CATHOLIC SECONDARY SCHOOLS' ASSOCIATION OF NEW SOUTH WALES

YEAR TWELVE FINAL TESTS 1993

MATHEMATICS

3/4 UNIT COMMON PAPER

(i.e. 3 UNIT COURSE – ADDITIONAL PAPER: 4 UNIT COURSE – FIRST PAPER)

Afternoon session | Friday 13th August 1993.

Time Allowed — Two Hours

EXAMINERS

Glenn Abrahams, Patrician Brothers' College, Fairfield
Graham Arnold, John Paul II Senior High, Marayong
Sandra Hayes, All Saints Catholic Senior High, Casula.

DIRECTIONS TO CANDIDATES :

ALL questions may be attempted.

ALL questions are of equal value.

All necessary working should be shown in every question.

Full marks may not be awarded for careless or badly arranged work.

Approved calculators may be used.

Standard integrals are printed on a separate page.

Students are advised that this is a Trial Examination only and cannot in any way guarantee the content or the format of the Higher School Certificate Examination. However, the committees responsible for the preparation of these 'Trial Examinations' do hope that they will provide a positive contribution to your preparation for the final examinations.

QUESTION 1

(a) Find $\int \frac{1}{\sqrt{9 - x^2}}\, dx$

(b) If $\sin \alpha = \frac{3}{4}$, $0 < \alpha < \frac{\pi}{2}$

and $\sin \beta = \frac{2}{3}$, $\frac{\pi}{2} < \beta < \pi$

find the exact values of

(i) $\tan 2\alpha$

(ii) $\cos (\alpha - \beta)$

(c) (i) Find the centre and the radius of the circle C whose equation is

$$x^2 + y^2 - 4x + 6y - 12 = 0.$$

(ii) Find, in terms of the constant k, the length of the perpendicular from the centre of C to the line L whose equation is $3x + 4y = k$.

(ii) Hence find the values of k for which L is a tangent to C.

Cont...

QUESTION 2

(a) If $f(x) = x^3 + 3x^2 - 10x - 24$ calculate $f(-2)$ and express $f(x)$ as the product of three linear factors.

(b) Two of the roots of the equation $x^3 + px^2 + qx + r = 0$ are equal in magnitude but opposite in sign.

(i) Show that $x = -p$ is the other root.

(ii) Show that $r = pq$.

(c) Solve the inequality $$\frac{x}{x^2 - 1} > 0.$$

Cont...

QUESTION 3

(a) Find the exact value of $\sin(2\tan^{-1}\frac{1}{2})$

(b) Use the substitution $u = 2 - x$, to evaluate $\int_{-1}^{2} x\sqrt{2-x}\,dx$.

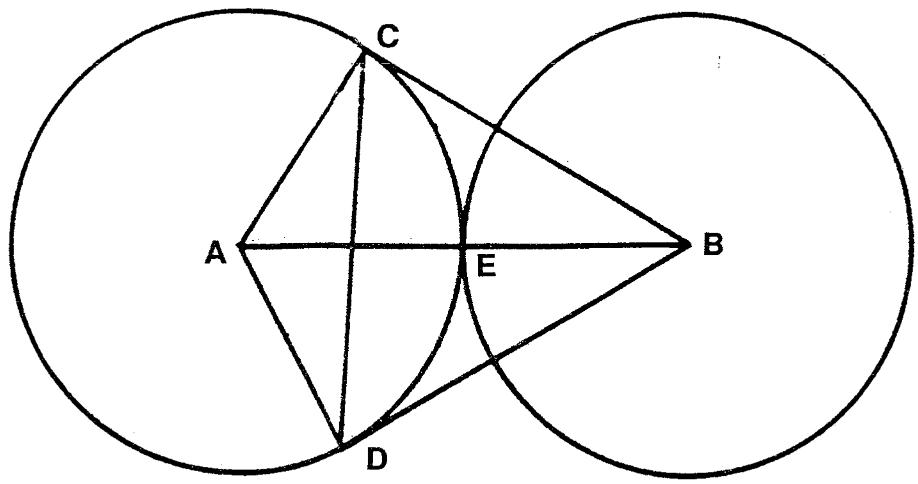
(c) $P(2p, p^2)$ and $Q(2q, q^2)$ are two points on the parabola $x^2 = 4y$.
The chord PQ subtends a right angle at the origin O.

(i) Show that $pq = -4$.

(ii) If M is the mid point of PQ find the locus of M as P and Q move on the parabola.

