# DOONSIDE TECHNOLOGY HIGH SCHOOL
# MATHEMATICS FACULTY

Trial HSC
Examination
2001

# Mathematics Extension I

*Time Allowed : Two Hours*

**Directions to candidates:**

* Attempt ALL questions.
* All questions are of equal value.
* Show all necessary working.
* Marks may be deducted for careless or badly arranged work.
* Only Board-approved calculators may be used.
* Standard Integrals are printed on the last page. These may be removed for your convenience.
* Start each question on a new sheet of paper.



**Total marks (84)**
**Attempt Questions 1 – 7**
**All questions are of equal value**

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

**Marks**

**Question 1** ( 12 marks) Use a SEPARATE writing booklet.

(a) Differentiate $x^2 \cos^{-1} x$ **2**

(b) $x-3$ divides $x^3 - 3x^2 + px - 14$ with a remainder of 1. **2**

Find the value of p.

(c) Solve the simultaneous equations:- **3**

$|x - 3| < 4$

$|x - 1| > 1$

(d) The point P(5,7) divides the interval joining the points A(-1,1) and B(3,5) <u>externally</u> in the ratio k : 1. **2**

Find the value of k.

(e) (i) Write $x^2 + 6x + 13$ in the form $(x + b)^2 + c$ **2**

(ii) Hence find **1**

$$\int \frac{dx}{x^2 + 6x + 13}$$

Marks

**Question 2** (12 marks) Use a SEPARATE writing booklet.

(a) Find the acute angle, to the nearest minute, between the curve $y = x^2$ and the line $5x - y - 6 = 0$ at the point of intersection $(3,9)$. **2**

(b) (i) Show that the equation $e^x = x + 2$ has a solution in the interval $1 < x < 2$. **2**

(ii) Letting $x_1 = 1.5$, use one application of Newton's Method to approximate that solution, correct to 3 decimal places. **2**

(c) Evaluate $\lim_{x \to 0} \frac{\sin 3x}{5x}$ **1**

(d) Find the maximum value of $3 \cos x - 2 \sin x$ **2**

(e) Use the substitution $x = \ln u$ to find **3**

$$\int \frac{e^x}{\sqrt{1 - e^{2x}}}\, dx$$





Marks

**Question 3** (12 marks) Use a SEPARATE writing booklet.

(a) Show that $\sin^{-1} x$ is an odd function. **2**

(b) Use the method of Mathematical Induction to prove that **3**

$$9^{n+2} - 4^n$$

is divisible by 5, for all positive integers, n

(c) (i) Using $t = \tan x/2$ write expressions for $\sin x$ and $\cos x$ in terms of t. **1**

(ii) Hence, or otherwise, solve **3**

$$3\cos x + 5\sin x = 5 \qquad 0 < x < 360^\circ$$

to the nearest degree.

(d) Using the identity **3**

$$(1 + x)^{2n} = (1 + x)^n (1 + x)^n$$

and considering coefficients of $x^n$ show that

$${}^{10}C_5 = ({}^5C_0)^2 + ({}^5C_1)^2 + \ldots\ldots + ({}^5C_5)^2$$

Marks

**Question 4** (12 marks) Use a SEPARATE writing booklet.

(a) By expressing $\cos^2 x$ in terms of $\cos 2x$ find the primitive of $\cos^2 x$. **2**

(b) An 8 person committee is to be formed from a group of 10 women and 15 men.

In how many ways can the committee be chosen if :-

(i) the committee must contain 4 men and 4 women. **1**

(ii) there must be more women than men. **2**

(iii) there must be at least 2 women, **2**

(c) (i) Sketch the function **1**

$$f(x) = |x - 1|$$

over its natural domain.

