CATHOLIC SECONDARY SCHOOLS' ASSOCIATION OF NEW SOUTH WALES

YEAR TWELVE FINAL TESTS 1995

MATHEMATICS

2/3 UNIT

Morning session

Wednesday 9th August 1995.

Time Allowed — Three Hours
(Plus 5 minutes reading time)

EXAMINERS

B. Cosgrove
M. Donaghy
C. Fernandes
J. Howard
A. Kollias
R. Pantua
E. Rainert
P. Rockett
S. Thomson
J. Wheatley

DIRECTIONS TO CANDIDATES :

* **ALL** questions may be attempted.
* **ALL** questions are of equal value.
* All necessary working should be shown in every question.
* Full marks may not be awarded for careless or badly arranged work.
* Approved slide rules or calculators may be used.
* Table of standard integrals is provided.
* The answers to the ten questions in this paper are to be returned in separate writing sheets clearly marked **QUESTION 1, QUESTION 2** etc. on the cover.
* If required, additional writing sheets may be obtained from the examinations supervisor upon request.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

QUESTION 1 *(Begin a new sheet)*

(a) (2 marks)

Simplify $5x - (2 - 3x)$

(b) (1 mark)

Factorise $2x^2 + 5x - 12$

(c) (2 marks)

Find the sum of the first twenty terms of the series:

$$10 + 7 + 4 + \ldots$$

(d) (2 marks)

In the diagram AB || CE, ∠ABF = 75° and ∠BFE = 35°.

Find the size of θ giving reasons.

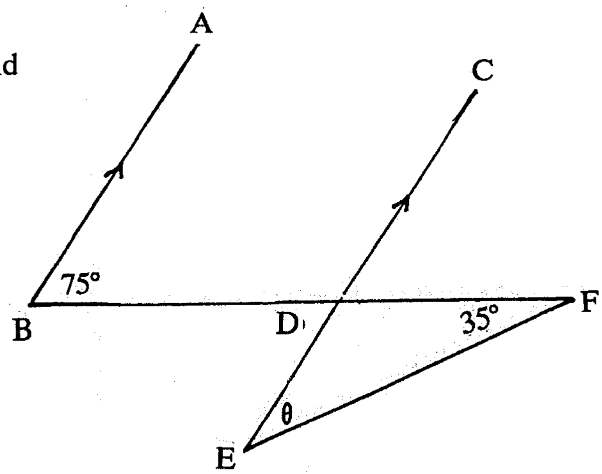


FIGURE NOT TO SCALE

(e) (2 marks)

At Octopus Communications' annual sale, all mobile phones were discounted by 40%. Cedric paid $630 for a mobile phone at the sale. What was the original price of the phone?

(f) (3 marks)

In ΔPQR, PR = 15.2cm, QR = 12.4cm and ∠PRQ = 107°.

Use the cosine rule to find the length of PQ correct to 1 decimal place.

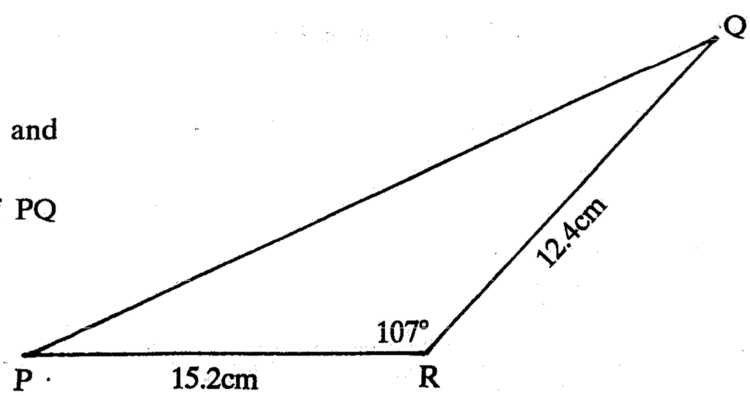
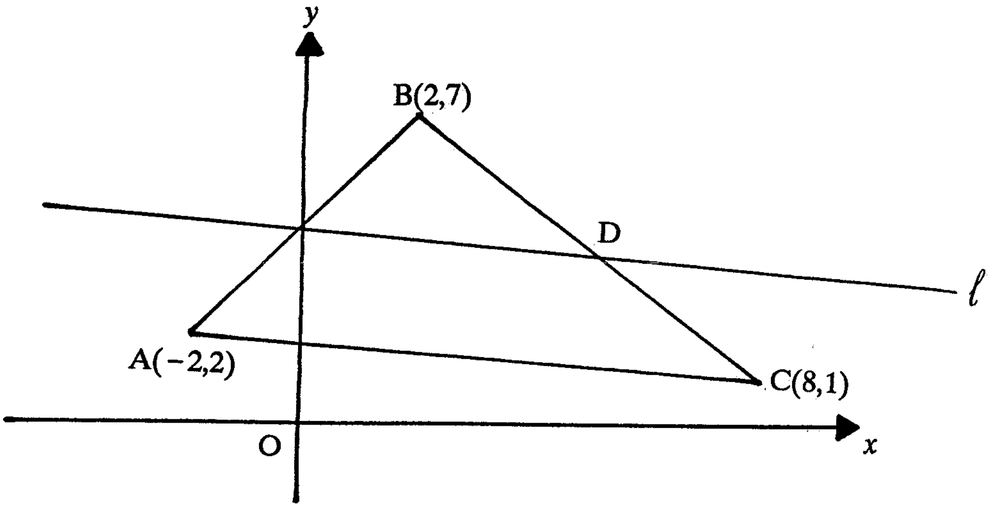


