Centre Number

Student Number

CSSA

CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NEW SOUTH WALES

2012

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Morning Session
Friday, 10 August 2012

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen. Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided on a separate sheet
- Show all necessary working for Questions 11-14
- Write your Centre Number and Student Number at the top of this page and page 6

Total marks – 70

Section I Pages 2–5

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 6–12

60 marks

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a \neq 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Section I

10 marks

Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Let $P(x) = 2x^3 - x^2 + 2$. Find the remainder when $P(x)$ is divided by $x + 1$.

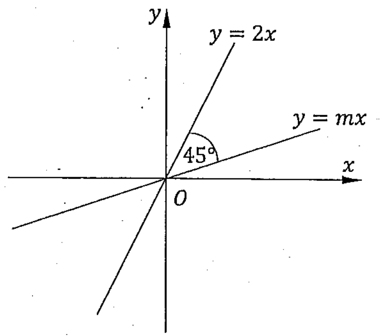
(A) -1

(B) 1

(C) 3

(D) 5

2 The angle between the lines $y = mx$ and $y = 2x$ is $45°$, where $m > 0$, as shown in the diagram below.



NOT TO SCALE

Find the value of m.

(A) $\frac{1}{3}$

(B) $\frac{1}{2}$

(C) 1

(D) 3

3 Let $t = \tan\frac{\theta}{2}$ where $0 < \theta < \pi$. Which of the following gives the correct expression for $\sin\theta + \cos\theta$?

(A) $\dfrac{3t - 1}{1 + t}$

(B) $\dfrac{2t - 1 + t^2}{1 + t^2}$

(C) $\dfrac{1 + 2t - t^2}{1 + t^2}$

(D) $\dfrac{t^2 - 1 - 2t}{1 + t^2}$

4 Let A be the point $(-2, 3)$ and B be the point $(3, -4)$. Find the coordinates of the point which divides AB externally in the ratio $3 : 2$.

(A) $\left(0, \frac{1}{5}\right)$

(B) $\left(1, -\frac{6}{5}\right)$

(C) $(-12, 17)$

(D) $(13, -18)$

5 From six girls and four boys, a committee of 3 girls and 2 boys is to be chosen. How many different committees can be formed?

(A) 26

(B) 120

(C) 252

(D) 1440

6 Consider the function $f(x) = \frac{2x}{x+1}$ and its inverse function $f^{-1}(x)$.
Evaluate $f^{-1}(3)$.

(A) -3

(B) $\frac{2}{3}$

(C) $\frac{3}{2}$

(D) 3

7 The equation of the normal to the parabola $x^2 = 4ay$ at the variable point $P(2ap, ap^2)$ is given by $x + py = 2ap + ap^3$.

How many different values of p are there such that the normal passes through the focus of the parabola?

(A) 0

(B) 1

(C) 2

(D) 3

8 An advertisement claims that '8 out of 10 people prefer Winky Chocolate Bars'. If the advertisement's claim is accurate and a sample of six people is interviewed, what is the probability that at least five people prefer Winky Chocolate Bars?

(A) $1 - (0.8)^6$

(B) $2(0.8)^5$

(C) $5(0.2)^5$

(D) $(0.8)^5(0.2) + (0.8)^6$

9 What is the coefficient of x^2 in the expansion of $\left(x^2 + \frac{2}{x}\right)^7$?

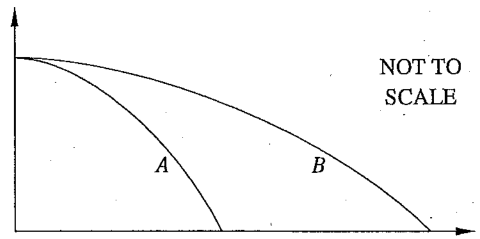
(A) 1

(B) 16

(C) 35

(D) 560

10 Two balls, A and B, are rolled horizontally off a 10 metre cliff at $10\ \text{ms}^{-1}$ and $20\ \text{ms}^{-1}$ respectively.



Which of the following statements is FALSE?

(A) A and B are in the air for the same length of time.

(B) A and B are travelling with the same vertical speed on impact.

(C) B is travelling at twice the speed of A on impact with the ground.

(D) B lands twice as far from the base of the cliff as A.

CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NSW
2012 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Centre Number

Student Number

Section II
60 marks

Attempt Questions 11–14
All questions are of equal value.
Allow about 1 hour and 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\lim_{x \to 0} \frac{\sin 2x}{x}$. **1**

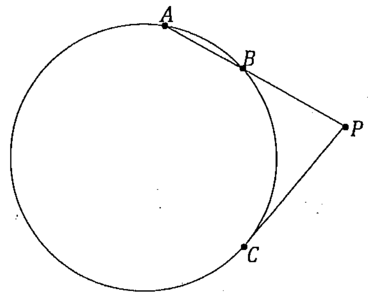
(b) Use the table of standard integrals to evaluate $\int_0^4 \frac{dx}{\sqrt{x^2+9}}$. **2**

(c) Use the substitution $u = x - 8$ to find $\int_8^{8.5} \frac{dx}{\sqrt{(7-x)(x-9)}}$. **3**

(d) Solve $\frac{2t}{1-t} \geq t$. **3**

Question 11 continues on page 7

(e)



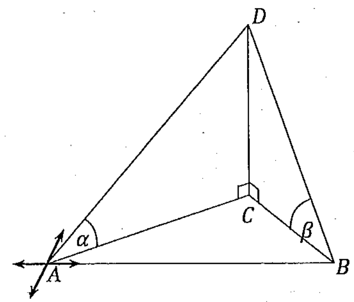
In the diagram the points A, B and C lie on the circle and AB produced meets the tangent from C at the point P.