(ii) Explain why $f(x)$ does not have an inverse over this domain. **1**

(iii) If $f_1(x)$ is the restriction of $f(x)$ to the domain $x \geqslant 1$ find $f_1^{-1}(x)$, stating its domain and range. **2**

(iv) What is $f_2^{-1}(x)$ if the domain of $f(x)$ is restricted to $x < 1$? **1**

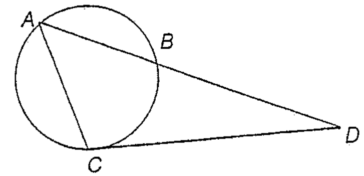




Marks

**Question 5** (12 marks) Use a SEPARATE writing booklet.

(a)

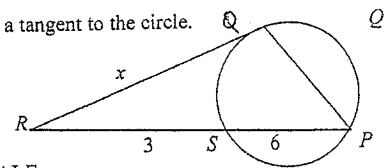



(i) Copy the diagram above and prove that $\Delta BCD \parallel\!| \Delta CAD$. **3**

(ii) Hence prove that $CD = \sqrt{BD.AD}$ **2**

(iii) Use this result to find the value of $x$ in the diagram below. **1**

QR is a tangent to the circle.




NOT TO SCALE

(b) When a particle is $x$ metres from the origin, its velocity, $v$ ms$^{-1}$, is given by **3**

$$v = \sqrt{8 - 2x^2}$$

Find the acceleration when the particle is 2 metres from the origin.

**Question 5 is continued on the next page.**

**Question 5** (continued)

(c) 30 girls, including Miss Australia, enter a Miss World competition.

The first 6 places are announced.

(i) How many different announcements are possible? **1**

(ii) How many different announcements are possible if Miss Australia is assured of a place in the first 6 ? **2**

**End of Question 5**

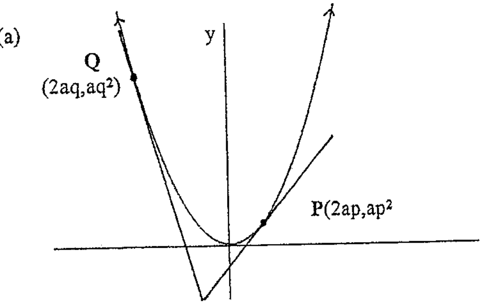




**Marks**

**Question 6** (12 marks) Use a SEPARATE writing booklet.

(a)




The points $P(2ap,ap^2)$ and $Q(2aq,aq^2)$ lie on the parabola $x^2 = 4ay$

(i) Show that the equation of the tangent at P is given by **2**

$$y = px - ap^2$$

(ii) If the tangent at P and the tangent at Q intersect at 45° show that **1**

$$|p - q| = |1 + pq|$$

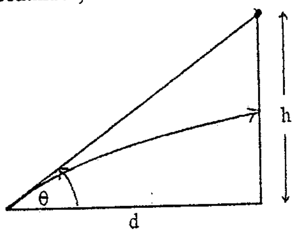
(iii) If q =2 find p, using the result above. **2**

**Question 6 continues on the next page**

Marks

**Question 6** (continued)

(b)




A target is hung on a wall at a height of h metres.

A small cannon, which fires a lead slug, is located on the floor, d metres from the wall.

The muzzle velocity, V, of the cannon is adjustable

The cannon is aimed at the bulls-eye on the target, at an angle of elevation of θ degrees.

At the instant the cannon is fired the target is released and falls vertically downwards under the force of gravity, g.

Given that $\ddot{x} = 0$ and $\ddot{y} = -g$

(i) Show that after time t **2**

$$x = tV\cos\theta \quad \text{and} \quad y = \frac{-gt^2}{2} + tV\sin\theta$$

(ii) Show that the slug hits the wall at a vertical height of **2**

$$H = \frac{-g\,d^2\sec^2\theta}{2V^2} + d\tan\theta$$

(iii) Experiments with the cannon show that the slug always hits the bulls-eye regardless of the muzzle velocity.

