CSSA

CATHOLIC SECONDARY SCHOOLS ASSOCIATION OF NSW

Centre Number

Student Number

2015

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

Morning Session
Friday August 7 2015

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen. Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided on a SEPARATE sheet
- In Questions 11-14, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and the Student Number at the top of this page

Total marks – 70

Section I Pages 2-5

10 marks

- Attempt Questions 1-10
- Allow 15 minutes for this section

Section II Pages 6 - 11

60 marks

- Attempt Questions 11-14
- Allow about 1 hour and 45 minutes for this section

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\ x > 0$

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 What is the remainder when the polynomial $p(x) = x^3 + 2x^2 - 5x - 6$ is divided by $(x-2)$?

(A) -12

(B) -6

(C) 0

(D) 4

2 What is the acute angle between the tangents drawn to the curve $y = \log_e x$ at the points where $x = 2$ and $x = 3$? Give your answer to the nearest degree.

(A) 8°

(B) 17°

(C) 36°

(D) 45°

3 The point P divides the interval joining $A(1, 3)$ to $B(4, 1)$ externally in the ratio 1:3. What are the coordinates of P?

(A) $\left(-\frac{7}{4}, \frac{5}{2}\right)$

(B) $\left(\frac{13}{4}, \frac{3}{2}\right)$

(C) $\left(-\frac{1}{2}, 4\right)$

(D) $\left(\frac{11}{2}, 0\right)$

4 What is the derivative of $\sin^{-1}\frac{2x}{3}$?

(A) $\frac{2}{\sqrt{3-2x^2}}$

(B) $\frac{2}{\sqrt{9-4x^2}}$

(C) $-\frac{2}{\sqrt{3-2x^2}}$

(D) $-\frac{2}{\sqrt{9-4x^2}}$

5 Find $\int \frac{4}{25+16x^2}\,dx$.

(A) $\frac{1}{5}\tan^{-1}\frac{5x}{4}+C$

(B) $\frac{1}{5}\tan^{-1}\frac{4x}{5}+C$

(C) $5\tan^{-1}\frac{5x}{4}+C$

(D) $5\tan^{-1}\frac{4x}{5}+C$

6 A coach, manager and six players sit around a circular table to discuss tactics. In how many ways can they sit if the coach and the manager are not to sit together?

(A) 3600

(B) 4320

(C) 38880

(D) 39600

7 If $\sin\theta=\frac{1}{4}$ and $\cos\theta<0$, what is the exact value of $\tan 2\theta$?

(A) $-\frac{15}{7\sqrt{15}}$

(B) $\frac{15}{7\sqrt{15}}$

(C) $-\frac{16}{7\sqrt{15}}$

(D) $\frac{16}{7\sqrt{15}}$

8 The volume of a cube is increasing at a constant rate of 100 cm^3 per second. At what rate is the total surface area of the cube increasing when the side length of the cube is 10 cm?

(A) $\frac{5}{6}$ cm^2 per second

(B) $\frac{1}{3}$ cm^2 per second

(C) 40 cm^2 per second

(D) 750 cm^2 per second

9 Consider the equation $\frac{\sin\theta\cos\theta}{2\cos^2\theta-1}=-\frac{\sqrt{3}}{2}$.
How many solutions does the equation have in the domain $0\le\theta\le 2\pi$?

(A) Two

(B) Three

(C) Four

(D) Five

10 At Euclid High School the Year 12 grade consists of n boys and n girls. A committee of 4 is to be chosen from Year 12 students.

How many different committees can be formed containing 2 boys and 2 girls?

(A) $n^2(n^2-2n+1)$

(B) $\frac{n^2-n}{2}$

(C) n^2-n

(D) $\frac{n^2(n^2-2n+1)}{4}$

Section II

60 marks
Attempt Questions 11 - 14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11-14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Solve $\frac{2}{1+3x} \leq 1$. **3**

(b) (i) Differentiate $\tan^{-1} 2x$ with respect to x. **1**

(ii) Find the coordinates of the points on the curve $y = \tan^{-1} 2x$ where the tangent is perpendicular to the line $3x+3y-1=0$. **3**

(c) (i) Find the values of A and α so that $\cos\theta - \sqrt{3}\sin\theta = A\cos(\theta+\alpha)$ where $A > 0$ and $0 < \alpha < \frac{\pi}{2}$. **2**

(ii) Hence, or otherwise, solve $\cos\theta - \sqrt{3}\sin\theta = 1$ for $0 \leq \theta \leq 2\pi$. **3**

(d) The equation $\log_e x = x^2 - 5x$ has a solution near $x = 5$. **3**

Use one application of Newton's method to find another approximation to the solution. Leave your answer correct to 1 decimal place.