(i) Given that $PC = 12$, $AB = 7$ and $PB = x$, find x. **2**

(ii) BC is the diameter of the circle passing through P, B and C. **1**

Find the length of BC.

(f) **3**



A vertical pole, CD, is positioned so that the angles of elevation of the top of the pole from the points A and B on the ground are α and β respectively.

The ground is a level horizontal surface and the triangle ABC is right-angled at C. Point B is due east of point A and point C is on a bearing of 060°T from point A.

Show that $\frac{\tan \alpha}{\tan \beta} = \frac{1}{\sqrt{3}}$.

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ where a, b, c and d are real numbers and $a \neq 0$. Let α, β and γ be zeros of $f(x)$.

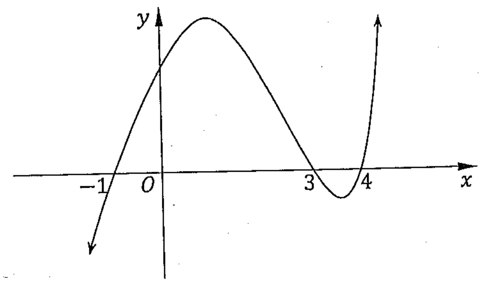
(i) Write down an expression for $\alpha + \beta + \gamma$. **1**

All cubic polynomial functions have a single point of inflexion when the second derivative is equal to zero.

(ii) Using part (i), or otherwise, show that the x-coordinate of the point of inflexion on the curve $y = f(x)$ is given by **2**

$$x = \frac{\alpha + \beta + \gamma}{3}.$$

(iii) The cubic polynomial below has x-intercepts at -1, 3 and 4. Find the x-coordinate of the point of inflexion of the cubic polynomial. **1**



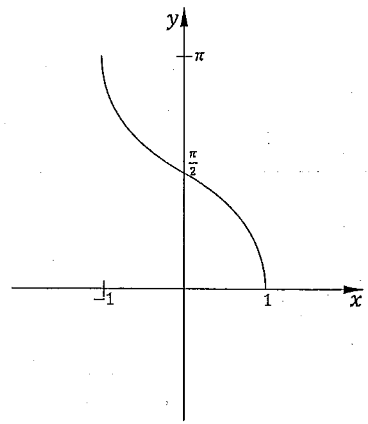
Question 12 continues on page 9

Question 12 (continued)

(b) (i) Given $\sin^{-1}x = \alpha$ and $\cos^{-1}x = \frac{\pi}{2} - \alpha$, solve $\sin^{-1}x = \cos^{-1}x$. **2**

(ii) The curve $y = \cos^{-1}x$ is sketched below. **1**

Copy the diagram into your writing booklet and draw a sketch of the curve $y = \sin^{-1}x$ on the same set of axes. Clearly show the point of intersection of the two curves.



The region bounded by $y = \sin^{-1}x$, $y = \cos^{-1}x$ and the y-axis is rotated about the y-axis to form a solid of revolution.

(iii) Explain why the volume V of the solid formed is given by: **2**

$$V = 2\pi \int_0^{\frac{\pi}{4}} \sin^2 y \, dy.$$

(iv) Hence, find the volume V of the solid formed. **2**

(c) (i) Use mathematical induction to prove that for $n \geq 2$ **3**

$$\left(1 - \frac{1}{2^2}\right) \times \left(1 - \frac{1}{3^2}\right) \times \left(1 - \frac{1}{4^2}\right) \times \dots \times \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}.$$

(ii) Hence evaluate $\frac{3}{4} \times \frac{8}{9} \times \frac{15}{16} \times \dots \times \frac{9999}{10000}$. **1**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) When an egg is placed in a pot of boiling water the rate at which the temperature, T (in degrees Celsius), of the egg increases after t minutes is given by $\frac{dT}{dt} = 0.2(100 - T)$.

(i) Show that $T = 100 - Ae^{-0.2t}$ satisfies this equation, for some constant A. **1**

(ii) Find the value of the constant A, for an egg taken from the refrigerator with an initial temperature of 4°C. **1**

(iii) It is known from experience that it will take $4\frac{1}{2}$ minutes for an egg taken from the refrigerator to cook. **1**

Determine the temperature of the egg after $4\frac{1}{2}$ minutes. Give your answer correct to 3 significant figures.