FIGURE NOT TO SCALE

QUESTION 2 *(Begin a new sheet)*



(FIGURE NOT TO SCALE)

A(−2,2) B(2,7) and C(8,1) are the vertices of ΔABC and line ℓ passes through point D, as shown in the diagram.

(a) (1 mark)

Copy the diagram onto your answer page.

(b) (1 mark)

D is the midpoint of BC. Show the coordinates of D are (5,4).

(c) (1 mark)

What is the gradient of line AC?

(d) (2 marks)

Line ℓ was drawn through point D and parallel to AC.

Show the equation of line ℓ is $x + 10y = 45$.

(e) (3 marks)

Line ℓ and side AB intersect at point E.

(i) Show the position of E on your diagram.

(ii) Prove ΔABC and ΔEBD are similar.

(f) (2 marks)

Hence, or otherwise, write down the coordinates of point E.

(g) (2 marks)

Explain how you know the interval AC is twice as long as the interval ED.

QUESTION 3 *(Begin a new sheet)*

(a) (3 marks)

Express $\frac{\sqrt{6}}{\sqrt{6} - \sqrt{3}}$ in the form $a + b\sqrt{2}$.

(b) (5 marks)

Differentiate:

(i) $ln\,(x^3 + 7)$

(ii) $x^2 e^{3x}$

(iii) $\frac{\sin x}{x}$

(c) (4 marks)

(i) Find $\int (2 + \sqrt{x})\, dx$

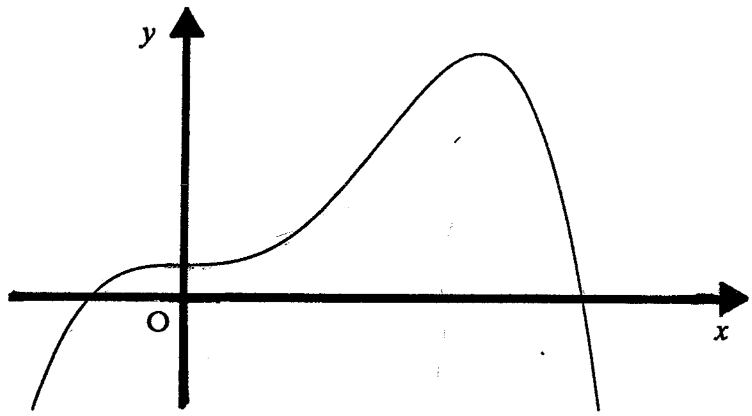
(ii) Evaluate $\int_0^{\pi/4} \sec^2 3x\, dx$

Cont....

QUESTION 4 *(Begin a new sheet)*

(a) (2 marks)

The graph given below shows $y = f(x)$



(i) Copy the graph onto your answer sheet.

(ii) On the same axes, draw the graph of its gradient function $y = f'(x)$

(b) (6 marks)

The function $y = x^3 - 3x^2 - 9x + 1$ is defined in the domain $-4 \leq x \leq 5$

(i) Find the coordinates of any turning points and determine their nature.

