Exercises – COMPLEX NUMBERS I

1. Let $z = 2 + 3i$, $\omega = -1 + 2i$
   Calculate: (a) $3z$ (b) z^2 (c) $z + 2\omega$ (d) z / ω (e) ω / z

2. Write the following expressions in the form $a + ib$ ("cartesian form")
   (a) $\frac{1+i}{1+2i}$ (b) $\frac{2-i}{3+i} - \frac{3-i}{2+i}$

3. If $z = a + ib$, express the following in Cartesian form:
   (a) z^2 (b) $\frac{1}{z}$ (c) $\frac{z+1}{z-1}$

4. Use the quadratic formula to find all the complex roots of the following polynomials:
   (a) $x^2 + x + 1$ (b) $x^2 + 2x + 3$ (c) $x^2 - 6x + 10$ (d) $-2x^2 + 6x - 3$
   (e) $x^4 + 5x^2 + 4$

5. Simplify: $\left(\sqrt{3+4i} + \sqrt{3-4i}\right)^2$

6. Find Re (z), Im (z) and $\bar{z}$ for:
   (a) $z = -1 + i$ (b) $z = 2 + 3i$ (c) $z = 2 - 3i$ (d) $\frac{2-i}{1+i}$ (e) $\frac{1}{(1+i)^2}$

7. Prove that for any 2 complex numbers z and ω :
   (a) $\text{Im}(z) = \frac{1}{2i}(z - \bar{z})$ (b) $2\,\text{Re}(z) = z + \bar{z}$ (c) $\overline{(z - \omega)} = \bar{z} - \bar{\omega}$
   (d) $\overline{\left(\frac{1}{z}\right)} = \frac{1}{\bar{z}}$ (e) $\overline{z\omega} = \bar{z}\bar{\omega}$ (f) $\overline{\left(\frac{z}{\omega}\right)} = \frac{\bar{z}}{\bar{\omega}}$

8. By evaluating each side of the equations, check that (a) $\overline{z\omega} = \bar{z}\bar{\omega}$ and (b) $\overline{\left(\frac{z}{\omega}\right)} = \frac{\bar{z}}{\bar{\omega}}$ are satisfied by the complex numbers: $z = 2 + 3i$ and $z = -1 + 2i$

9. Show that: $\left[(\sqrt{3} + 1) + (\sqrt{3} - 1)i\right]^3 = 16(i + 1)$

10. Simplify: $\left(\frac{a+bi}{a-bi}\right)^2 - \left(\frac{a-bi}{a+bi}\right)^2$, where a and b are real numbers, not both zero.

11. (a) If $ax^2 + bx + c = 0$ has **real** co-efficients, use the properties of complex numbers (found in quest. 7) to show that: If the complex number α is a root of the quadratic equation, then so is $\bar{\alpha}$.

    (b) Write down the monic quadratic polynomial with real co-efficients which has $3 - 2i$ as one of its roots.

12. If $z = 1 + i$, calculate the powers of of z^j for $j = 1, 2, 3, \ldots, 10$ and plot them on the Argand diagram. What is the smallest positive integer, n, such that z^n is a real number?

Answers – Complex No. I

Q1. (a) $6+9i$ (b) $-5+12i$ (c) $7i$ (d) $-2+10i$ (e) $\frac{1}{5}(4-7i)$ (f) $\frac{1}{13}(4+7i)$

Q2. (a) $\frac{1}{5}(3-i)$ (b) $-\frac{1}{2}(1-i)$

Q3. (a) a^2-b^2+2abi (b) $\dfrac{a}{a^2+b^2}-\dfrac{ib}{a^2+b^2}$ (c) $\dfrac{1}{(a-1)^2+b^2}\left[(a^2-1+b^2)-2bi\right]$

Q4. (a) $\frac{1}{2}(-1\pm\sqrt{3}i)$ (b) $-1\pm\sqrt{2}i$ (c) $3\pm i$ (d) $\frac{1}{2}(3\pm\sqrt{3})$ (e) $\pm i$, $\pm 2i$

Q5. 10

Q6. (a) Re(z) = -1 , Im(z) = 1 , $\bar{z}=-1-i$ (b) Re(z) = 2 , Im(z) = 3 , $\bar{z}=2-3i$
(c) Re(z) = 2 , Im(z) = -3 , $\bar{z}=2+3i$ (d) Re(z) = ½ , Im(z) = $-\frac{3}{2}$, $\bar{z}=\frac{1}{2}(1+3i)$
(e) Re(z) = 0 , Im(z) = - ½ , $\bar{z}=\frac{1}{2}i$

Q7. Check with tutor

Q8. Check with tutor.

Q9. Hint: use $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$, { where: $a=\sqrt{3}+1$, $b=(\sqrt{3}-1)i$ } to simplify this.

Q10. Hint: use the difference of 2 squares, $X^2 - Y^2 = (X-Y).(X+Y)$ and simplify.

Q11. (a) Hint: Let $\alpha = x+iy$ and $\beta = p+iq$;
use the fact that $\alpha\beta = -\frac{b}{a} \in R$ to show that the imaginary parts are "opposites" ($q = -y$)
and use $\alpha+\beta = \frac{c}{a} \in R$ to show that the real parts are equal.
(b) $z^2+6z+13$

Q12. $z = 1+i$ $z^2 = 2i$ $z^3 = -2+2i$ $z^4 = -4$ $z^5 = -4-4i$
$z^6 = -8i$ $z^7 = 8-8i$ $z^8 = 16$ $z^9 = 16+16i$ $z^{10} = 32i$
$n = 4$