(iv) If an egg is initially at room temperature, the temperature of the egg can be modelled by the equation $T = 100 - 79e^{-0.2t}$. **2**

How much less time will it take for an egg initially at room temperature to reach the temperature of part (iii)?

(b) A particle moves in Simple Harmonic Motion. Initially, the particle is 4 metres to the right of the origin, moving with a velocity of $-8\sqrt{3}$ ms^{-1}. The displacement x is given by $x = A\cos(2t + \alpha)$ for some constants A and α.

(i) Find the values of A and α. **3**

(ii) When does the particle first reach the centre of motion? **2**

(c) The acceleration of a particle P is given by the equation $\frac{d^2x}{dt^2} = 2x^3 + 18x$, where x is its displacement in metres from the origin after t seconds. Initially the particle is at the origin and has velocity 9 ms^{-1}.

(i) Show that the velocity of the particle is given by $v = x^2 + 9$. **3**

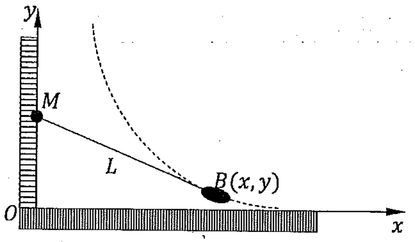
(ii) Hence, find an expression for x as a function of t. **2**

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) The equation $3\sin x = \ln x$ has a number of positive solutions, with the smallest solution being close to $x = 3$. **3**

Use ONE application of Newton's Method to find another approximation to the smallest positive solution. Give this approximation correct to TWO decimal places.

(b) A man M walks along a pier, represented by the positive y-axis, pulling on a boat B by a rope of length L. The man is initially at the origin O and the boat is initially on the x-axis, L metres from O. The man keeps the rope taut and the path followed by the boat is such that the rope is always tangent to the curve tracing its path.



(i) Let the path followed by the boat be the graph of the function $y = f(x)$. By considering the gradient of the line MB, show that **1**

$$\frac{dy}{dx} = \frac{-\sqrt{L^2 - x^2}}{x}.$$

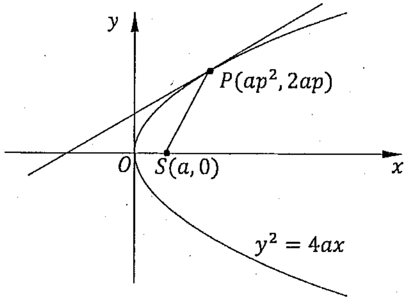
(ii) The man walks along the pier such that the boat moves in the y direction at a constant rate of 3 ms^{-1}. **2**

Find the rate $\frac{dx}{dt}$ at which the boat approaches the pier, when it is a distance $\frac{L}{2}$ metres horizontally from the pier.

Question 14 continues on page 12

Question 14 (continued)

(c) The tangent at the point $P(ap^2, 2ap)$ on the parabola $y^2 = 4ax$, with the focus $S(a, 0)$ is shown in the diagram below. **3**



The gradient function is given by $\frac{dy}{dx} = \frac{2a}{y}$. (Do NOT prove this.)

Prove that the tangent to the parabola at P is equally inclined to the axis of the parabola and the focal chord through P.

(d) (i) Show that $\binom{n}{r} = \binom{n}{n-r}$ for $r = 0, 1, 2, \dots, n$. **1**

(ii) Let $f(r) = \binom{n}{0}\binom{n}{r} + \binom{n}{1}\binom{n}{r+1} + \dots + \binom{n}{n-r}\binom{n}{n}$ for $r = 0, 1, 2, \dots, n$. **3**

By considering the coefficient of x^{n-r} in the expansions of $(1+x)^{2n}$ and $(1+x)^n(1+x)^n$ show that

$$f(r) = \binom{2n}{n-r}$$

(iii) Show that $\binom{n}{0}f(0) + \binom{n}{1}f(1) + \dots + \binom{n}{n}f(n) = \binom{3n}{n}$. **2**

End of Paper

Examiners

Gerry Sozio (Convenor)	St Mary Star of the Sea College, Wollongong
Jenny Bell	St Joseph's Catholic High School, Albion Park
Frank Reid	University of New South Wales, Australian Catholic University
Thanom Shaw	SCEGGS, Darlinghurst
Greg Wagner	Kesser Torah College, Dover Heights

CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NSW

2012 TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

MATHEMATICS EXTENSION 1

Section I
10 marks

Questions 1–10 (1 mark each)

Question 1 (1 mark)
Outcomes Assessed: PE2
***Targeted Performance Bands*: E2**

Solution	Answer	Mark
$P(-1) = -1$	A	1

Question 2 (1 mark)
Outcomes Assessed: H5, HE7
***Targeted Performance Bands*: E2**

Solution	Answer	Mark
$\left\|\frac{2-m}{1+2m}\right\| = \tan 45°, m > o$ $m = \frac{1}{3}$	A	1