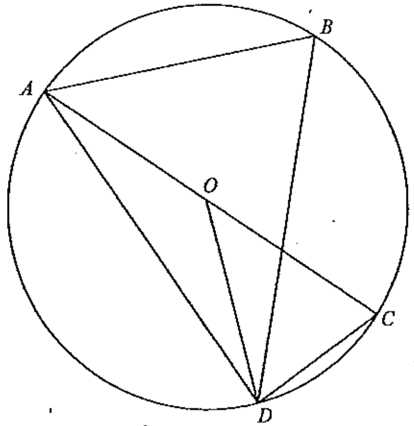
End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\int_0^{\frac{\pi}{2}} \cos^2 2x + \sin^2 \frac{x}{2}\, dx$. **3**

(b) Use the substitution $u = 2t+1$ or otherwise to evaluate $\int_0^4 \frac{2t}{\sqrt{2t+1}}\, dt$. **3**

(c) Consider the circle below where O is the centre and AC is a diameter. The points A, B, C and D all lie on the circumference of the circle. **2**



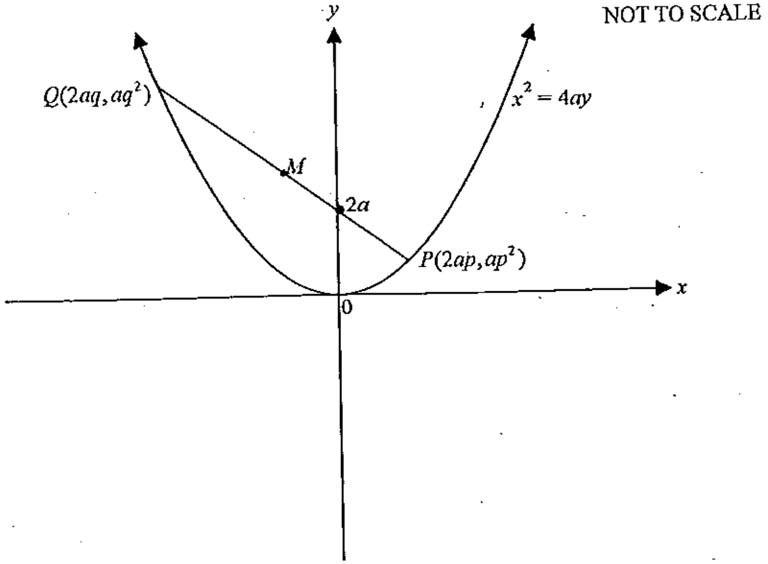
NOT TO SCALE

Prove $\angle DCA = 90^\circ - \angle DBC$.

Question 12 continues on page 8

Question 12 (continued)

(d)



$P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are points on the parabola with equation $x^2 = 4ay$ and the chord PQ passes through the point $(0, 2a)$.
M is the midpoint of PQ.

(i) Given that the equation of the chord PQ is $y - ap^2 = \frac{(p+q)}{2}(x - 2ap)$, show that $pq = -2$. **1**

(ii) Find the equation of the locus of M as P and Q move on the parabola. **3**

(e) Find the coefficient of x^8 in the expansion $\left(\frac{x}{3} - \frac{3}{x}\right)^{12}$. **3**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Use mathematical induction to prove that **3**

$$1^2 + 3^2 + 5^2 + \ldots\ldots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3} \text{ for all positive integers } n.$$

(b) Let $f(x) = \log_e\left(\frac{2-x}{x}\right)$ for $0 < x < 2$.

(i) By considering the derivative function, explain why $f(x)$ has an inverse function. **2**

(ii) Find the equation of the inverse function in terms of x. **2**

(c) A particle is moving in simple harmonic motion. The acceleration equation of the particle in ms^{-2} is given by $\ddot{x} = 5 - x$, where x is the displacement of the particle from the origin.

The particle is initially at $x = 4$ and moving with a velocity of $\sqrt{3}$ ms^{-1}.

Find

(i) the centre of motion. **1**

(ii) the amplitude of the motion. **2**

(iii) the maximum speed reached by the particle. **2**

(d) A bag contains two balls marked with a 6 and three balls marked with a 4. A ball is drawn from the bag, its number noted and then placed back into the bag. Altogether 4 balls are drawn, repeating the same process. **3**

What is the probability that the sum of the four balls drawn is greater than 20?

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the function $f(x) = 2\sin^{-1}\left(\frac{1-x}{3}\right) + \frac{\pi}{2}$.

(i) Evaluate $f(-2)$. **1**

(ii) Find the domain of $f(x)$. **1**

(iii) Find the range of $f(x)$. **1**

(iv) Hence sketch the graph of $y = f(x)$. **2**

(b) The equations of motion of a projectile fired from the origin with initial velocity U ms^{-1} at angle α to the horizontal are given by

$x = Ut\cos\alpha$ and $y = -5t^2 + Ut\sin\alpha$, where t is in seconds.