(ii) Find the coordinates of any points of inflexion.

(iii) Draw a neat sketch of the curve.

(iv) Determine the minimum value of the function y, in the domain $-4 \leq x \leq 5$

(c) (4 marks)

After heavy rain the flood gate of a dam was opened. Water was released from the dam at a rate of $31.8t$ litres per second where t is measured in seconds after the gate was opened.

(i) Explain why the total volume of water, V, released from the dam in k seconds can be found by evaluating the expression

$$V = \int_0^k 31.8t\, dt$$

(ii) Calculate the time it took for 10^9 litres of water to be released from the dam. Express your answer to the nearest second.

Cont....

QUESTION 5 *(Begin a new sheet)*

(a) (4 marks)

Justin is a talented soccer goalkeeper. The probability that he can stop a penalty shot at goal is $\frac{3}{4}$. During a match the opposition had 2 penalty shots at goal. What is the probability Justin stopped:

(i) both shots.

(ii) at least 1 shot.

(b) (2 marks)

When Jim started working his annual salary was \$18 500. Each year his salary increased by \$670. What was his annual salary during his tenth year of working?

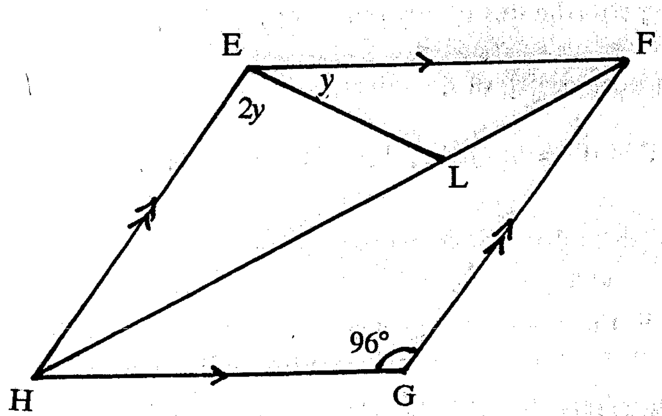
(c) (3 marks)

Use Simpson's Rule with the five function values given in the table below to evaluate correct to 1 decimal place:

$$\int_1^5 f(x)\,dx$$

x	1	2	3	4	5
$f(x)$	0	1.39	3.30	5.55	8.05

(d) (3 marks)



(FIGURE NOT TO SCALE)

The diagram shows a rhombus EFGH. A line EL is drawn through E so that $\angle HEL = 2 \times \angle FEL$.

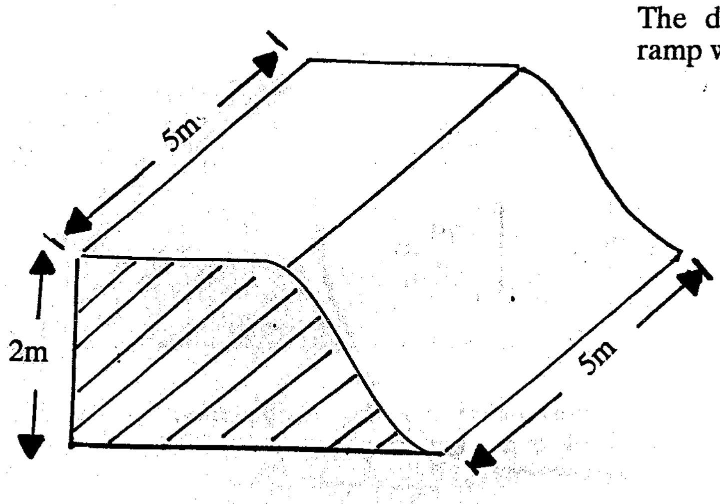
(i) Copy the diagram onto your answer page.

(ii) $\angle FGH = 96°$, find the size of $\angle ELF$ giving reasons.

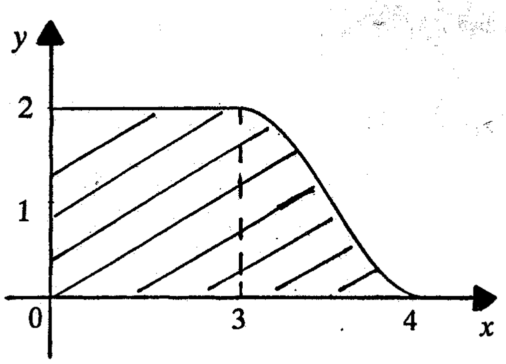
QUESTION 6 *(Begin a new sheet)*

(a) (6 marks)

The diagram shows a sketch of a skateboard ramp which is 2 metres high and 5 metres wide.