Question 3 (1 mark)
Outcomes Assessed: H5
***Targeted Performance Bands*: E2**

Solution	Answer	Mark
$\sin\theta = \frac{2t}{1+t^2}$ and $\cos\theta = \frac{1-t^2}{1+t^2}$ $\sin\theta + \cos\theta = \frac{1+2t-t^2}{1+t^2}$	C	1

Question 4 (1 mark)
Outcomes Assessed: P4, PE2
***Targeted Performance Bands*: E2**

Solution	Answer	Mark
$(x,y) = \left(\frac{mx_2+nx_1}{m+n}, \frac{my_2+ny_1}{m+n}\right)$ $= \left(\frac{3\times3+(-2)\times(-2)}{3+(-2)}, \frac{3\times(-4)+(-2)\times3}{3+(-2)}\right)$ $= (13, -18)$	D	1

Question 5 (1 mark)
Outcomes Assessed: PE3
***Targeted Performance Bands*: E2**

Solution	Answer	Mark
$\binom{6}{3} \times \binom{4}{2} = 120$	B	1

Question 6 (1 mark)
Outcomes Assessed: HE4
***Targeted Performance Bands*: E3**

Solution	Answer	Mark
$f^{-1}(3) = x$ $f(x) = 3$ $\frac{2x}{x+1} = 3$ $x = -3$	A	1

DISCLAIMER
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Question 7 (1 mark)
Outcomes assessed: PE3, PE4
Targeted Performance Bands: E2

Solution	Answer	Mark
$x + py = 2ap + ap^3$ When $x = 0, y = a$: $ap = 2ap + ap^3$ $ap(1 + p^2) = 0$ $p = 0$ is the only solution. Therefore there exists only one solution.	B	1

Question 8 (1 mark)
Outcomes Assessed: HE3, HE7
Targeted Performance Bands: E3

Solution	Answer	Mark
Let $p = 0.8$ and $q = 0.2$ $P(X \geq 5) = P(X = 5) + P(X = 6)$ $= \binom{6}{5}(0.2)(0.8)^5 + (0.8)^6$ $= (0.8)^5(6 \times 0.2 + 0.8)$ $= 2(0.8)^5$	B	1

Question 9 (1 mark)
Outcomes Assessed: HE3
Targeted Performance Bands: E3

Solution	Answer	Mark
$T_{k+1} = \binom{7}{k}(x^2)^{7-k}\left(\frac{2}{x}\right)^k$ $= \binom{7}{k}2^k x^{14-3k}$ When $14 - 3k = 2, k = 4$. The coefficient of $x^2 = \binom{7}{4}2^4$ $= 560.$	D	1

Question 10 (1 mark)
Outcomes Assessed: HE3
Targeted Performance Bands: E3, E4

Solution	Answer	Mark
The FALSE statement is C	C	1

Section II
60 marks

Question 11 (15 marks)
(a) (1 mark)
Outcomes assessed: H5
Targeted Performance Bands: E2

Criteria	Mark
• Correct answer	1

Sample answer:

$$\lim_{x \to 0} \frac{\sin 2x}{x} = 2 \lim_{x \to 0} \frac{\sin 2x}{2x}$$
$$= 2$$

(b) (2 marks)
Outcomes assessed: HE6
Targeted Performance Bands: E2

Criteria	Marks
• Correct answer	2
• Recognising the correct primitive	1

Sample answer:

$$\int_0^4 \frac{dx}{\sqrt{x^2 + 9}} = \left[\ln\left(x + \sqrt{x^2 + 9}\right)\right]_0^4$$
$$= \ln 9 - \ln 3$$
$$= \ln 3$$

(c) (3 marks)
Outcomes assessed: HE4, HE6
Targeted Performance Bands: E2–E3

Criteria	Marks
• Correct answer	3
• Significant progress towards solution	2
• Correct integral with correct limits, after applying given substitution	1

Sample answer:

Let $u = x - 8$
$du = dx$
$x = 8.5 \Rightarrow u = 0.5$ and $x = 8 \Rightarrow u = 0$.

Upon substituting: $\int_0^{0.5} \frac{du}{\sqrt{(7-(u+8))(u+8-9)}}$

$= \int_0^{0.5} \frac{du}{\sqrt{1-u^2}}$

$= [\sin^{-1}u]_0^{0.5}$

$= \frac{\pi}{6}$

(d) (3 marks)
Outcomes assessed: PE3
Targeted Performance Bands: E2–E3

Criteria	Marks
• Correct range of solutions given	3
• Significant progress towards solution (e.g. inequality if using solution presented below or points of intersection if using graphical techniques)	2
• Some progress towards answer (considering technique used e.g. graphical or algebraic)	1

Sample answer:

$\frac{2t}{1-t} \geq t \qquad t \neq 1$

$2t(1-t) \geq t(1-t)^2$

$t(1-t)(1+t) \geq 0$

$\therefore t \leq -1, 0 \leq t < 1$

(e) (i) (2 marks)
Outcomes assessed: PE2, PE3
Targeted Performance Bands: E2

Criteria	Marks
• Correct answer	2
• Recognises the relationship between the tangent and the intercepts from an external point	1

Sample answer:

$PC^2 = AP.BP$
$12^2 = (x+7).x$
$x = 9 \quad (\text{since } x > 0)$

(e) (ii) (1 mark)
Outcomes assessed: PE3
Targeted Performance Bands: E2

Criteria	Mark
• Correct answer	1

Sample answer:
$\angle BPC = 90°$ since BC is the diameter of the circle passing through P, B and C.
$\therefore \Delta BPC$ is a right-angled triangle.