[Do NOT prove this]

(i) Show that $y = \frac{-5x^2}{U^2}\sec^2\alpha + x\tan\alpha$ **2**

(ii) A ball is projected to just clear 2 walls of height h metres at distances of a metres and b metres $(b > a)$ from the point of projection. **3**

Show that $\tan\alpha = \frac{h(a+b)}{ab}$.

(iii) If the two walls are 20 metres high each, at distances of 40 metres and 80 metres from the point of projection, find the angle of projection if the ball just clears the two walls. Answer to the nearest degree. **1**

Question 14 continues on Page 11

Question 14 (continued)

(c) (i) Let m and n be positive integers. You are given the identity **2**

$$(1+x)^n + (1+x)^{n+1} + \dots\dots + (1+x)^{n+m} = \frac{(1+x)^{n+m+1} - (1+x)^n}{x} \quad \text{where } x \neq 0.$$

[Do NOT prove this]

Show that ${}^{n}C_n + {}^{n+1}C_n + \dots\dots + {}^{n+m}C_n = {}^{n+m+1}C_{n+1}$.

(ii) Using (i) or otherwise, show that **2**

$$\sum_{r=5}^{n+4} r(r-1)(r-2)(r-3) = 24({}^{n+5}C_5 - 1).$$

End of Examination

EXTENSION 1 CATHOLIC TRIAL 2015 SOLUTIONS

Outcome/Band

$P(x) = x^3 + 2x^2 - 5x - 6$

$P(2) = (2)^3 + 2(2)^2 - 5(2) - 6$

$P(2) = 0$ C   PE3 / E2

2. $y = \ln x$

$\frac{dy}{dx} = \frac{1}{x}$

$m_1 = \frac{1}{2}$

$m_2 = \frac{1}{3}$

$\tan\theta = \frac{|m_1 - m_2|}{1 + m_1 m_2}$

$\tan\theta = \frac{\left|\frac{1}{2} - \frac{1}{3}\right|}{1 + \left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}$

$\tan\theta = \frac{1}{7}$

$\theta = \tan^{-1}\left(\frac{1}{7}\right)$

$\theta \doteq 8°$ A   PE2 / E2

3. $P = \left(\frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}\right)$ $A(1, 3)$ $B(4, 1)$ $m : n$ $1 : -3$ (External)

$P = \left(\frac{1(4) - 3(1)}{1 - 3}, \frac{1(1) - 3(3)}{1 - 3}\right)$

$P = \left(-\frac{1}{2}, 4\right)$ C   PE3/E2

4. $\frac{d}{dx} \sin^{-1}\frac{2x}{3}$

$= \frac{1}{\sqrt{1 - \left(\frac{2}{3}x\right)^2}} \cdot \frac{2}{3}$

$= \frac{1}{\sqrt{1 - \frac{4}{9}x^2}} \cdot \frac{2}{3}$

$= \frac{1}{\sqrt{\frac{9 - 4x^2}{9}}} \cdot \frac{2}{3}$

$= \frac{2}{\cancel{3}\frac{\sqrt{9 - 4x^2}}{\cancel{3}}}$

$= \frac{2}{\sqrt{9 - 4x^2}}$ B   HE4 / E3

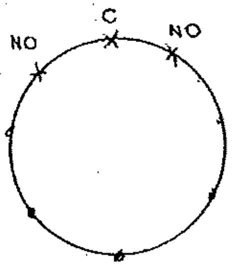
5. $\displaystyle\int \frac{4}{25+16x^2}\,dx$

$= 4\displaystyle\int \frac{1}{16\left(\frac{25}{16}+x^2\right)}\,dx$

$= \dfrac{1}{4}\displaystyle\int \frac{1}{\left(\frac{5}{4}\right)^2+x^2}\,dx$

$= \dfrac{1}{4}\cdot\dfrac{1}{\frac{5}{4}}\tan^{-1}\dfrac{x}{\frac{5}{4}} + C$

$= \dfrac{1}{5}\tan^{-1}\dfrac{4x}{5} + C$ (B) HE4 / E2-E3



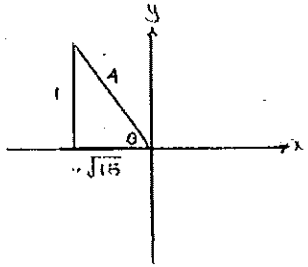
Coach = 1

Manager = 5 possible seats

Six Players = 6!

Possible arrangements $= 1 \times 5 \times 6!$

$= 3600$ (A) PE3 / E2-E3

7. 