Explain why this is always so. **3**

**End of Question 6**

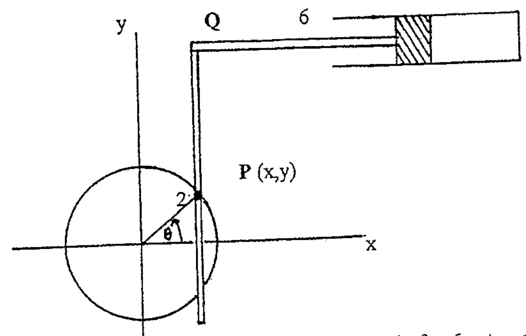




Marks

**Question 7** (12 marks) Use a SEPARATE writing booklet.

(a)




A piston moves back and forth on the end of a 6metre shaft.

The other end is attached at Q to a vertical slotted arm fitted to a peg P on the rim of a wheel of radius 2metres.

Suppose the wheel begins with point P at $\theta = \frac{\pi}{4}$

when t = 0 and rotates anticlockwise at 5 radians per second.

(i) Show that $\theta = 5t + \frac{\pi}{4}$ **1**

(ii) Hence find an expression for x as a function of t and show that the motion of the piston is simple harmonic. **2**

(iii) State the amplitude and period of the motion. **2**

(iv) Find the initial velocity of the piston. **1**

**Question 7 continues on the next page**

**Marks**

**Question 7** (continued)

(b) A water tank is generated by rotating the curve

$$y = \frac{x^4}{16}$$

around the y - axis

(i) Show that the volume of water, $V$ as a function of its depth h, is given by: **2**

$$V = \frac{8}{3}\pi . h^{\frac{3}{2}}$$

(ii) Water drains from the tank through a small hole at the bottom. **4**

The rate of change of the volume of water in the tank is proportional to the square root of the water's depth.

Use this fact to show that the water level in the tank falls at a constant rate.

**End of paper**

Doonside Tech HS 2001 3u trial Dr Ong.

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1.(a) $\frac{d}{dx}(x^2\cos^{-1}x)$

$u = x^2$ $v = \cos^{-1}x$ ✓

$u' = 2x$ $v' = \frac{-1}{\sqrt{1-x^2}}$

$\frac{d}{dx} = \frac{-x^2}{\sqrt{1-x^2}} + 2x\cos^{-1}x$ ✓

(b). $P(x) = x^3 - 3x^2 + px - 14$

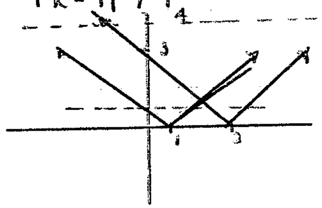
$P(3) = 27 - 27 + 3p - 14 = 1$ ✓

$3p = 15$

$p = 5$ ✓

(c). $|x-3| < 4$.

$|x-1| > 1$



$|x-3| < 4$.

$\therefore -1 < x < 7$ ✓

$|x-1| > 1$

$\therefore x > 0 \cap x > 2$. ✓

$\therefore -1 < x < 0 \cap 2 < x < 7$ ✓

(d). $P(5, 7)$ $A(-1, 1)$ $B(3, 7)$ $k:1$

$\therefore 5 = \frac{3k+1}{k-1}$ ✓ $7 = \frac{5k-1}{k-1}$

$k = 3$.

Only ONE equation needed to solve for k!

(12)

(e)(i) $x^2 + 6x + 13$.

$= x + 6x + 9 + 4$.

$= (x+3)^2 + 4$. ✓

(ii) $\int \frac{dx}{x^2+6x+13}$.

$= \frac{1}{2}\tan^{-1}\left(\frac{x+3}{2}\right) + c$. ✓

2.(a). $y = x^2$

$\frac{dy}{dx} = 2x$

at $x = 3$.

$m_1 = 6$. ✓

$y = 5x - 6$.

$m_2 = 1$.

$\Longrightarrow \therefore \tan\theta = \left|\frac{6-5}{1+6\times5}\right|$ ✓

$= \frac{1}{31}$

(b)(i) $e^x = x + 2$.