QUESTION 4

(a)



Two circles of equal radius and with centres at A and B respectively touch each other externally at E. BC and BD are tangents from B to the circle with centre A.

(i) Copy the diagram.

(ii) Show that BCAD is a cyclic quadrilateral.

(iii) Show that E is the centre of the circle which passes through B, C, A and D.

(iv) Show that $\angle CBA = \angle DBA = 30°$.

(v) Show that triangle BCD is equilateral.

(b) A is the point $(-2, -1)$, B is the point $(1,5)$. Find the co-ordinates of the point Q which divides AB externally in the ratio $5:2$.

Cont...

QUESTION 5

(a) (i) A ball is thrown from a point 0 on the edge of a cliff which is 20 metrres above a beach. The ball is thrown with speed $15\sqrt{2}\ \text{ms}^{-1}$ at an angle of 45° above the horizontal.

Taking $g = 10\ \text{ms}^{-2}$ show that the ball hits the beach at a point 60 metres along the beach.

(ii) A second ball is thrown horizontally from 0 and hits the beach at the same point as the first ball.

Taking $g = 10\ \text{ms}^{-2}$ find the speed of projection of the second ball.

(Standard results about projectile motion can be quoted without proof.)

(b) A sector of a circle with centre O and radius r cm is bounded by the radii OP and OQ, and by the arc PQ. The angle POQ is θ radians.

(i) Given that r and θ vary in such a way that the area of the sector POQ has a constant value of 100 cm^2, show that $\theta = \dfrac{200}{r^2}$

(ii) Given also that the radius is increasing at a constant rate of 0.5 cm^{-1}, find the rate at which the angle POQ is decreasing when r = 10 cm.

Cont...

QUESTION 6

(a) Find the term independent of x in the expansion of

$$\left(2x + \frac{1}{x}\right)^{10}$$

(b) The letters of the word CALCULUS are arranged in a row.

(i) How many different arrangements are there?

(ii) If one of these arrangements is selected at random, what is the probability that it begins with 'U' and ends in 'U' ?

(c) Use the method of mathematical induction to show that :

$2 \times 1! + 5 \times 2! + 10 \times 3! + \dots + (n^2 + 1)n! = n(n + 1)!$ for all positive integers $n \geq 1$.

QUESTION 7

(a) Use the result $\lim_{x \to 0} \frac{\sin x}{x} = 1$

to find $\lim_{x \to 0} \frac{1 - \cos 2x}{x^2}$

(b) A particle moves in a straight line. At time t its displacement from a fixed point O on the line is x, its velocity is v and its acceleration is a.

(i) Show that $a = \frac{d}{dx}\left(\frac{1}{2} v^2\right)$.

(ii) If $a = 4x - 4$ and when $t = 0$, $x = 6$ and $|v| = 8$ show that $v^2 = 4x^2 - 8x - 32$.

(iii) Use the expression for v^2 to find the set of possible values of x.

(iv) Describe the motion of the particle in each of the cases

(α) when $t = 0$, $x = 6$ and $v = 8$.

(β) when $t = 0$, $x = 6$ and $v = -8$.

[1][a] $\sin^{-1}\left(\frac{x}{3}\right)+c$

[b] [i] $-3\sqrt{7}$

[ii] $\frac{1}{12}\left(6-\sqrt{35}\right)$

[c] [i] Centre $(2,-3)$ Radius 5

[ii] $\frac{|k+6|}{5}$

[iii] $k=-31$ or 19

[2] [a] $(x+2)(x+4)(x-3)$

[b] [i] Proof [ii] Proof

[c] $-1<x<0$ or $x>1$

[3] [a] $\frac{4}{5}$

[b] $\frac{2\sqrt{3}}{5}$

[c] [i] Proof

[ii] Locus is $y=\frac{x^2+8}{2}$

[4] [a] Proofs

[b] $Q(3,9)$

[5] [a] [i] Proof

[ii] 60 ms^{-1}

[b] [i] Proof

[ii] 0.2 rad/sec

[6] [a] 8064

[b] [i] 5040 [ii] $\frac{180}{5040}=\frac{1}{28}$

[c] Proof

[7] [a] 2

[b] [i] , [ii] Proofs

[iii] $x\geq 4$

[iv] [α] The particle starts at 6 units to the right of O and moves to the right with an increasing speed.

[β] The particle starts at 6 units to the right of O. It is moving to the left and slowing down. It moves to the left until it is 4 units to the right of O where it comes to rest. It then moves to the right and speeds up and continues to move to the right.