$\sin\theta > 0$

$\cos\theta < 0$

$\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

$= 2\left(\dfrac{1}{-\sqrt{15}}\right) \div \left(1-\left(\dfrac{-1}{\sqrt{15}}\right)^2\right)$

$= -\dfrac{2}{\sqrt{15}} \div \left(\dfrac{15-1}{15}\right)$

$= -\dfrac{\cancel{2}}{\cancel{\sqrt{15}}} \times \dfrac{\sqrt{15}}{14_{\,7}}$ (A) PE2 / E3

$S = 6x^2$

$\dfrac{dS}{dx} = 12x$

$\dfrac{dS}{dt} = \dfrac{dS}{dx}\cdot\dfrac{dx}{dV}\cdot\dfrac{dV}{dt}$

$\dfrac{dS}{dt} = 12x\cdot\dfrac{1}{3x^2}\cdot 100$

Given: $\dfrac{dV}{dt} = 100$

$V = x^3$

$\dfrac{dV}{dx} = 3x^2$

$x = 10$:

$\dfrac{dS}{dt} = 12(10)\cdot\dfrac{1}{3(10)^2}\cdot 100$

$\dfrac{dS}{dt} = 40\text{ cm}^2/\text{second}$ (C) HE3 / E3-E4

$\dfrac{\sin\theta\cos\theta}{2\cos^2\theta-1} = -\dfrac{\sqrt{3}}{2}$ $\quad 0 \le \theta \le 2\pi$

$\dfrac{\frac{1}{\cancel{2}}\sin 2\theta}{\cos 2\theta} = \dfrac{-\sqrt{3}}{\cancel{2}}$

Quadrant 2 and 4

$\tan 2\theta = \dfrac{-\sqrt{3}}{1}$ $\quad 0 \le 2\theta \le 4\pi$

$2\theta = \pi-\dfrac{\pi}{3},\ 2\pi-\dfrac{\pi}{3},\ 3\pi-\dfrac{\pi}{3},\ 4\pi-\dfrac{\pi}{3}$

$2\theta = \dfrac{2\pi}{3},\ \dfrac{5\pi}{3},\ \dfrac{8\pi}{3},\ \dfrac{11\pi}{3}$

$\theta = \dfrac{2\pi}{6},\ \dfrac{5\pi}{6},\ \dfrac{8\pi}{6},\ \dfrac{11\pi}{6}$

$\theta = \dfrac{\pi}{3},\ \dfrac{5\pi}{6},\ \dfrac{4\pi}{3},\ \dfrac{11\pi}{6}$ (C) HE7 / E3-E4

Q10. Boys $= {}^nC_2$

Girls $= {}^nC_2$

Number of Committees Boys and Girls

$$= {}^nC_2 \times {}^nC_2$$

$$= \frac{n!}{2!(n-2)!} \times \frac{n!}{2!(n-2)!}$$

$$= \frac{n(n-1)\cancel{(n-2)\ldots 3.2.1}}{2\,\cancel{(n-2)\ldots 3.2.1}} \times \frac{n(n-1)\cancel{(n-2)\ldots 3.2.1}}{2\,\cancel{(n-2)\ldots 3.2.1}}$$

$$= \frac{n^2(n-1)^2}{4}$$

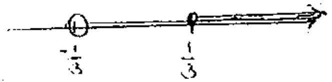
$$= \frac{n^2(n^2-2n+1)}{4}$$

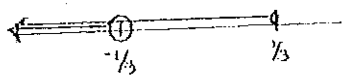
(D) PE3/E3-E4

Section II

11(a) $\dfrac{2}{1+3x} \le 1$

$1+3x > 0$	$1+3x < 0$
$3x > -1$	$3x < -1$
$\boxed{x > -\frac{1}{3}}$	$\boxed{x < -\frac{1}{3}}$
$2 \le 1(1+3x)$	$2 \ge 1(1+3x)$
$2 \le 1+3x$	$2 \ge 1+3x$
$1 \le 3x$	$1 \ge 3x$
$\frac{1}{3} \le x$	$\frac{1}{3} \ge x$
$x \ge \frac{1}{3}$	$x \le \frac{1}{3}$





$x \ge \frac{1}{3}$ or $x < -\frac{1}{3}$ PE3/E2

(b)(i) $\dfrac{d}{dx} \tan^{-1}\dfrac{2x}{1}$

$$= \frac{1}{1^2 + (2x)^2} \cdot 2$$

$$= \frac{2}{1+4x^2}$$

HE4/E2

Q11(b)(ii) $\frac{dy}{dx} = \frac{2}{1+4x^2}$

$m_1 = -\frac{b}{a}$

$m_1 = -\frac{3}{3} = -1$

$m_2 = -\frac{1}{-1} = 1$

$\therefore 1 = \frac{2}{1+4x^2}$

$1+4x^2 = 2$

$4x^2 = 1$

$x^2 = \frac{1}{4}$

$x = \pm\sqrt{\frac{1}{4}}$

$x = \pm\frac{1}{2}$

$x = \frac{1}{2}$; $y = \tan^{-1} 2\left(\frac{1}{2}\right)$

$y = \tan^{-1} 1$

$y = \frac{\pi}{4}$

$x = -\frac{1}{2}$; $y = \tan^{-1} 2\left(-\frac{1}{2}\right)$

$y = \tan^{-1}(-1)$

$y = -\frac{\pi}{4}$

$\therefore$ Coordinates are: $\left(\frac{1}{2}, \frac{\pi}{4}\right)$ and $\left(-\frac{1}{2}, -\frac{\pi}{4}\right)$

