Extension 2 Assessment Integration 31-5-04

Name________________

1. Find the following indefinite integrals: [3m each]

(i) $\int \frac{dx}{3x^2+9x+12}$

(ii) $\int \frac{\sec^2 2x}{1+\tan\ 2x}\,dx$

(iii) $\int \cos^5 x\ \sin^3 x\ dx$

(iv) $\int \ln(a^2+x^2)\,dx$

(v) $\int \frac{7x-9}{\sqrt{7-5x-3x^2}}\,dx$

2. Evaluate the following integrals: [3m each]

(i) $\int_0^3 \sqrt{9-x^2}\ dx$

(ii) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1-\operatorname{cosec} x)^2 \operatorname{cosec} x \cot x\ dx$

(iii) $\int_{-1}^{1} \frac{(1+\cos 2x)^3}{\tan^{-1} x}\,dx$

(iv) $\int_0^{\frac{\pi}{2}} \frac{\sec^3 x}{\sec^3 x+\operatorname{cosec}^3 x}\,dx$

(v) $\int_0^{2\pi} \frac{\cot x}{\cos^2 x+\sin^4 x}\,dx$

3. (a) Show that $\int_{5\frac{1}{2}}^{6\frac{1}{2}} \frac{dx}{\sqrt{(x-5)(7-x)}} = \frac{\pi}{3}$. [4m]

(b) Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ [2m]

(c) Find $\int \cos 2x \cos x\,dx$ [3m]

(d) If $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \sin^2 x\,dx$ for $n \geq 0$ show that

$I_n = \frac{n-1}{n+2} I_{n-2}$ for $n \geq 2$. [4m]

Hence or otherwise evaluate $\int_0^{\frac{\pi}{2}} \cos^4 x \sin^2 x\,dx$. [2m]

4. (a) Find $\int \sec^3 x\,dx$ [3m]

(b) If $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x\,dx$ for $n \geq 0$ show that [3m]

$I_n + n(n-1)I_{n-2} = n(\frac{\pi}{2})^{n-1}$ for $n \geq 2$.

Find I_1 and hence evaluate $\int_0^{\frac{\pi}{2}} x^3 \sin x\,dx$. [3m]

(c) Evaluate: $\int_0^{\frac{\pi}{4}} \frac{dx}{2\sin 2x + \cos x}$ [6m]

Extension 2 Assessment Integration 31-5-04

1) (i) $I = \int \frac{dx}{3x^2+9x+12}$

$= \int \frac{dx}{3(x^2+3x+4)}$

$= \frac{1}{3} \int \frac{dx}{(x+\frac{3}{2})^2 - \frac{9}{4} + 4}$

$= \frac{1}{3} \int \frac{dx}{(x+\frac{3}{2})^2 + \frac{7}{4}}$

$= \frac{1}{3} \cdot \frac{1}{\sqrt{7}/2} \tan^{-1}\left(\frac{x+\frac{3}{2}}{\sqrt{7}/2}\right) + c$

$= \frac{2}{3\sqrt{7}} \tan^{-1}\left(\frac{2x+3}{\sqrt{7}}\right) + c$

(ii) $I = \int \frac{\sec^2 2x}{1+\tan 2x}\, dx$

let $u = 1+\tan 2x$

$\therefore \frac{du}{dx} = 2\sec^2 2x$

$\therefore I = \int \frac{\frac{du}{2}}{u}$

$= \frac{1}{2} \ln|u| + c$

$= \frac{1}{2} \ln|1+\tan 2x| + c$

(iii) $I = \int \cos^5 x \sin^3 x \, dx$

$= \int \cos^5 x \sin x (1-\cos^2 x)\, dx$

let $u = \cos x$ $\therefore \frac{du}{dx} = -\sin x$

$\therefore I = -\int u^5(1-u^2)\, du$

$= -\int u^5 - u^7 \, du$

$= \frac{u^8}{8} - \frac{u^6}{6} + c$

$= \frac{1}{8}\cos^8 x - \frac{1}{6}\cos^6 x + c$

(iv) $I = \int \ln(a^2+x^2)\, dx$

$= \int \ln(a^2+x^2) . 1 \, dx$

Let $u = \ln(a^2+x^2)$ $dv = 1\, dx$

$\therefore I = x\ln(a^2+x^2) - \int \frac{2x^2}{a^2+x^2}\, dx$

$$\begin{array}{r} 1 \\ x^2+a^2 \overline{)\, x^2 \quad} \\ -(x^2+a^2) \\ \hline -a^2 \end{array}$$