$BC^2 = BP^2 + PC^2$
$BC^2 = 9^2 + 12^2$
$BC = 15$

(f) (3 marks)
Outcomes assessed: PE2, PE6, HE7
Targeted Performance Bands: E3

Criteria	Marks
• Correct proof logically presented	3
• Progress towards solution	2
• Correctly identifying a relevant trigonometric relationship in one of the triangles	1

Sample answer:
Let $CD = h$

In ΔADC, $\tan\alpha = \frac{h}{AC}$
In ΔBDC, $\tan\beta = \frac{h}{BC}$
$\therefore \frac{\tan\alpha}{\tan\beta} = \frac{BC}{AC}$

ΔABC is right-angled at C and $\angle CAB = 90° - 60° = 30°$ (from the bearings given), hence

$\frac{BC}{AC} = \tan 30° = \frac{1}{\sqrt{3}}$

$\therefore \frac{\tan\alpha}{\tan\beta} = \frac{1}{\sqrt{3}}$

Question 12 (15 marks)

(a) (i) (1 mark)

Outcomes assessed: PE3

Targeted Performance Bands: E2

Criteria	Mark
• Correct answer	1

Sample answer:

$$\alpha+\beta+\gamma=\frac{-b}{a}$$

(a) (ii) (2 marks)

Outcomes assessed: H5, HE7

Targeted Performance Bands: E3–E4

Criteria	Marks
• Correctly using the result in part (i) to obtain the result	2
• Solves $f''(x)=0$	1

Sample answer:

$$f'(x)=3ax^2+2bx+c$$
$$f''(x)=6ax+2b$$

Solving $f''(x)=0$

$$6ax+2b=0$$
$$x=\frac{-b}{3a}$$
$$=\frac{\alpha+\beta+\gamma}{3}$$

(a) (iii) (1 mark)

Outcomes assessed: HE7

Targeted Performance Bands: E3

Criteria	Mark
• Correct answer	1

Sample answer:

$$x=\frac{\alpha+\beta+\gamma}{3}$$
$$=\frac{-1+3+4}{3}$$
$$=2$$

(b) (i) (2 marks)

Outcomes assessed: HE4, HE7

Targeted Performance Bands: E3

Criteria	Marks
• Correct value of x	2
• Correct value of α	1

Sample answer:

$\sin^{-1}x=\alpha,\ \cos^{-1}x=\frac{\pi}{2}-\alpha$

Now, $\sin^{-1}x=\cos^{-1}x$

i.e. $\alpha=\frac{\pi}{2}-\alpha$

$\alpha=\frac{\pi}{4}$

$\therefore \sin^{-1}x=\frac{\pi}{4}$

$x=\sin\frac{\pi}{4}$

$\therefore$ The solution is $x=\frac{\sqrt{2}}{2}$.

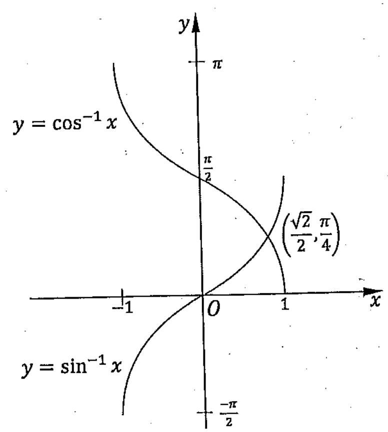
(b) (ii) (1 marks)

Outcomes assessed: HE4, HE7

Targeted Performance Bands: E2

Criteria	Marks
• Sketches curves correctly, showing the point of intersection	1

Sample answer:



(b) (iii) (2 marks)
Outcomes assessed: HE2
Targeted Performance Bands: E3

Criteria	Mark
• Correct explanation	2
• Reference to symmetry of region	1

Sample answer:

The region is symmetrical about the line $y = \frac{\pi}{4}$.
Therefore the volume of the rotated region bounded by $y = \sin^{-1} x$ between $y = 0$ and $y = \frac{\pi}{4}$ (given by $\pi \int_0^{\frac{\pi}{4}} \sin^2 y\, dy$) is equal to the volume of the rotated region bounded by $y = \cos^{-1} x$ between $y = \frac{\pi}{4}$ and $y = \frac{\pi}{2}$ (given by $\pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 y\, dy$).

$$\therefore V = \pi \int_0^{\frac{\pi}{4}} \sin^2 y\, dy + \pi \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^2 y\, dy$$
$$= 2\pi \int_0^{\frac{\pi}{4}} \sin^2 y\, dy$$

