EXERCISES 1.6 FUNCTIONS & GRAPHS

Find the vertices of the parabolas given by the following equations.

1. $y = 2x^2 + 3x + 1.$

2. $y = 4x - x^2.$

3. $y = 3 - x - 3x^2.$

4. $y = 4x^2 + 16x + 4.$

Find the domains and ranges of the following functions and sketch their graphs:

5. $f(x) = \sqrt{4 - x^2}.$

6. $f(x) = 2 - \sqrt{9 - x^2}.$

7. $g(x) = -\sqrt{3 - x}.$

8. $f(x) = \sqrt{x - 2}.$

9. $f(x) = \dfrac{1}{x}.$

10. $f(x) = \dfrac{-3}{x - 2}.$

11. $f(x) = x^3.$

12. $f(x) = 1 - x^3.$

13. $f(x) = 2 - |x|.$

14. $g(x) = |x| + 3.$

15. $f(x) = |x + 3|.$

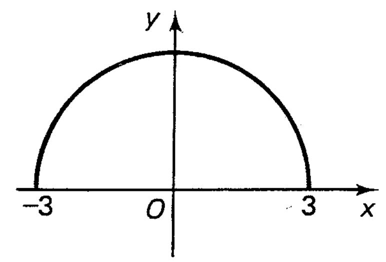
16. $F(x) = -|x - 2|.$

17. $f(x) = \dfrac{|x - 3|}{x - 3}.$

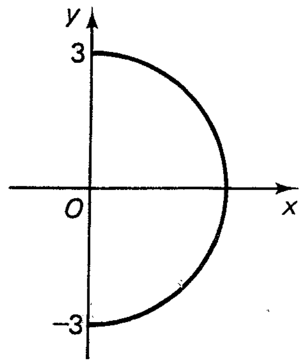
18. $G(x) = \dfrac{2 - x}{|x - 2|}.$

19. Which of the following half circles represent the graphs of functions? In each case where the answer is positive, determine the equation for the function from the graph.

(a)



(b)



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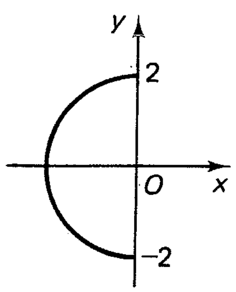
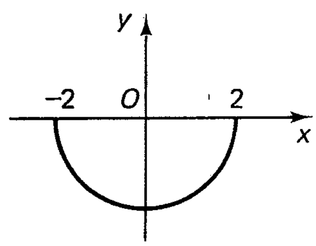


(d)



Determine the equation of each of the circles whose radii and centers are given below:

20. Radius 5, center $(0, -2)$.

21. Radius 3, center (2, 5).

22. Radius 2, center (0, 0).

23. Radius 4, center (−3, 0).

24. Find the equation of a circle that lies in the first quadrant, has a radius of 2 units, and touches the coordinate axes.

25. Find the equation of a circle with its center at (3, −3), and that touches the coordinate axes.

26. Find the two numbers x and y that satisfy the condition that $3x - y = 18$ and for which their product xy is as small as possible.

27. A farmer has 400 yd of fencing with which to build a rectangular paddock. What is the largest area he can enclose?

28. The yield of apples from each tree in an orchard is $(500 - 5x)$ lb, where x is the density with which the trees are planted (i.e., number of trees per acre). Find the value of x that makes the total yield per acre a maximum.

29. If rice plants are sown at a density of x plants per square foot, the yield of rice from a certain location is $x(10 - 0.5x)$ bushels per acre. What value of x maximizes the yield per acre?

30. A bird flies a distance of 1000 mi over the ocean, its route passing over two islands, one after 200 mi and the other after 700 mi. If x is the distance along the route to a given point $(0 \leq x \leq 1000)$, determine the function $f(x)$ that is equal to the distance of that point from the nearest land. Sketch its graph.

31. In the previous exercise the function $g(x)$ is equal to the distance of the point x from the nearest land *ahead* of the bird. Write algebraic expressions for $g(x)$.