The cross section of the ramp was determined by using the graph of $y = f(x)$



$$f(x) = \begin{cases} 2 & 0 \le x < 3 \\ 1 - \cos \pi x & 3 \le x \le 4 \end{cases}$$

(FIGURE NOT TO SCALE)

(i) Find the area of the cross section of the ramp, the shaded area in the diagrams.

(ii) The ramp is solid concrete. How much concrete was used to make the ramp? Give your answer to the nearest m^3.

(b) (6 marks)

The rabbit population P, in a park was growing exponentially and could be calculated using the formula $P = 500e^{kt}$ where t was the time in weeks after the rabbits were first counted.

(i) At the end of the first week there were 550 rabbits in the park. Calculate the value of k correct to 3 decimal places.

(ii) How many weeks did it take for the rabbit population to reach 1500?

(iii) At what rate was the colony increasing after 2 weeks?

Cont....

QUESTION 7 *(Begin a new sheet)*

(a) (4 marks)

Find the values of m for which the expression below is always positive.

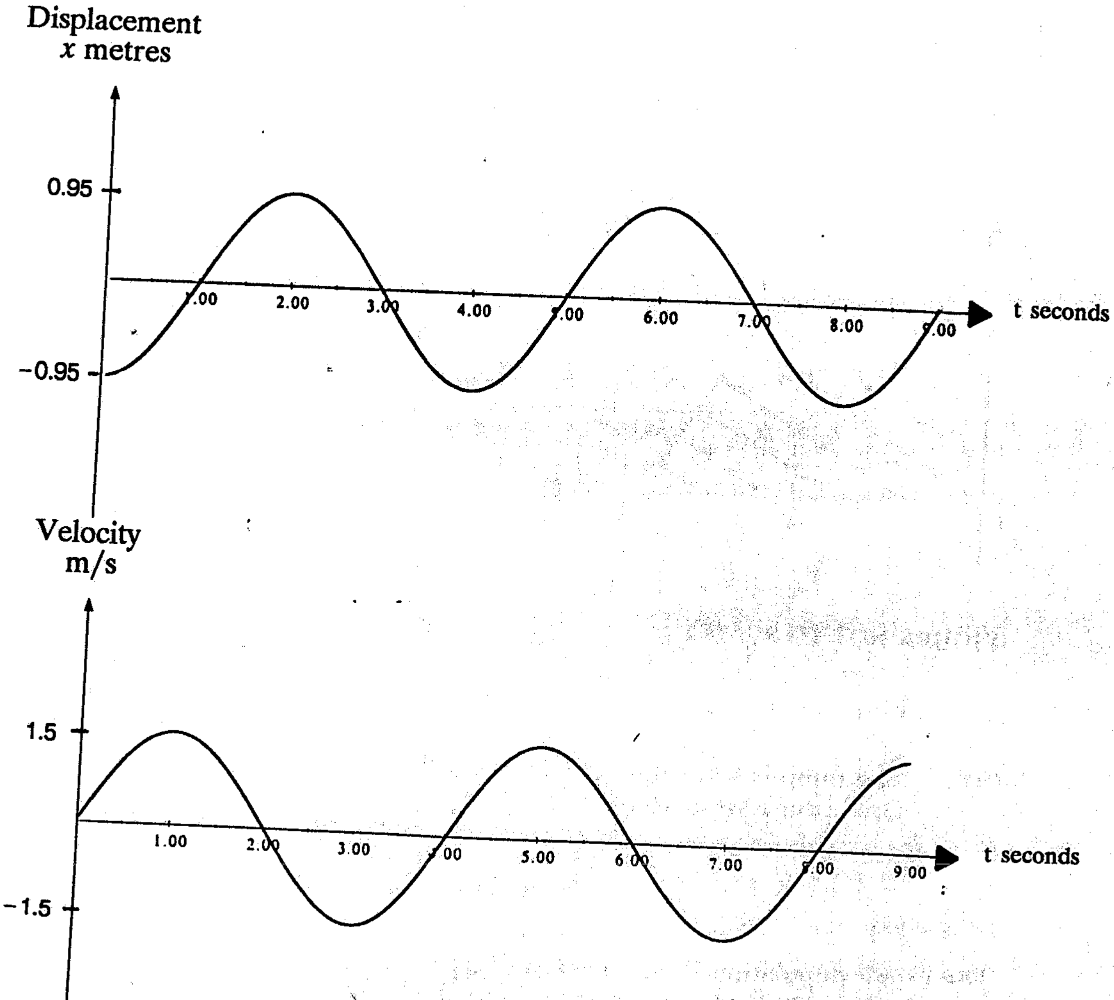
$$x^2 + 2mx + (3m - 2)$$

(b) (4 marks)

Differentiate $y = \frac{d}{dx}(ln\,x)^2$ and hence evaluate $\int_1^2 \frac{lnx}{x}\,dx$

(c) (4 marks)

A particle moves in a straight line. The diagrams below are the displacement and velocity graphs of this particle during the first 9 seconds of its motion.



Use these graphs to answer the following questions.

(i) Write down the initial displacement and velocity of the particle.

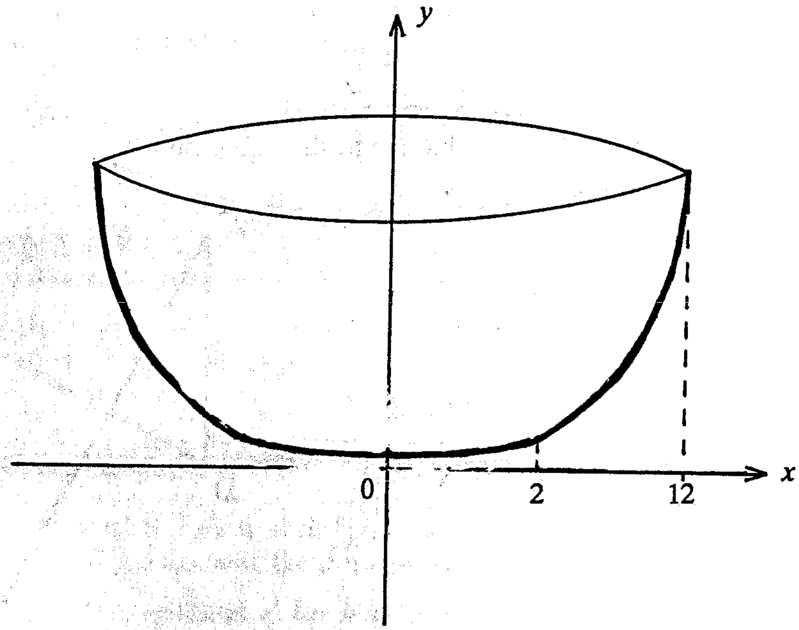
(ii) When does the particle change direction for the first time?

(iii) Where does the particle reach its maximum velocity?

Cont....

QUESTION 8 *(Begin a new sheet)*

(a) (4 marks)



(FIGURE NOT TO SCALE)

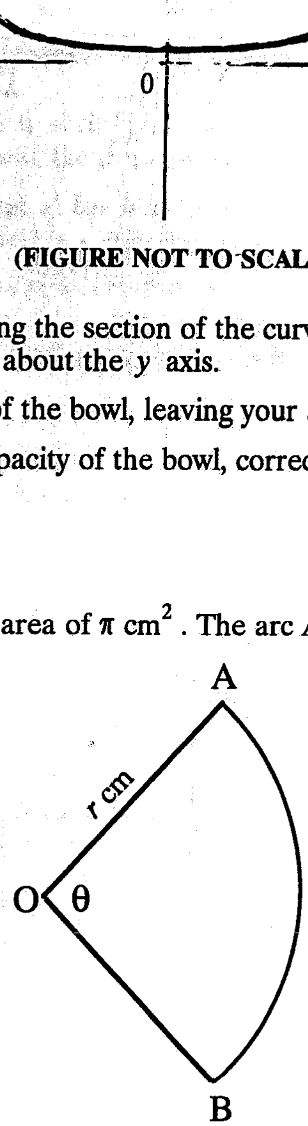
A bowl was designed by rotating the section of the curve $y = \frac{1}{4}x^2$ between $x = 2$ and $x = 12$ centimetres, about the y axis.