$P(x) = e^x - x - 2 = 0$.

$P(1) = e - 3$

$= -0.28$

$= \text{the } -ve$.

$P(2) = e^2 - 4$.

$= 3.38$ ✓

$= +ve$.

$\therefore$ Root between $1 < x < 2$ ✓

(ii). $x_1 = 1.5$

$f'(x) = e^x - 1$.

$x_2 = 1.5 - \frac{f(1.5)}{f'(1.5)}$ ✓

$= 1.219$ ✓

(c). $\lim_{x\to 0} \frac{\sin 3x}{3x} \times \frac{3}{5}$ ✓

$= 1 \times \frac{3}{5}$.

$= \frac{3}{5}$

(d). $3\cos x - 2\sin x$

$= \sqrt{13}\cos(\theta + \alpha)$

$y = \sqrt{13}\cos(\theta + 33°41')$

$y = -\sqrt{13}\sin(\theta + 33°41') = 0$.

$\sin(\theta + 33°41') = 0$.

$\theta = $ ~~326°19~~

$\tan\alpha = 2/3$

$\alpha = 33°41'$

$y_{max} = \sqrt{13}$

(11)

(e). $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$

$x = \ln u$. $dx = \frac{du}{u}$ ✓

$\int \frac{e^{\ln u}}{\sqrt{1-e^{2\ln u}}}$

$= \int \frac{u}{\sqrt{1-u^2}}\,\frac{du}{u}$ ✓

$= \sin^{-1}(u) + c$. ✓

$= \sin^{-1}(e^x) + c$.

3(a) $f(x) = \sin^{-1}x$

$f(-x) = \sin^{-1}(-x)$ ✓

$= -\sin^{-1}(x)$

$= -f(x)$ ✓

$\therefore$ odd.

) $9^{n+2} - 4^n \div 5$.

Let n=1

$9^3 - 4$.

$= 725$ True.

Let n=k

$9^{k+2} - 4^k = 5M$ ✓

$9^{k+2} - 5M = 4^k$

Prove for n=k+1

$9^{k+3} - 4^{k+1}$

$= 9^{k+3} - 4[9^{k+2} - 5M]$

$= 9^{k+3} - 4.9^{k+2} + 20M$ ✓

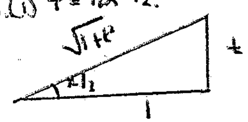
$= 9^{k+2}[9-4] + 20M$

$= 9^{k+2} \times 5 - 20M$

$= 5(9^{k+2} - 4M]$ ✓ ∴ 5N.

If true for n=k, and proving n=k+1, by the principles of Mathematical Induction, true for all values of n.

(c)(i) $t = \tan x/2$.




$\sin x = 2\cos x/2 \sin x/2$

$= \frac{2t}{1+t^2}$ ✓

$\cos x = \cos^2 x/2 - \sin^2 x/2$

$= \frac{1}{1+t^2} - \frac{t^2}{1+t^2}$

$= \frac{1-t^2}{1+t^2}$ ✓

(ii). $\frac{3(1-t^2)}{1+t^2} + \frac{10t}{1+t^2} = 5$. $0 < x < 360°$

$-3t^2 + 10t + 3 = 5(1+t^2)$

$-3t^2 + 10t + 3 = 5 + 5t^2$

$8t^2 - 10t + 2 = 0$.

$t = 1$, $t = 1/4$. ✓

∴ $\tan\frac{x}{2} = 1$

$\frac{x}{2} = \pi/4$ ✓

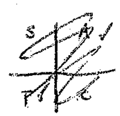
$x = \pi/2 = 90°$

$\tan\frac{x}{2} = \frac{1}{4}$ ✓

$\frac{x}{2} = 14°2'$

$x = 28°4'$ ✓

$= 28°$ (to nearest deg)