EXERCISES 1.7 — ANSWERS – (odd No. quests)

1. $(f \pm g)(x) = x^2 \pm \frac{1}{x-1}$; $(fg)(x) = \frac{x^2}{x-1}$; $\left(\frac{f}{g}\right)(x) = x^2(x-1)$; $\left(\frac{g}{f}\right)(x) = \frac{1}{x^2(x-1)}$.
$D_{f+g} = D_{f-g} = D_{fg} = D_{f/g} = \{x \mid x \neq 1\}$; $D_{g/f} = \{x \mid x \neq 0, 1\}$.

3. $(f \pm g)(x) = \sqrt{x-1} \pm \frac{1}{x+2}$; $(fg)(x) = \frac{\sqrt{x-1}}{x+2}$;
$\left(\frac{f}{g}\right)(x) = \sqrt{x-1}(x+2)$; $\left(\frac{g}{f}\right)(x) = \frac{1}{\sqrt{x-1}(x+2)}$.
$D_{f+g} = D_{f-g} = D_{fg} = D_{f/g} = \{x \mid x \geq 1\}$; $D_{g/f} = \{x \mid x > 1\}$.

5. $(f \pm g)(x) = (x+1)^2 \pm \frac{1}{x^2-1}$; $(fg)(x) = \frac{x+1}{x-1}$;
$\left(\frac{f}{g}\right)(x) = (x+1)^3(x-1)$; $\left(\frac{g}{f}\right)(x) = \frac{1}{(x+1)^3(x-1)}$.
$D_{f+g} = D_{f-g} = D_{fg} = D_{f/g} = D_{g/f} = \{x \mid x \neq \pm 1\}$.

7. $\sqrt{8}$.

9. $\sqrt{3}$.

11. Not defined.

13. 0.

15. $f \circ g(x) = |x| + 1$; $g \circ f(x) = (\sqrt{x} + 1)^2$. $D_{f \circ g} = \{x \mid x \text{ real}\}$; $D_{g \circ f} = \{x \mid x \geq 0\}$.

17. $f \circ g(x) = 2 + |x - 2|$; $g \circ f(x) = x$. $D_{f \circ g} = \{x \mid x \text{ real}\}$; $D_{g \circ f} = \{x \mid x \geq 0\}$.

19. $g(x) = 1 + x^4$.

21. $g(x) = x - 1$.

23. $f(x) = x^3$, $g(x) = x^2 + 1$ is the "simplest" answer.

EXERCISES 1.7

Find the sum, difference, product, and quotient of the two functions f and g in each of the following examples. Determine the domains of the resulting functions.

1. $f(x) = x^2, \quad g(x) = \frac{1}{x-1}.$
2. $f(x) = x^2 + 1, \quad g(x) = \sqrt{x}.$
3. $f(x) = \sqrt{x-1}, \quad g(x) = \frac{1}{x+2}.$
4. $f(x) = 1 + \sqrt{x}, \quad g(x) = \frac{2x+1}{x+2}.$
5. $f(x) = (x+1)^2, \quad g(x) = \frac{1}{x^2-1}.$

Given $f(x) = x^2$ and $g(x) = \sqrt{x-1}$, evaluate

6. $(f \circ g)(5)$.
7. $(g \circ f)(3)$.
8. $(f \circ g)(\frac{5}{4})$.
9. $(g \circ f)(-2)$.
10. $(f \circ g)(\frac{1}{2})$.
11. $(g \circ f)(\frac{1}{3})$.
12. $(f \circ g)(2)$.
13. $(g \circ f)(1)$.

Determine $(f \circ g)(x)$ and $(g \circ f)(x)$ in the following examples. In each case determine the domains of $f \circ g$ and $g \circ f$.