$\therefore I = x\ln(a^2+x^2) - 2\int 1 - \frac{a^2}{x^2+a^2}\, dx$

$= x\ln(a^2+x^2) - 2\left[x - a^2 \cdot \frac{1}{a} \tan^{-1}\frac{x}{a}\right] + c$

$= x\ln(a^2+x^2) - 2x + 2a\tan^{-1}\frac{x}{a} + c$

(v) $I = \int \frac{7x-9}{\sqrt{7-5x-3x^2}}\, dx$

$= \int \frac{-\frac{7}{6}(-6x-5) - 14\frac{5}{6}}{\sqrt{7-5x-3x^2}}\, dx$

$= -\frac{7}{6}\underbrace{\int \frac{-6x-5}{\sqrt{7-5x-3x^2}}\, dx} - \frac{89}{6}\int \frac{dx}{\sqrt{-3[x^2+\frac{5x}{3}-\frac{7}{3}]}}$

let $u = 7-5x-3x^2$

$\therefore \frac{du}{dx} = -5-6x$

$= -\frac{7}{6}\int u^{-\frac{1}{2}}\, du - \frac{89}{6}\int \frac{dx}{\sqrt{-3[(x+\frac{5}{6})^2 - \frac{25}{36} - \frac{7}{3}]}}$

$= -\frac{7}{6}\,\frac{u^{\frac{1}{2}}}{\frac{1}{2}} - \frac{89}{6}\int \frac{dx}{\sqrt{3}\sqrt{\frac{109}{36} - (x+\frac{5}{6})^2}}$

$= -\frac{7}{3}\sqrt{7-5x-3x^2} - \frac{89}{6\sqrt{3}} \sin^{-1}\left(\frac{x+\frac{5}{6}}{\frac{\sqrt{109}}{6}}\right) + c$

$= -\frac{7}{3}\sqrt{7-5x-3x^2} - \frac{89}{6\sqrt{3}} \sin^{-1}\left(\frac{6x+5}{\sqrt{109}}\right) + c$

4(a) $I = \int \sec^3 x\, dx$

$= \int \sec^2 x \sec x \, dx$

let $u = \sec x$ $dv = \sec^2 x\, dx$

$\therefore \frac{du}{dx} = \sec x \tan x$ $v = \tan x$

$\therefore I = \sec x \tan x - \int \sec x \tan^2 x\, dx$

$= \sec x \tan x - \int \sec x(\sec^2 x - 1)\, dx$

$= \sec x \tan x - \int \sec^3 x\, dx + \int \sec x\, dx$

$\therefore 2I = \sec x \tan x + \ln|\sec x + \tan x|$

2) (i) $I=\int_0^3 \sqrt{9-x^2}\,dx$

Let $x=3\sin\theta$ when $x=0$ $\theta=0$

$\frac{dx}{d\theta}=3\cos\theta$ $\quad x=3$ $\theta=\frac{\pi}{2}$

$$\therefore I=\int_0^{\frac{\pi}{2}} \sqrt{9-9\sin^2\theta}\,.\,3\cos\theta\,d\theta$$

$$=9\int_0^{\frac{\pi}{2}} \cos^2\theta\,d\theta$$

$$=\frac{9}{2}\int_0^{\frac{\pi}{2}} 1+\cos2\theta\,d\theta$$

$$=\frac{9}{2}\left[\theta+\frac{\sin2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$=\frac{9}{2}\left[\left(\frac{\pi}{2}+0\right)-(0+0)\right]$$

$$=\frac{9\pi}{4}$$

(ii) $I=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1-\operatorname{cosec}x)^2 \operatorname{cosec}x\cot x\,dx$

Let $u=1-\operatorname{cosec}x$ $\quad$ when $x=\frac{\pi}{4}$ $u=1-\sqrt{2}$

$\frac{du}{dx}=\operatorname{cosec}x\cot x$ $\quad x=\frac{\pi}{2}$ $u=0$

$$\therefore I=\int_{1-\sqrt{2}}^{0} u^2\,.\,du$$

$$=\left[\frac{u^3}{3}\right]_{1-\sqrt{2}}^{0}$$

$$=\left[0-\frac{(1-\sqrt{2})^3}{3}\right]$$

$$=-\frac{1}{3}\left[1-3\sqrt{2}+6-2\sqrt{2}\right]$$

$$=-\frac{1}{3}\left[7-5\sqrt{2}\right]$$

$$=\frac{1}{3}(5\sqrt{2}-7)$$

(iii) $I=\int_{-1}^{1} \frac{(1+\cos2x)^3}{\tan^{-1}x}\,dx$

Let $f(x)=(1+\cos2x)^3$

$\therefore f(-x)=(1+\cos(-2x))^3$

$=(1+\cos2x)^3$

$=f(x)$ $\quad \therefore (1+\cos2x)^3$ is an even fn.