(d). $(1+x)^{2n} = (1+x)^n (1+x)^n$

$\binom{2n}{n}x^n = \binom{n}{0}\binom{n}{n} + \binom{n}{1}\binom{n}{n-1} + \binom{n}{2}\binom{n}{n-2} + \ldots$

But $\binom{n}{r} = \binom{n}{n-r}$

∴ $\binom{2n}{n}x^n = \binom{n}{0}^2 + \binom{n}{1}^2 + \ldots \binom{n}{n}^2$

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4.(a). $\cos 2x = \cos^2 x - \sin^2 x$

$= \cos^2 x - (1-\cos^2 x)$

$= 2\cos^2 x - 1$ ✓

$\cos^2 x = \frac{\cos 2x + 1}{2}$.

$\int \cos^2 x$

$= \frac{1}{2}\int \cos 2x + 1 \, dx$

$= \frac{1}{2}\left[\frac{\sin 2x}{2} + x\right] + c$ ✓

$= \frac{\sin 2x}{4} + \frac{x}{2} + c$.

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(b). (i) $\binom{15}{4}\binom{10}{4}$

$= 286\,650$. ✓

(ii). Case 1: 5W, 3M $\binom{10}{5}\binom{15}{3}$ + Case 2: 6W, 2M $\binom{10}{6}\binom{15}{2}$ + Case 3: 7W, 1M $\binom{10}{7}\binom{15}{1}$ + Case 4: 8W $\binom{10}{8}$

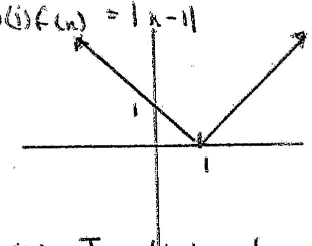
$= 138555$.

(iii) $P(All) - \binom{15}{8} - \binom{15}{7}\binom{10}{1}$

$= \binom{25}{8} - \binom{15}{8} - \binom{15}{7}\binom{10}{1}$

$= 1010\,740$.

(c)(i) $f(x) = |x-1|$




(ii). Two x values for each y value. for $y = f^{-1}(x)$.

(iii). $f(x) = x-1$ D: $x \geq 1$ Range $y \geq 0$.

$f^{-1}(x) = x = y-1$

$y = x+1$

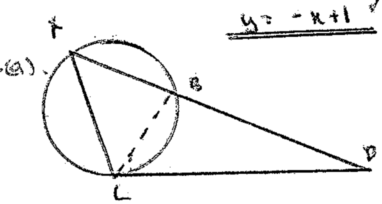
∴ $f^{-1}(x)$: D $x \geq 0$, R $y \geq 1$ ✓

(iv). $f_2(x) = -x+1$

$f_2^{-1}(x)$ $x = -y+1$

$y = -x+1$ ✓

5.(a).




In Δ's BCD & CAD. ✓

$A\hat{D}C$ is common.

$B\hat{C}D = C\hat{A}B$ (Angles in the alternate seg)

∴ Δ BCD ||| Δ CAD.

i). ~~$CD^2 = AB \times AD$~~ (secant rule)

~~$CD = \sqrt{AD \times AD}$~~

ii). $x = \sqrt{6 \times 3}$

$= 3\sqrt{2}$.

Use similar Δ as a proof

$\frac{CD}{AD} = \frac{BD}{CD}$

$\therefore CD^2 = AD \times BD$

$CD = \sqrt{AB \times BD}$

2). $V = \sqrt{8 - 2x^2}$

$V^2 = 8 - 2x^2$

$\frac{v^2}{2} = 4 - x^2$

$\frac{d}{dx}\left(\frac{v^2}{2}\right) = -2x$ ✓

$\ddot{x} = -2x$

At $x = 2$. ✓

$\ddot{x} = -4$. ✓ $m/s^2$

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5(a)(i) $\binom{30}{6} 6!$

$= 427\,518\,000$ ✓

(ii). $\binom{29}{5} \times 6!$

$= 342\,014\,400$ ✓

6.(a).(i) $x^2 = 4ay$

$\frac{x^2}{4a} = y$

$\frac{dy}{dx} = \frac{2x}{4a}$

At $x = 2ap$

$m = \frac{4ap}{4a}$ ✓

$= p$.