14. $f(x) = x^2, \quad g(x) = 1 + x.$
15. $f(x) = \sqrt{x} + 1, \quad g(x) = x^2.$
16. $f(x) = \frac{1}{x+1}, \quad g(x) = \sqrt{x} + 1.$
17. $f(x) = 2 + \sqrt{x}, \quad g(x) = (x-2)^2.$

Determine $g(x)$ if

18. $f(x) = x^2$ and $(f \circ g)(x) = (1+x)^2$.
19. $f(x) = \sqrt{x-1}$ and $(f \circ g)(x) = x^2$.
20. $f(x) = \frac{1}{x+2}$ and $(g \circ f)(x) = x + 2$.
21. $f(x) = \frac{x+1}{x-2}$ and $(g \circ f)(x) = \frac{3}{x-2}.$
22. $f(x) = \sqrt{x}$ and $(f \circ g)(x) = |x|$.

Determine $f(x)$ and $g(x)$ if the composite function $f \circ g$ is as follows: (The answer is not unique.)

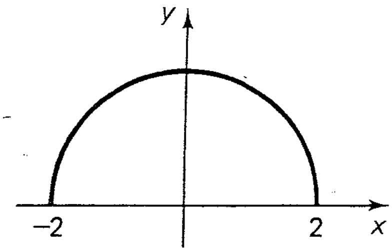
23. $(f \circ g)(x) = (x^2 + 1)^3$.
24. $(f \circ g)(x) = \sqrt{2x+3}$.

EXERCISES 1.6 — FUNCTIONS & GRAPHS — Odd No. Answers

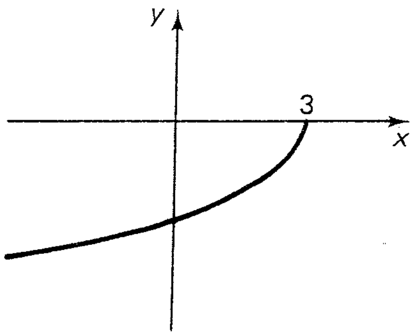
1. $(-\frac{3}{4}, -\frac{1}{8})$.

3. $(-\frac{1}{6}, \frac{37}{12})$.

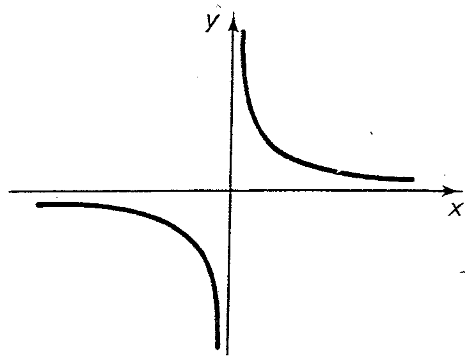
5. $D_f = \{x \mid -2 \leq x \leq 2\}$; $R_f = \{y \mid 0 \leq y \leq 2\}$.



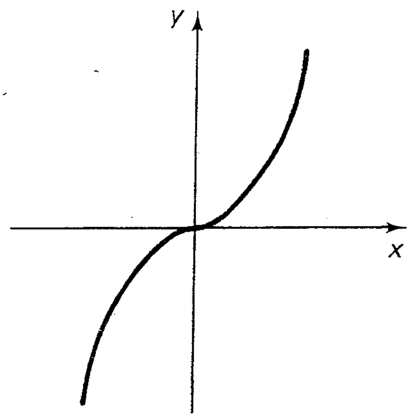
7. $D_f = \{x \mid x \leq 3\}$; $R_f = \{y \mid y \leq 0\}$.



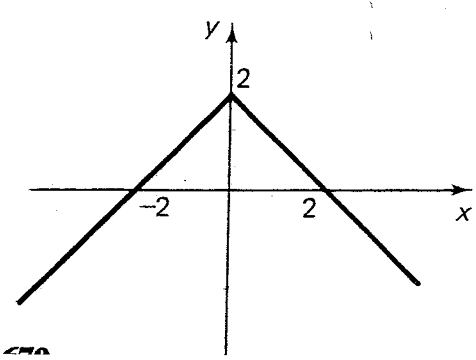
9. $D_f = \{x \mid x \neq 0\}$; $R_f = \{y \mid y \neq 0\}$.