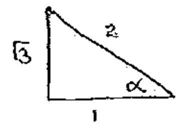
HE4/E3

Q11 (c)(i) $1\cos\theta - \sqrt{3}\sin\theta = A\cos(\theta+\alpha)$

$A[\cos\alpha\cos\theta - \sin\alpha\sin\theta] = A\cos\theta\cos\alpha + \sin\theta\sin\alpha$

$2\left(\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right) = 2\cos(\theta+\alpha)$

$2\left(\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right) = 2\cos\left(\theta+\frac{\pi}{3}\right)$



$\cos\alpha = \frac{1}{2}$

$\alpha = \cos^{-1}\left(\frac{1}{2}\right)$

$\alpha = \frac{\pi}{3}$

HE7/E2 - E3

(c)(ii) $\cos\theta - \sqrt{3}\sin\theta = 1$

$2\cos\left(\theta+\frac{\pi}{3}\right) = 1$

$\cos\left(\theta+\frac{\pi}{3}\right) = \frac{1}{2}$ Quad 1 and 4

$0 \le \theta \le 2\pi$

$0+\frac{\pi}{3} \le \theta+\frac{\pi}{3} \le 2\pi+\frac{\pi}{3}$

$\frac{\pi}{3} \le \theta+\frac{\pi}{3} \le \frac{7\pi}{3}$

$\theta+\frac{\pi}{3} = \cos^{-1}\left(\frac{1}{2}\right)$

$\theta+\frac{\pi}{3} = \frac{\pi}{3}, \frac{5\pi}{3}, \frac{7\pi}{3}$

$\theta = \frac{\pi}{3}-\frac{\pi}{3}, \frac{5\pi}{3}-\frac{\pi}{3}, \frac{7\pi}{3}-\frac{\pi}{3}$

$\theta = 0, \frac{4\pi}{3}, 2\pi$

HE7/E3

2 11(d) $\log_e x = x^2 - 5x$

$P(x) = x^2 - 5x - \ln x$

$P'(x) = 2x - 5 - \frac{1}{x}$

• $\underline{x = 5:}$ $P(5) = 5^2 - 5(5) - \ln 5$

$P(5) = -\ln 5$

$P'(5) = 2(5) - 5 - \frac{1}{5} = 4\frac{4}{5}$

$$\boxed{x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}}$$

$x_1 = 5 - \frac{-\ln 5}{4.8}$

$x_1 \doteq 5.3$

PE3 / E2 - E3

2 12(a) $\int_0^{\pi/2} \cos^2 2x + \sin^2 \frac{x}{2} \, dx$

$\cos^2\theta = \frac{1}{2}(1 + \cos 2\theta)$

$\sin^2\theta = \frac{1}{2}(1 - \cos 2\theta)$

$= \int_0^{\pi/2} \frac{1}{2}(1 + \cos 4x) + \frac{1}{2}(1 - \cos x) \, dx$

$= \frac{1}{2}\int_0^{\pi/2} 2 + \cos 4x - \cos x \, dx$

S
C
-S
-C

$= \frac{1}{2}\left[2x + \frac{\sin 4x}{4} - \sin x\right]_0^{\pi/2}$

$= \frac{1}{2}\left[2\left(\frac{\pi}{2}\right) + \frac{1}{4}\sin 4\left(\frac{\pi}{2}\right) - \sin\frac{\pi}{2}\right] - \frac{1}{2}\left[2(0) + \frac{1}{4}\sin 4(0) - \sin 0\right]$

$= \frac{1}{2}\left[\pi + 0 - 1\right]$

$= \frac{1}{2}\left[\pi - 1\right]$

HE6 / E2 - E3

(b) $\frac{1}{2}\int_0^4 \frac{2t}{\sqrt{2t+1}} 2\, dt$

Let $u = 2t + 1$

$\frac{du}{dt} = 2$

$du = 2\,dt$

• $t = 0: u = 1$

$t = 4: u = 9$

• $u - 1 = 2t$

$= \frac{1}{2}\int_1^9 \frac{(u-1)\,du}{u^{1/2}}$

$= \frac{1}{2}\int_1^9 \frac{u^1}{u^{1/2}} - \frac{1}{u^{1/2}} \, du$

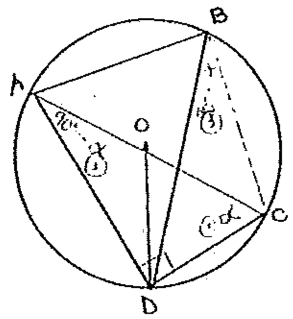
$= \frac{1}{2}\int_1^9 u^{1/2} - u^{-1/2} \, du$

$= \frac{1}{2}\left[\frac{u^{3/2}}{3/2} - \frac{u^{1/2}}{1/2}\right]_1^9$

$= \frac{1}{2}\left[\left(\frac{2}{3} 9^{3/2} - 2.9^{1/2}\right) - \left(\frac{2}{3} 1^{3/2} - 2(1)^{1/2}\right)\right]$

$= 6\frac{2}{3}$

Q12(c)



Join BC

Let $\angle DCA = \alpha$

$\angle ADC = 90°$ ($\angle$ at circumference).