(i) Calculate the volume of the bowl, leaving your answer in terms of π.

(ii) Hence calculate the capacity of the bowl, correct to the nearest litre. (1 litre = 1000 cm^3)

(b) (4 marks)

The sector OAB below has an area of π cm^2. The arc AB has length $\pi/2$ cm.



By solving a pair of equations simultaneously find the exact values of r and θ.

(c) (4 marks)

Find the coordinates of the point on the curve $y = e^{3x}$ where the tangent is perpendicular to the line $y = -\frac{1}{6}x + 4$.

Cont....

QUESTION 9 *(Begin a new sheet)*

(a) (3 marks)

(i) Sketch $y = \tan \pi x$ for $0 \leq x \leq 2$.

(ii) On the same diagram sketch $y = 1 - x$.

(iii) Hence determine the number of solutions to the equation

$$\tan \pi x = 1 - x \text{ for } 0 \leq x \leq 2.$$

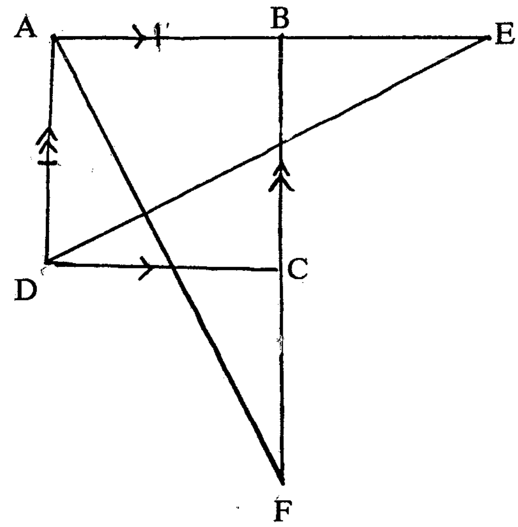
(b) (4 marks)

In the diagram ABCD is a square. AB is produced to E so that AB = BE and BC is produced to F so that BC = CF.

(i) Copy the diagram onto your answer page.

(ii) Prove $\Delta AED \equiv \Delta BFA$

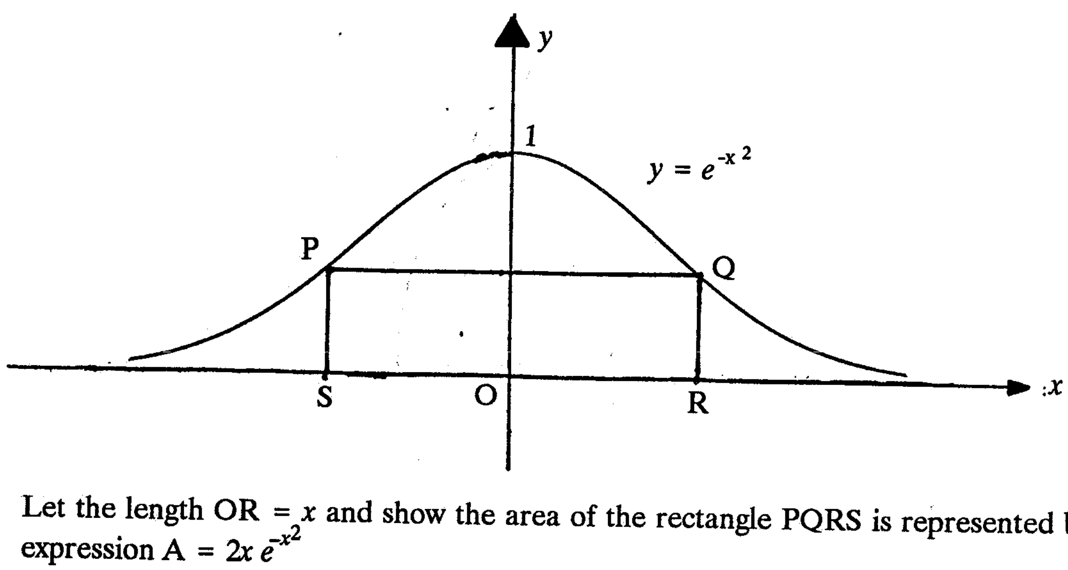
(iii) Hence prove $\angle AED = \angle BFA$



(FIGURE NOT TO SCALE)

(c) (5 marks)

The diagram shows a rectangle PQRS where P and Q are on the curve $y = e^{-x^2}$ and R and S are on the x axis. The point O is the origin and lengths OS and OR are equal.