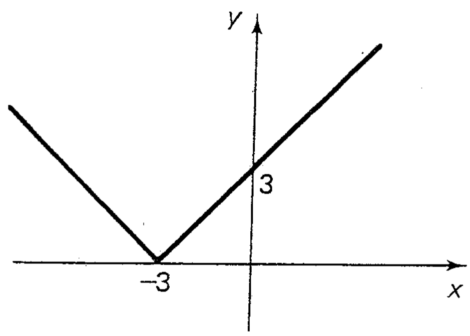
11. $D_f = R_f =$ set of all real numbers.



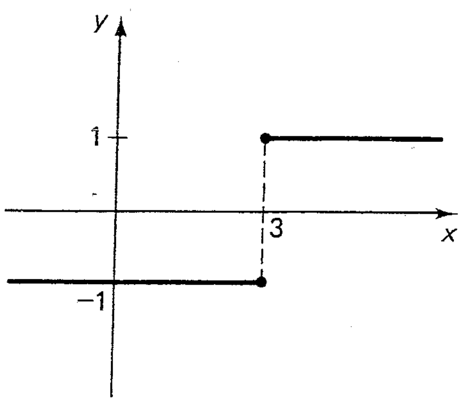
13. $D_f =$ set of all real numbers; $R_f = \{y \mid y \leq 2\}$



15. $D_f =$ set of all real numbers; $R_f = \{y \mid y \geq 0\}$.



17. $D_f = \{x \mid x \neq 3\}$; $R_f = \{1, -1\}$.



19. (a) Yes; $y = \sqrt{9 - x^2}$; (b) No; (c) No; (d) Yes; $y = -\sqrt{4 - x^2}$.

21. $x^2 + y^2 - 4x - 10y + 20 = 0$.

23. $x^2 + y^2 + 6x - 7 = 0$.

25. $x^2 + y^2 - 6x + 6y + 9 = 0$.

27. 10,000 yd².

29. $x = 10$.

31. $g(x) = \begin{cases} 200 - x \text{ if } 0 < x \leq 200 \\ 700 - x \text{ if } 200 < x \leq 700 \\ 1000 - x \text{ if } 700 < x \leq 1000. \end{cases}$

REVIEW EXERCISES

1. Are the following statements true or false? If false, give the corresponding correct statements.

(a) Every real number is a rational number.
(b) Every natural number is a real number.
(c) All integers are rational numbers.
(d) A given curve is the graph of a function if any vertical line meets the curve in at least one point.
(e) If f and g are two functions such that the composite functions $f \circ g$ and $g \circ f$ are defined, then $f \circ g = g \circ f$.
(f) Every equation in x and y expresses y as a function of x.
(g) The graph of a linear equation $ax + by + c = 0$ is a straight line for all values of the constants a, b, and c.
(h) The graph of $ax + by + c > 0$ $(b \neq 0)$ is the half-plane above the line $ax + by + c = 0$.
(i) The domain of $f(x) = |x - 2|$ is the set of all real numbers greater than or equal to 2.
(j) $(x^2 - 9)/(x - 3) = x + 3$ for all real values of x.
(k) If f, g are two functions, then $f + g$, $f \cdot g$ and f/g have the same domain.
(l) $-2y > 4x - 6$ is equivalent to $y > -2x + 3$.
(m) $\sqrt{x^2} = |x|$ for all real numbers x.

2. For each of the following, give an example of a function f that satisfies the following property for all values of x and y.

(a) $f(x) = f(-x)$ [such a function is called an even function].
(b) $f(-x) = -f(x)$ [such a function is called an odd function].
(c) $f(x + y) = f(x) + f(y)$ [a function with this property is called a linear function].

3. If $f(x) = |x|$ and $g(x) = x^2$, determine $f \circ g$ and $g \circ f$ and their domains. Is $f \circ g = g \circ f$?