$\angle CAD = 90° - \alpha$ ($\angle$ sum ΔADC)

$\angle CBD = \angle CAD = 90° - \alpha$ ($\angle$'s on same arc DC)

RHS $= 90° - \angle DBC$

$= 90° - (90° - \alpha)$

$= \alpha$

$= \angle DCA$

$=$ LHS.

Q12(d)(i) $y - ap^2 = \frac{(p+q)}{2}(x - 2ap)$

$(0, 2a)$ $2a - ap^2 = \frac{(p+q)}{2}(0 - 2ap)$

$2a - ap^2 = \frac{p+q}{\cancel{2}}(-\cancel{2}ap)$

$2a - a\cancel{p^2} = -a\cancel{p^2} - apq$

$2a = -apq$

$\frac{2\cancel{a}}{-\cancel{a}} = pq$

$\boxed{-2 = pq}$

(ii) $M = \left(\frac{x_2 + x_1}{2}, \frac{y_1 + y_2}{2}\right)$

$M = \left(\frac{2ap + 2aq}{2}, \frac{ap^2 + aq^2}{2}\right)$

$M = \left(\frac{\cancel{2}a(p+q)}{\cancel{2}}, \frac{a(p^2+q^2)}{2}\right)$

$x = a(p+q)$ ——— I

$y = \frac{a(p^2+q^2)}{2}$ ——— II

From I:

$\frac{x}{a} = p + q$

$\left(\frac{x}{a}\right)^2 = (p+q)^2$

$$\frac{x^2}{a^2} = p^2 + q^2 + 2pq$$

$$\frac{x^2}{a^2} - 2pq = p^2 + q^2$$

$$\frac{x^2}{a^2} - 2(-2) = p^2 + q^2$$

$$\frac{x^2}{a^2} + 4 = p^2 + q^2 \quad \text{———— III}$$

From II

$$\frac{2y}{a} = p^2 + q^2$$

$$\therefore \frac{x^2}{a^2} + 4 = \frac{2y}{a}$$

$$\frac{x^2}{a^2} = \frac{2y}{a} - 4$$

$$x^2 = 2ay - 4a^2$$

$$x^2 = 2a(y - a)$$

② $\left(\frac{x}{3} - \frac{3}{x}\right)^{12} = \left(\frac{x}{3} - 3x^{-1}\right)^{12}$

$$T_{r+1} = {}^{n}C_r\, a^{n-r}\, b^r$$

$$T_{r+1} = {}^{12}C_r \left(\frac{x}{3}\right)^{12-r} \left(-3x^{-1}\right)^r$$

$$T_{r+1} = {}^{12}C_r \left(\frac{1}{3}\right)^{12-r} (-3)^r \,(x^1)^{12-r}.(x^{-1})^r$$

$$T_{r+1} = {}^{12}C_r \left(\frac{1}{3}\right)^{12-r} (-3)^r\, x^{12-r-r}$$

$$T_{r+1} = {}^{12}C_r \left(\frac{1}{3}\right)^{12-r} (-3)^r\, x^{12-2r}$$

$$12 - 2r = 8$$

$$-2r = -4$$

$$r = 2$$

$$\therefore \text{Coefficient of } x^8 = {}^{12}C_2 \left(\frac{1}{3}\right)^{12-2} (-3)^2$$

$$= \frac{22}{2187}$$

Q13 (a) $1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$

• Prove true for $n=1$

$$T_1 = [2(1)-1]^2$$

$$T_1 = 1$$

$$S_1 = \frac{1(2(1)-1)(2(1)+1)}{3}$$

$$S_1 = 1$$
$$= T_1$$

$\therefore$ true for $n=1$

• Assume true for $n=k$

$$1^2 + 3^2 + 5^2 + \ldots (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3}$$