$g(x)=\tan^{-1}x$ is an odd fn

$\therefore \frac{(1+\cos2x)^3}{\tan^{-1}x}$ is an odd fn.

$\therefore I=0$, as the integration of an odd function about symmetrical limits is zero.

(iv) $I=\int_0^{\frac{\pi}{2}} \frac{\sec^3x}{\sec^3x+\operatorname{cosec}^3x}\,dx$

$$=\int_0^{\frac{\pi}{2}} \frac{\sec^3\left(\frac{\pi}{2}-x\right)}{\sec^3\left(\frac{\pi}{2}-x\right)+\operatorname{cosec}^3\left(\frac{\pi}{2}-x\right)}\,dx$$

$$=\int_0^{\frac{\pi}{2}} \frac{\operatorname{cosec}^3x}{\operatorname{cosec}^3x+\sec^3x}\,dx$$

$$\therefore 2I=\int_0^{\frac{\pi}{2}} \frac{\sec^3x+\operatorname{cosec}^3x}{\operatorname{cosec}^3x+\sec^3x}\,dx$$

$$=\int_0^{\frac{\pi}{2}} 1\,dx$$

$$=\left[x\right]_0^{\frac{\pi}{2}}$$

$$=\left[\frac{\pi}{2}-0\right]$$

$$\therefore I=\frac{\pi}{4}$$

(v) $I=\int_0^{2\pi} \frac{\cot x}{\cos^2x+\sin^4x}\,dx$

$$=\int_0^{2\pi} \frac{\cot(2\pi-x)}{\cos^2(2\pi-x)+\sin^4(2\pi-x)}\,dx$$

$$=\int_0^{2\pi} \frac{-\cot x}{\cos^2x+\sin^4x}$$

$$=-I$$

$$\therefore 2I=0$$

$$\therefore I=0$$

3 (a) $I = \int_{5\frac{1}{2}}^{6\frac{1}{2}} \frac{dx}{\sqrt{(x-5)(7-x)}}$

let $u = x-5$ when $x = 5\frac{1}{2}$ $u = \frac{1}{2}$

$\therefore x = u+5$ $x = 6\frac{1}{2}$ $u = 1\frac{1}{2}$

$\frac{du}{dx} = 1$

$$\therefore I = \int_{\frac{1}{2}}^{1\frac{1}{2}} \frac{du}{\sqrt{u(2-u)}}$$

$$= \int_{\frac{1}{2}}^{1\frac{1}{2}} \frac{du}{\sqrt{-u^2+2u}}$$

$$= \int_{\frac{1}{2}}^{1\frac{1}{2}} \frac{du}{\sqrt{-[u^2-2u]}}$$

$$= \int_{\frac{1}{2}}^{1\frac{1}{2}} \frac{du}{\sqrt{-[(u-1)^2-1]}}$$

$$= \int_{\frac{1}{2}}^{1\frac{1}{2}} \frac{du}{\sqrt{1-(u-1)^2}}$$

$$= \left[\sin^{-1}(u-1)\right]_{\frac{1}{2}}^{1\frac{1}{2}}$$

$$= \sin^{-1}\tfrac{1}{2} - \sin^{-1}\left(-\tfrac{1}{2}\right)$$

$$= 2\sin^{-1}\tfrac{1}{2}$$

$$= 2 \times \frac{\pi}{6}$$

$$= \frac{\pi}{3}$$

ie $\int_{5\frac{1}{2}}^{6\frac{1}{2}} \frac{dx}{\sqrt{(x-5)(7-x)}} = \frac{\pi}{3}$.

(b) TO PROVE: $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

PROOF: Let $x = a-t$ when $x=0$ $t=a$

$\therefore \frac{dx}{dt} = -1$ $x=a$ $t=0$

$$\therefore LHS = \int_0^a f(x)\,dx$$

$$= \int_a^0 f(a-t).-dt$$

$$= \int_0^a f(a-t)\,dt$$

$$= \int_0^a f(a-x)\,dx$$

[reverting to the variable x]

