+2} - 4^n$ is divisible by 5 for all positive integers n.

Cont...

QUESTION 7

$\begin{cases} x = 2t \\ y = t^2 \end{cases}$



(i) Find the equations of the normals to the parabola

$\begin{cases} x = 2t \\ y = t^2 \end{cases}$ at the points $P(2p, p^2)$ and $Q(2q, q^2)$,

where $p \neq q$.

Hence show that these normals intersect at the point $R(X,Y)$ where $X = -pq(p + q)$ and $Y = (p + q)^2 - pq + 2$.

(ii) If the chord PQ has gradient m and passes through the point $A(0, -2)$ find, in terms of m, the equation of PQ.
Hence show that p and q are the roots of the equation $t^2 - 2mt + 2 = 0$.

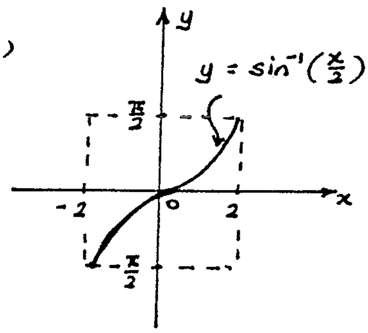
(iii) By considering the sum and the product of the roots of this quadratic equation show that the point R lies on the original parabola.

(iv) Find the least value of m^2 for which p and q are real.
Hence find the set of possible values of the y coordinate of R.

(1) (a) $0°, 120°, 240°$ or $360°$

(b) $21\frac{1}{3}$

(c) (i)



(ii) D: $|x| \leq 2$ or $-2 \leq x \leq x$

R: $|y| \leq \frac{\pi}{2}$ or $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

(iii) $2\sqrt{3}x - 6y + \pi - 2\sqrt{3} = 0$

(2) (a) $k = \frac{\sqrt{3}}{18}$

(b) (i) $\alpha + \beta + \gamma = -2$

$\alpha\beta + \beta\gamma + \gamma\alpha = -3$

$\alpha\beta\gamma = -5$

(ii) -5

(c) (i) $\frac{8\pi}{3}$ cm s^{-1}

(ii) $\frac{16\pi^2}{9}$ cm s^{-2}

(iii) $\frac{4\sqrt{3}}{3}\pi$ cm s^{-1}

(3) (a) $x < -7$ or $x > 4$

(b) (i) $f(1) > 0$

$f(1.1) < 0$

$\therefore$ root exists.

(ii) $f'(1) = 0$

Newton's method fails because $x_0 = 1$ lies at a stationary point of $f(x)$

(c) (i) Proof

(ii) $k = \frac{1}{12}\ln\left(\frac{1315}{985}\right)$

It cools to 60°C after a <u>further</u> 139 mins.

(4) (a) (i) Copy diagram

(ii) Show that $\angle CDE + \angle CBE = 180°$

(b) (i) $\frac{\pi}{6}$ or $\frac{3\pi}{2}$

(ii) $\frac{\pi}{2}(4n+3)$ for $n = 0, \pm 1, \pm 2, \ldots$

(5) (a) (i) R (8, 7)

(ii) Proof

(b) (i) $^{12}C_5 = 792$

(ii) $1 \times 4! = 24$

(6) (a) (i) $1 + nax + \frac{n(n-1)a^2x^2}{2} + \ldots$

(ii) (α) $a = -\frac{3}{2}$

$n = 8$

(β) $-189x^3$

(b) Proof

(7) (i) Proof

(ii) Proof

(iii) Proof

(iv) Least value of m^2 is 2

$\therefore$ the y-coordinate of $R = 4m^2 \geq 8$