• Prove true for $n=k+1$

$$S_{k+1} = S_k + T_{k+1}$$

$$S_{k+1} = \frac{k(2k-1)(2k+1)}{3} + [2(k+1)-1]^2$$

$$S_{k+1} = \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2$$

$$S_{k+1} = \frac{(2k+1)}{3}\left[k(2k-1) + 3(2k+1)\right]$$

$$S_{k+1} = \frac{(2k+1)}{3}\left[2k^2 + 5k + 3\right]$$

$$S_{k+1} = \frac{(2k+1)}{3}\left[\frac{(2k+2)(2k+3)}{2}\right]$$

$$= \frac{(2k+1)}{3}\left[(k+1)(2k+3)\right]$$

$$= \frac{(k+1)[2(k+1)-1][2(k+1)+1]}{3}$$

HE2/E3

$\therefore$ By principle of mathematical induction true for all positive int. n

(b) $f(x) = \log_e\left(\dfrac{2-x}{x}\right)$ for $0 < x < 2$

(i) $f(x) = \ln(2-x) - \ln x$

$$f'(x) = \frac{-1}{2-x} - \frac{1}{x}$$

$$f'(x) = \frac{-x-(2-x)}{x(2-x)}$$

$$f'(x) = \frac{-2}{x(2-x)}$$

Now: $\boxed{0 < x < 2}$

$$x(2-x) > 0$$

$\therefore f'(x) < 0$ for this domain

ie. monotonic decreasing

HE4/E3

(ii) $y = \log_e \frac{(2-x)}{x}$

Inverse:

$$x = \log_e \frac{(2-y)}{y}$$

$$e^x = \frac{2-y}{y}$$

$$ye^x = 2 - y$$

$$y + ye^x = 2$$

$$y(1+e^x) = 2$$

$$y = \frac{2}{1+e^x}$$

HE4/E3

9) (i) $\ddot{x} = 5 - x$

Centre of motion when $\ddot{x} = 0$

$$0 = 5 - x$$

$$x = 5$$

HE3/E3

(ii) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 5 - x$

$$\frac{1}{2}v^2 = 5x - \frac{x^2}{2} + C$$

$$v^2 = 10x - x^2 + C$$

$x = 4$, $v = \sqrt{3}$

$$(\sqrt{3})^2 = 10(4) - 4^2 + C$$

$$3 = 24 + C$$

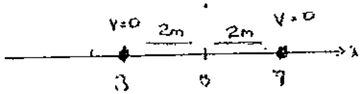
$$v^2 = 10x - x^2 - 21$$

Amplitude = max/min · extremes $\Rightarrow v = 0$

$$0 = 10x - x^2 - 21$$

$$x^2 - 10x + 21 = 0$$

$$(x-7)(x-3) = 0$$



centre of motion $x = 5$.

$\therefore$ Maximum displacement (amplitude) = 2 m

HE3/E3

(iii) Max speed $x = 5$

$$v^2 = 10(5) - 5^2 - 21$$

$$v^2 = 4$$

$$v = \pm\sqrt{4}$$

$$v = 2\text{ m/s}$$

HE3/E3

Q13(d) $P(6) = \frac{2}{5}$ $P(4) = \frac{3}{5}$

- Sum > 20

$6, 6, 6, 6 = 24$

$6, 6, 6, 4 = 22$

$6, 6, 4, 4 = 20$ ✗

$P(\text{sum} > 20) = {}^4C_0\left(\frac{2}{5}\right)^4\left(\frac{3}{5}\right)^0 + {}^4C_1\left(\frac{2}{5}\right)^3\left(\frac{3}{5}\right)^1$

$= \frac{112}{625}$

HE3/7 / E3-E4

Q14 (a) $f(x) = 2\sin^{-1}\left(\frac{1-x}{3}\right) + \frac{\pi}{2}$

(i) $f(-2) = 2\sin^{-1}\left(\frac{1+2}{3}\right) + \frac{\pi}{2}$

$f(-2) = 2\sin^{-1}(1) + \frac{\pi}{2}$

$f(-2) = 2\left(\frac{\pi}{2}\right) + \frac{\pi}{2}$

$f(-2) = \frac{3\pi}{2}$

HE4 / E3-E4

(ii) Domain = $-1 \le x \le 1$

$-1 \le \frac{1-x}{3} \le 1$

$-3 \le 1-x \le 3$

$-4 \le -x \le 2$

$4 \ge x \ge -2$

$\therefore \boxed{-2 \le x \le 4}$

HE4 / E3-E4

(iii) Range = $-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$

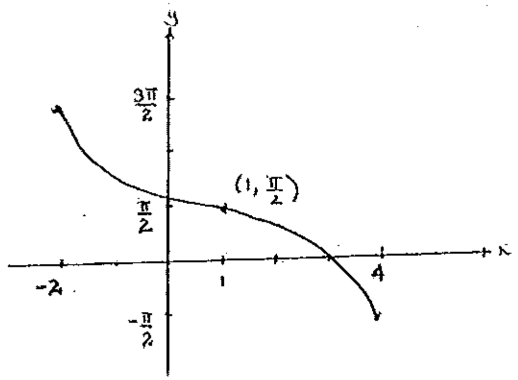
$-\frac{\pi}{2} \le \sin^{-1}\left(\frac{1-x}{3}\right) \le \frac{\pi}{2}$