Eqn: $y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$ ✓

$y = px - ap^2$

(ii). $\tan 45 = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$ ✓

$1 = \left|\frac{p - q}{1 + pq}\right|$ ✓

$|1 + pq| = |p - q|$ ✓

(iii). $|p - 2| = |1 + 2p|$

$p - 2 = 1 + 2p$ or $-p + 2 = 1 + 2p$

$-p = 3$. ✓ $1 = 3p$ ✓

$p = -3$ $p = \frac{1}{3}$

(b) $\ddot{x} = 0$.

$\dot{x} = c_1$

At $t = 0$, $\dot{x} = V\cos\alpha$

$\therefore \dot{x} = V\cos\alpha$

$x = Vt\cos\alpha + c_2$

At $x = 0$, $t = 0$ ✓

$\therefore x = Vt\cos\theta$

$\ddot{y} = -g$

$\dot{y} = -gt + c_1$.

At $t = 0$, $\dot{y} = V\sin\alpha$

$\dot{y} = -gt + V\sin\alpha$

$y = -\frac{gt^2}{2} + Vt\sin\alpha$

At $x = 0$, $t = 0$ ✓

$y = Vt\sin\theta - \frac{gt^2}{2}$

(ii). $t = \frac{x}{V\cos\theta}$

$\therefore y = \frac{V x\sin\theta}{V\cos\theta} - \frac{g\left(\frac{x^2}{V^2\cos^2\theta}\right)}{2}$ ✓

$= x\tan\theta - \frac{g x^2 \sec^2\theta}{2V^2}$

At $x = a$, $y = H$ ✓

$H = a\tan\theta - \frac{ga^2\sec^2\theta}{2V^2}$

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(iii) ~~[struck-through text] try again.~~

target ↓

$\ddot{y} = -g$

$\dot{y} = -gt$

$y = -\frac{gt^2}{2}$

At $t = \frac{d}{V\cos\theta}$

$y = -\frac{g}{2}\left(\frac{d}{V\cos\theta}\right)^2$

Also $-\frac{gd^2}{2V^2\cos^2\theta} + d\tan\theta - \frac{gd^2}{2V^2\cos^2\theta} = h$

$\therefore$ $\tan\theta = \frac{h}{d}$ which is ~~[struck-through]~~ independent of $V$.

7(a)(i) $\frac{d\theta}{dt} = 5$.

$\int \frac{d\theta}{dt}\,dt = \int 5\,dt$

$\theta = 5t + c$.

At $t = 0$, $\theta = \pi/4$. ✓

$\theta = 5t + \frac{\pi}{4}$.

(ii) $x = a\cos(5t + \pi/4)$ ✓

$a = 2$. (given) ✓

$x = 2\cos(5t + \pi/4)$.

(iii). $T = \frac{2\pi}{n}$ Amp = 2. ✓

$= \frac{2\pi}{5}$ ✓

(iv). $V = -10\sin(5t + \pi/4)$

At $t = 0$,

$= \frac{10}{\sqrt{2}}$. ✓

(b).

$y = \frac{x^4}{16}$

$16y = x^4$

$4\sqrt{y} = x^2$

$V = 4\pi \int_0^h \sqrt{y} \, dy$

$= 4\pi \left[ \frac{2}{3} y^{3/2} \right]_0^h$

$= \frac{8\pi h^{3/2}}{3}$

$\frac{dV}{dt} = -k\sqrt{h}$

constant k.

$V = \frac{8}{3}\pi h^{3/2}$

$\frac{dV}{dh} = 4\pi h^{1/2}$

$\therefore \frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$

$= \frac{-k\sqrt{h}}{4\pi\sqrt{h}}$

$= \frac{-k}{4\pi}$

(12)