4. Two functions f and g are said to be equal if $f(x) = g(x)$ for all x in the domain and $D_f = D_g$. Use this criterion to determine which of the following functions are equal to $f(x) = (2x^2 + x)/x$.

(a) $g(x) = 2x + 1$.
(b) $h(x) = \sqrt{1 + 4x + 4x^2}$.
(c) $F(x) = \dfrac{2x^3 + x^2}{x^2}$.
(d) $G(x) = \dfrac{(x^3 + 2x)(1 + 2x)}{x(x^2 + 2)}$.

5. Determine the equation of a straight line

(a) whose slope is 3 and y-intercept is 4 units.
(b) whose slope is 3 and passes through the point (7, 5).

6. Find the equation of the straight line passing through the points (3, 9) and (4, 8); show that the point (5, 7) lies on this line.

7. Determine the equation of the circle whose center is $(-1, 2)$ and whose radius is 4.

8. Determine the equation of the circle whose center is $(-2, 4)$ and that passes through the point (1, 0).

9. Find the vertex of the parabola $y = 3x^2 + 2x + 1$.

10. Find the vertex of the parabola $y = 4 - 2x - 2x^2$ and sketch its graph.

11. Draw the graph of the region in the xy-plane that satisfies the inequalities

$$x + y < 5, \qquad 2x + y \geq 6, \qquad x - y \leq 2.$$

Evaluate the limits

12. $\lim_{x\to\infty} \frac{(x + 1)(2x + 3)}{(x + 2)(3x + 4)}$.

13. $\lim_{x\to\infty} \sqrt{\frac{2 + 3x}{6x - 1}}$.

14. If $f(x) = x/(1 + x)$ and $g(x) = 1/(x - 1)$, evaluate $f \circ g(x)$ and find the domain of this function.

15. In exercise 14, evaluate $g \circ f(x)$ and find the domain of this function.

REVIEW EXERCISES [redacted] ANSWERS (odd No. only)

1. (a) False; for example, $\sqrt{2}$ is a real number that is not rational; (b) True; (c) True;
   (d) False; a curve is the graph of a function if any vertical line meets the curve in *at most* one point;
   (e) False; in general $f \circ g \neq g \circ f$;
   (f) False; for example, the equation $x^2 + y^2 = 4$ does not express y as a function of x;
   (g) False; the statement is true provided that a and b are not both zero;
   (h) False; if $b < 0$ the graph is the half-plane below the line $ax + by + c = 0$;
   (i) False; D_f is the set of all real numbers;
   (j) False; $(x^2 - 9)/(x - 3) = x + 3$ only for $x \neq 3$;
   (k) False; the domain of f/g may differ from that of $f + g$ and fg;
   (l) False; $-2y > 4x - 6$ is equivalent to $y < -2x + 3$. (Dividing an inequality by a negative number changes the direction of the inequality);
   (m) True.

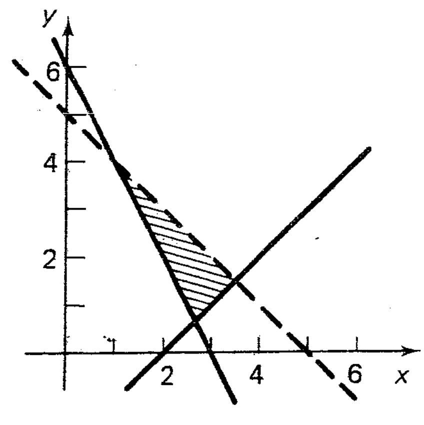
3. $f \circ g(x) = g \circ f(x) = x^2$. The domain of $f \circ g$ and $g \circ f$ are both the set of all real numbers.

5. (a) $y = 3x + 4$; (b) $y = 3x - 16$.

7. $x^2 + y^2 + 2x - 4y - 11 = 0$.

9. $(-\frac{1}{3}, \frac{2}{3})$.

11.



13. $1/\sqrt{2}$.

15. $g \circ f(x) = -(1 + x)$. $D_{g\circ f} = \{x \mid x \neq -1\}$.