$-\pi \le 2\sin^{-1}\left(\frac{1-x}{3}\right) \le \pi$

$-\frac{\pi}{2} \le 2\sin^{-1}\left(\frac{1-x}{3}\right) + \frac{\pi}{2} \le \frac{3\pi}{2}$

HE4 / E3-E4

(iv)



HE7 / E3 · E4

Q14 (b) (i) $x = Ut\cos\alpha$.

$$\frac{x}{U\cos\alpha} = t.$$

$$y = -5t^2 + Ut\sin\alpha$$

$$y = -5\left(\frac{x}{U\cos\alpha}\right)^2 + U\left(\frac{x}{U\cos\alpha}\right)\sin\alpha$$

$$y = \frac{-5x^2}{U^2\cos^2\alpha} + \frac{x\sin\alpha}{\cos\alpha}$$

$$y = \frac{-5x^2}{U^2}\sec^2\alpha + x\tan\alpha.$$

HE3/E3

(ii) $x = a,\ h = y$

$$h = \frac{-5a^2\sec^2\alpha}{U^2} + a\tan\alpha \quad \text{———} \quad \mathrm{I}$$

$x = b,\ h = y$

$$h = \frac{-5b^2\sec^2\alpha}{U^2} + b\tan\alpha \quad \text{———} \quad \mathrm{II}$$

Note: can't let I = II as h disappears from our proof

$\mathrm{I} \times b^2$: $$b^2h = \frac{-5a^2b^2}{U^2}\sec^2\alpha + ab^2\tan\alpha \quad \text{———} \quad \mathrm{III}$$

$\mathrm{II} \times a^2$: $$a^2h = \frac{-5a^2b^2}{U^2}\sec^2\alpha + a^2b\tan\alpha \quad \text{———} \quad \mathrm{IV}$$

$\mathrm{III} - \mathrm{IV}$: $$(b^2 - a^2)h = (ab^2 - a^2b)\tan\alpha$$

$$\frac{(b^2-a^2)h}{(ab^2 - a^2b)} = \tan\alpha$$

$$\frac{\cancel{(b-a)}(b+a)h}{ab\cancel{(b-a)}} = \tan\alpha$$

$$\frac{(b+a)h}{ab} = \tan\alpha$$

HE3/E4

ie. $$\tan\alpha = \frac{h(a+b)}{ab}$$

(iii) $h = 20,\ a = 40,\ b = 80$

$$\tan\alpha = \frac{20(40+80)}{(40)(80)}$$

$$\tan\alpha = \frac{3}{4}$$

$$\alpha = \tan^{-1}\left(\frac{3}{4}\right)$$

$$\alpha \doteqdot 37^\circ$$

HE3/E2

214(c)(i) LHS $= (1+x)^n + (1+x)^{n+1} + \dots + (1+x)^{n+m}$

Terms of $x^n \Rightarrow$ $\quad {}^{n}C_n x^n + {}^{n+1}C_n x^n + \dots + {}^{n+m}C_n x^n$

Coefficients of $x^n \Rightarrow$ $\quad {}^{n}C_n + {}^{n+1}C_n + \dots + {}^{n+m}C_n$.

$$\text{RHS} = \frac{(1+x)^{n+m+1} - (1+x)^n}{x^1}$$

$$= \frac{(1+x)^{n+m+1}}{x^1} - \frac{(1+x)^n}{x^1}$$

$$= (1+x)^{n+m} - (1+x)^{n-1}$$

has no x^n terms

can only come from here

Terms of $x^n \Rightarrow \dfrac{(1+x)^{n+m+1}}{x^1} = {}^{n+m+1}C_{n+1}\, x^n$

Coefficient of $x^n \Rightarrow {}^{n+m+1}C_{n+1}$

As LHS = RHS then $\quad {}^{n}C_n + {}^{n+1}C_n + \dots + {}^{n+m}C_n = {}^{n+m+1}C_{n+1}$

214(c)(ii). $m+n = m+4 \longrightarrow n = 4$.

From (i) $\quad {}^{4}C_4 + {}^{5}C_4 + \dots + {}^{m+4}C_4 = {}^{m+5}C_5$

$$1 + {}^{5}C_4 + \dots + {}^{m+4}C_4 = {}^{m+5}C_5$$

$${}^{5}C_4 + \dots + {}^{m+4}C_4 = {}^{m+5}C_5 - 1$$

$$\sum_{r=5}^{m+4} {}^{r}C_4 = {}^{m+5}C_5 - 1$$

$$\sum_{r=5}^{m+4} \frac{r!}{(r-4)!\,4!} = {}^{m+5}C_5 - 1$$

$$\sum_{r=5}^{m+4} \frac{r!}{(r-4)!} = 4!\left({}^{m+5}C_5 - 1\right)$$

$$\sum_{r=5}^{m+4} r(r-1)(r-2)(r-3) = 24\left({}^{m+5}C_5 - 1\right)$$