(i) Let the length OR $= x$ and show the area of the rectangle PQRS is represented by the expression $A = 2x\,e^{-x^2}$

(ii) Find the value of x for which PQRS has maximum area.

QUESTION 10 *(Begin a new sheet)*

(a) (3 marks)

By considering the recurring decimal $0.4\dot{5}$ (i.e. 0.45555555...) as the sum of an infinite geometric series, express $0.4\dot{5}$ in the form $\frac{a}{b}$.

(b) (3 marks)

Of the 4000 population of a small Scottish town, 2500 people are descendants of the Campbell clan, 1 900 are descendants of the Donald clan while 300 are not descendants of either clan.

A member of the town was selected at random. What is the probability he is a descendant of both the Donald and Campbell clans?

(c) (6 marks)

Over the years the statistics showed the population of a particular town was decreasing in such a way that at the end of each year the population could be determined in the following way:

10% of the population at the beginning of the year moved out and
500 new citizens moved in during the year.

At the beginning of 1990, before the 10% moved out, the population of the town was 10 000.

(i) Show that at the end of 1992 the population of the town was:

$$10\,000\,(0.9)^3 + 500\,(1 + 0.9 + 0.9^2)$$

(ii) This trend continued indefinitely. Find the population of the town at the end of the year 2009. (i.e. the end of the 20th year)

(1)(a) $8x-2$

(b) $(2x-3)(x+4)$

(c) -370

(d) $\theta = 40^0$

(e) $1050

(f) 22.2 cm

(2) (a) Diagram

(b) Midpoint formula

(c) $-\frac{1}{10}$

(d) Proof

(e) Proof (Equiangular)

(f) $E(0, 4\frac{1}{2})$

(g) Show that $ED : AC = 1 : 2$

(3)(a) $a = 2, b = 1$

(b) (i) $\frac{3x^2}{x^3+7}$

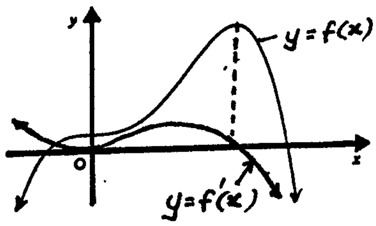
(ii) $xe^{3x}(3x+2)$

(iii) $\frac{x\cos x - \sin x}{x^2}$

(c) (i) $2x + \frac{2x^{\frac{3}{2}}}{3} + c$

(ii) $-\frac{1}{3}$

(4) (a) (i) & (ii)



(b) (i) $(3, -26)$ min.
$(-1, 6)$ max.

(ii) $(1, -10)$ Pt. of inflexion

(iii) Sketch

(iv) Min. $= -75$

(c)(i) Given : $\frac{dV}{dt} = 31.8t$
Therefore, integrating with respect to t gives the Volume.

(ii) 7931 secs.

(5) (a) (i) $\frac{9}{16}$

(ii) $\frac{15}{16}$

(b) $24 530

(c) 14.1 (to 1 d.p.)

(d) (i) Diagram

(ii) $\angle ELF = 106^0$

(6)
(a) (i) $3 \times 2 + \int_3^4 1 - \cos \pi x \, dx$
$= 7 \text{ m}^2$.

(ii) 35 m^3

(b)(i) $k = 0.095$ (to 3 d.p.)

(ii) 12 weeks

(iii) 58 rabbits / week

(7)(a) $1 < m < 2$

(b) $\frac{dy}{dx} = \frac{2\ln x}{x}$, 0.24

(c) (i) -0.95 m , 0 m/s

(ii) $t = 2$ sec

(iii) $x = 0$

(8) (a) (i) $2590\pi \text{ cm}^3$

(ii) 8 litres

(b) $r = 4$, $\theta = \frac{\pi}{8}$

(c) $(\frac{1}{3}\ln 2, 2)$

(9) (a)(i),(ii) Sketches

(iii) 3 solutions

(b) (i) Diagram

(ii), (iii) Proofs

(c) (i) Area $L \times B$

(ii) $x = \frac{1}{\sqrt{2}}$

(10) (a) $\frac{41}{90}$

(b) $\frac{7}{40}$

(c) Proof

(d) 5607