(b) (iv) (2 marks)
Outcomes assessed: H8, HE6
Targeted Performance Bands: E3

Criteria	Marks
• Correct answer	2
• Substantial progress towards integrating $\sin^2 y$	1

Sample answer:

$$\begin{aligned} V &= 2\pi \int_0^{\frac{\pi}{4}} \sin^2 y\, dy \\ &= \pi \int_0^{\frac{\pi}{4}} (1 - \cos 2y) dy \\ &= \pi \left[y - \frac{\sin 2y}{2} \right]_0^{\frac{\pi}{4}} \\ &= \frac{\pi}{4}(\pi - 2) \end{aligned}$$

$\therefore$ The volume is $\frac{\pi}{4}(\pi - 2)$ units^3.

(c) (i) (3 marks)
Outcomes assessed: HE2, HE7
Targeted Performance Bands: E3

Criteria	Marks
• Correct solution	3
• Establishes the induction step	2
• Verifies the result for n=2	1

Sample answer:

(1) When $n = 2$: $\quad \text{LHS} = \left(1 - \frac{1}{2^2}\right) = \frac{3}{4}$

$\text{RHS} = \frac{2+1}{2\times 2} = \frac{3}{4} = \text{LHS}$

Assume true for $n = k$: $\left(1 - \frac{1}{2^2}\right) \times \left(1 - \frac{1}{3^2}\right) \times \left(1 - \frac{1}{4^2}\right) \times \ldots \times \left(1 - \frac{1}{k^2}\right) = \frac{k+1}{2k}$

Prove true for $n = k + 1$:

i.e. $\left(1 - \frac{1}{2^2}\right) \times \left(1 - \frac{1}{3^2}\right) \times \left(1 - \frac{1}{4^2}\right) \times \ldots \times \left(1 - \frac{1}{k^2}\right) \times \left(1 - \frac{1}{(k+1)^2}\right) = \frac{k+2}{2(k+1)}$

Now $\quad \text{LHS} = \frac{k+1}{2k} \times \left(1 - \frac{1}{(k+1)^2}\right) \quad$ (using assumption)

$$\begin{aligned} &= \frac{k+1}{2k} \times \frac{(k+1)^2 - 1}{(k+1)^2} \\ &= \frac{k^2 + 2k}{2k(k+1)} \\ &= \frac{k+2}{2(k+1)} \\ &= \text{RHS} \end{aligned}$$

Hence, by the Principle of Mathematical Induction, the result holds true for all integers $n \geq 2$.

(c) (ii) (1 mark)
Outcomes assessed: HE2, HE7
Targeted Performance Bands: E2

Criteria	Mark
• Correct answer	1

Sample answer:

$n = 100$, hence the product equals $\frac{101}{200}$.

Question 13 (15 marks)
(a) (i) (1 mark)
Outcomes assessed: HE3, HE7
Targeted Performance Bands: E2–E3

Criteria	Mark
• Correct proof	1

Sample answer:

$$T = 100 - Ae^{-0.2t}$$
$$\frac{dT}{dt} = 0.2Ae^{-0.2t}$$
$$= 0.2(100 - T) \qquad \text{as } Ae^{-0.2t} = 100 - T.$$

(a) (ii) (1 mark)
Outcomes assessed: HE3, HE7
Targeted Performance Bands: E2–E3

Criteria	Mark
• Correct value for A	1

Sample answer:

When $t = 0, T = 4$.
$4 = 100 - Ae^{-0.2\times 0}$
$A = 96$

(a) (iii) (1 mark)
Outcomes assessed: HE3, HE7
Targeted Performance Bands: E2–E3

Criteria	Mark
• Correct value for temperature	1

Sample answer:

$T = 100 - 96e^{-0.2t}$
Substitute $t = 4.5$
$T = 100 - 96e^{-0.2\times 4.5}$
$T = 60.9693\ldots$

Therefore, the temperature correct to three significant figures is 61.0°C.

(a) (iv) (2 marks)
Outcomes assessed: HE3, HE7
Targeted Performance Bands: E2–E3

Criteria	Marks
• Correct answer	2
• Correct value of t for an egg boiled from room temperature	1

Sample answer:

$T = 100 - 79e^{-0.2t}$
Substituting $T = 60.9693\ldots$
$60.9693\ldots = 100 - 79e^{-0.2t}$
$t = \ln\left(\frac{100-60.9693\ldots}{79}\right) \div -0.2$
$= 3.525498\ldots \approx 3.5$ minutes

Therefore, the egg at room temperature will cook approximately one minute faster than an egg taken from the refrigerator.

(b) (i) (3 marks)
Outcomes assessed: HE3, HE7
Targeted Performance Bands: E3–E4

Criteria	Marks
• Correct answers for A and α	3
• Correct value for either A or α	2
• Significant progress towards correct equations involving A and α	1

Sample answer:

$x = A\cos(2t + \alpha)$

When $t = 0, x = 4$: $\quad A\cos\alpha = 4 \quad$ (i)

Now $\quad \frac{dx}{dt} = -2A\sin(2t + \alpha)$

When $t = 0, \frac{dx}{dt} = -8\sqrt{3}$ $\quad A\sin\alpha = 4\sqrt{3} \quad$ (ii)

From (i) and (ii) above: $\quad \frac{A\sin\alpha}{A\cos\alpha} = \frac{4\sqrt{3}}{4}$

$\tan\alpha = \sqrt{3}$

$\alpha = \frac{\pi}{3}$

Substitute $\alpha = \frac{\pi}{3}$ into (i): $\quad A\cos\frac{\pi}{3} = 4$

$A = 8$

Therefore, $A = 8$ and $\alpha = \frac{\pi}{3}$.

(b) (ii) (2 marks)
Outcomes assessed: HE3, HE7
Targeted Performance Bands: E3–E4

Criteria	Marks
• Correct answer	2
• Correct equation with ONE unknown, t	1

Sample answer:
Particle is at the centre of motion when $x = 0$:

$$8\cos\left(2t + \frac{\pi}{3}\right) = 0$$
$$2t + \frac{\pi}{3} = \frac{\pi}{2}, \text{ for the first time.}$$
$$t = \frac{\pi}{12}$$

Therefore, the particle arrives at the origin after $\frac{\pi}{12}$ seconds.

(c) (i) (3 marks)
Outcomes assessed: HE5, HE7
Targeted Performance Bands: E3–E4

Criteria	Marks
• Correctly shows the result	3
• Correct expression for v^2, including the value of the constant	2
• Progress towards expressing the result in terms of v and x, e.g. using the correction version of acceleration $\frac{d}{dx}\left(\frac{1}{2}v^2\right)$	1

Sample answer:

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2x^3 + 18x$$
$$\frac{1}{2}v^2 = \frac{1}{2}x^4 + 9x^2 + c$$

When $x = 0, v = 9$: $c = \frac{81}{2}$

Therefore,
$$\frac{1}{2}v^2 = \frac{1}{2}x^4 + 9x^2 + \frac{81}{2}$$
$$v^2 = x^4 + 18x^2 + 81$$
$$v^2 = (x^2 + 9)^2$$
$$v = \pm(x^2 + 9)$$

However, since initially the particle has a positive velocity and there are no solutions to $v = 0$, the particle must always travel in a positive direction.
Hence, $v = x^2 + 9$.

(c) (ii) (2 marks)
Outcomes assessed: HE5, HE7
Targeted Performance Bands: E3–E4

Criteria	Mar
• Correct equation for x in terms of t	2
• Progress towards an expression for t in terms of x	1

Sample answer:

$$\frac{dx}{dt} = x^2 + 9$$
$$\frac{dt}{dx} = \frac{1}{x^2+9}$$
$$t = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + c$$

When $x = 0$, $t = 0$, therefore $c = 0$.

$$t = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right)$$
$$x = 3\tan(3t)$$

Question 14 (15 marks)
(a) (3 marks)
Outcomes assessed: H5, HE7
Targeted Performance Bands: E2–E3

Criteria	Marks
• Correct answer	3
• Correct substitution into the formula of Newton's Method	2
• Establishing an appropriate function and determining its derivative	1

Sample answer:

$$3\sin x = \ln x$$
$$\text{Let } f(x) = 3\sin x - \ln x$$
$$f'(x) = 3\cos x - \frac{1}{x}$$
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$
$$= 3 - \frac{3\sin 3 - \ln 3}{3\cos 3 - \frac{1}{3}}$$
$$= 2.79558\ldots$$
$$= 2.80 \text{ (2 d.p.)}$$

(b) (i) (1 mark)
Outcomes assessed: H5
Targeted Performance Bands: E3, E4

Criteria	Mark
• Correct justification for result	1

Sample answer:
For the gradient of MB: applying Pythagoras' Theorem, rise $= \sqrt{L^2 - x^2}$; run $= x$.
The gradient of $MB = \frac{-\text{rise}}{\text{run}} = \frac{-\sqrt{L^2-x^2}}{x}$.
Since the rope is always tangent to the curve, the line MB is a tangent to $y = f(x)$

$$\therefore \frac{dy}{dx} = \frac{-\sqrt{L^2 - x^2}}{x}.$$

(b) (ii) (2 marks)
Outcomes assessed: HE5, HE7
Targeted Performance Bands: E3–E4

Criteria	Marks
• Correct answer	2
• Substantial progress towards expressing $\frac{dx}{dt}$ as a function of x	1

Sample answer:

$$\frac{dx}{dt} = \frac{dx}{dy} \times \frac{dy}{dt}$$
$$= \frac{-x}{\sqrt{L^2-x^2}} \times 3$$
$$= \frac{\frac{-L}{2}}{\sqrt{L^2 - \frac{L^2}{4}}} \times 3 \quad \text{when } x = \frac{L}{2}$$
$$= -\sqrt{3}$$

Therefore, the boat approaches the pier at a rate of $\sqrt{3}\ \text{ms}^{-1}$.

(c) (3 marks)
Outcomes assessed: PE3, PE4, HE2
Targeted Performance Bands: E3–E4

Criteria	Marks
• Correct proof	3
• Substantial progress towards determining both required distances or both required angles	2
• Recognising the gradient at P is given by $\frac{1}{p}$ and progress towards determining a relevant distance or a relevant angle	1

Sample answer:
The gradient of the tangent at $P = \frac{2a}{2ap} = \frac{1}{p}$.
The equation of the tangent at P is given by $y - 2ap = \frac{1}{p}(x - ap^2)$
Let the tangent at P intersect the x-axis at G.
$\therefore G$ has coordinates $(-ap^2, 0)$. P has coordinates $(ap^2, 2ap)$. S has coordinates $(a, 0)$.

$SG = ap^2 + a$

$$SP = \sqrt{(ap^2 - a)^2 + (2ap)^2}$$
$$= \sqrt{a^2p^4 + 2a^2p^2 + a^2}$$
$$= \sqrt{(ap^2 + a)^2}$$
$$= ap^2 + a$$

Since $SG = SP$, ΔSPG is isosceles.
$\therefore \angle SGP = \angle SPG$ (angles opposite equal sides of an isosceles triangle are equal), thus the angle of inclination to the axis of the parabola equals the angle between the tangent and the focal chord.
Hence the tangent to the parabola at P is equally inclined to the axis of the parabola and the focal chord through P.

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(d) (i) (1 mark)

Outcomes assessed: HE2, HE7,

Targeted Performance Bands: E3

Criteria	Mark
• Correct proof	1

Sample answer:

The coefficient of x^r in the expansion $(1+x)^n$ is $\binom{n}{r}$.

The coefficient of x^r in the expansion $(x+1)^n$ is $\binom{n}{n-r}$.

Since $(1+x)^n = (x+1)^n$, the expansions are the same. Therefore, $\binom{n}{n-r} = \binom{n}{r}$.

(d) (ii) (3 marks)

Outcomes assessed: HE2, HE7

Targeted Performance Bands: E4

Criteria	Marks
• Correct proof	3
• Correctly identifies and matches coefficients of x^{n-r} in the two binomial expansions	2
• Expansion of ONE relevant binomial expression	1

Sample answer:

$f(r) = \binom{n}{0}\binom{n}{r} + \binom{n}{1}\binom{n}{r+1} + \cdots + \binom{n}{n-r}\binom{n}{n}$.

Therefore, $f(r) = \binom{n}{0}\binom{n}{n-r} + \binom{n}{1}\binom{n}{n-r-1} + \cdots + \binom{n}{n-r}\binom{n}{0}$, using the result in part (i).

Now, consider $(1+x)^n(1+x)^n = (1+x)^{2n}$

$$(1+x)^n(1+x)^n = \left(\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n\right)\left(\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n\right)$$

The coefficient of x^{n-r} in this expansion is:

$$\binom{n}{0}\binom{n}{n-r} + \binom{n}{1}\binom{n}{n-r-1} + \binom{n}{2}\binom{n}{n-r-2} + \cdots + \binom{n}{n-r}\binom{n}{0} = f(r) \text{ from above}$$

But the coefficient of x^{n-r} in the expansion of $(1+x)^{2n}$ is $\binom{2n}{n-r}$.

Therefore, $f(r) = \binom{2n}{n-r}$, for $r = 0, 1, 2, \ldots, n$

(d) (iii) (2 marks)

Outcomes assessed: HE2, HE7

Targeted Performance Bands: E3–E4

Criteria	Mar
• Correct proof	2
• Correctly identifies and matches coefficients in relevant binomial expansions	1

Sample answer:

Consider $(1+x)^{3n} = (1+x)^{2n}(1+x)^n$

$$(1+x)^{2n}(1+x)^n = \left(\binom{2n}{0} + \binom{2n}{1}x + \binom{2n}{2}x^2 + \cdots + \binom{2n}{2n}x^{2n}\right)\left(\binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n\right)$$

The coefficient of x^n in this expansion is:

$$\binom{2n}{0}\binom{n}{n} + \binom{2n}{1}\binom{n}{n-1} + \binom{2n}{2}\binom{n}{n-2} + \cdots + \binom{2n}{n}\binom{n}{0}$$

$= f(n)\binom{n}{n} + f(n-1)\binom{n}{n-1} + f(n-2)\binom{n}{n-2} + \cdots + f(0)\binom{n}{0}$, as $f(r) = \binom{2n}{n-r}$ for $r = 0, 1, 2 \ldots,$

But the coefficient of x^n in the expansion of $(1+x)^{3n}$ is $\binom{3n}{n}$.

Therefore, $\binom{n}{0}f(0) + \binom{n}{1}f(1) + \cdots + \binom{n}{n}f(n) = \binom{3n}{n}$.