$= RHS.$

(c) $I = \int \cos 2x \cos x \, dx$

$$= \int (1-2\sin^2 x)\cos x\,dx$$

$$= \int \cos x\,dx - 2\underbrace{\int \sin^2 x \cos x\,dx}$$

let $u = \sin x$

$\therefore \frac{du}{dx} = \cos x$

$$= \sin x - 2\int u^2\,du$$

$$= \sin x - 2\frac{u^3}{3} + C$$

$$= \sin x - \frac{2}{3}\sin^3 x + C$$

(d) $I_n = \int_0^{\frac{\pi}{2}} \cos^n x \sin^2 x\,dx$, $n \geq 0$

$$= \int_0^{\frac{\pi}{2}} \cos^n x(1-\cos^2 x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} \cos^n x\,dx - \int_0^{\frac{\pi}{2}} \cos^{n+2} x\,dx$$

$$= U_n - U_{n+2}$$

Now $U_n = \underbrace{\int_0^{\frac{\pi}{2}} \cos^{n-1} x \cos x\,dx}$

let $u = \cos^{n-1} x$, $dv = \cos x\,dx$

$\therefore \frac{du}{dx} = (n-1)\cos^{n-2} x.-\sin x$ $v = \sin x$

$$\therefore U_n = \left[\sin x \cos^{n-1} x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1)\cos^{n-2} x.\sin^2 x\,dx$$

$$\therefore U_n = [0-0] + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2} x \sin^2 x\,dx$$

$$\therefore U_n = (n-1)I_{n-2}$$

But $I_n = U_n - U_{n+2}$

$$\therefore I_n = (n-1)I_{n-2} - [(n+2-1)I_{n+2-2}]$$

$$= (n-1)I_{n-2} - [(n+1)I_n]$$

$$\therefore I_n(1+n+1) = (n-1)I_{n-2}$$

$$\therefore I_n = \frac{n-1}{n+2} I_{n-2}, \ n \geq 2.$$

Now $\int_0^{\frac{\pi}{2}} \cos^4 x \sin^2 x \, dx = I_4$

$$\therefore I_4 = \frac{3}{6} I_2$$

$$= \frac{3}{6}\left[\frac{1}{4} I_0\right]$$

$$= \frac{1}{8} \int_0^{\frac{\pi}{2}} \sin^2 x \, dx$$

$$= \frac{1}{8} \cdot \frac{1}{2} \int_0^{\frac{\pi}{2}} 1 - \cos 2x \, dx$$

$$= \frac{1}{16}\left[x - \frac{\sin 2x}{2}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{16}\left[\left(\frac{\pi}{2} - 0\right) - 0\right]$$

$$= \frac{\pi}{32}.$$

4) (a)* $I = \int \sec^3 2x \, dx$

$$= \underbrace{\int \sec 2x \sec^2 2x \, dx}$$

let $u = \sec 2x$ $\quad dv = \sec^2 2x \, dx$

$\frac{du}{dx} = 2 \sec 2x \tan 2x \quad v = \frac{\tan 2x}{2}$

$$\therefore I = \frac{\sec 2x \tan 2x}{2} - \int \sec 2x \tan^2 2x \, dx$$

$$= \frac{\sec 2x \tan 2x}{2} - \int \sec 2x \left[\sec^2 2x - 1\right] dx$$

$$= \frac{\sec 2x \tan 2x}{2} + \int \sec 2x \, dx - \int \sec^3 2x \, dx$$

$$\therefore 2I = \frac{\sec 2x \tan 2x}{2} + \frac{\ln|\sec 2x + \tan 2x|}{2} + c$$

$$\therefore I = \frac{\sec 2x \tan 2x + \ln|\sec 2x + \tan 2x|}{4} + c_1$$

(b) $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx \quad , n \geq 0$

Let $u = x^n$, $dv = \sin x \, dx$

$\frac{du}{dx} = n x^{n-1}$, $v = -\cos x$

$$I_n = \left[-x^n \cos x\right]_0^{\frac{\pi}{2}} + n \int_0^{\frac{\pi}{2}} x^{n-1} \cos x \, dx$$

$$= \left[0 - 0\right] + n \underbrace{\int_0^{\frac{\pi}{2}} x^{n-1} \cos x \, dx}$$

let $u = x^{n-1}$, $dv = \cos x \, dx$

$\therefore \frac{du}{dx} = (n-1)x^{n-2}$ $\quad v = \sin x$

$$\therefore I_n = n\left[x^{n-1} \sin x\right]_0^{\frac{\pi}{2}} - n(n-1) \int_0^{\frac{\pi}{2}} x^{n-2} \sin x$$

$$= n\left[\left(\frac{\pi}{2}\right)^{n-1} - 0\right] - n(n-1) I_{n-2}$$

$$\therefore I_n + n(n-1) I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}, n \geq 2.$$

Now $I_1 = \int_0^{\frac{\pi}{2}} x \sin x \, dx$

let $u = x$ $\quad dv = \sin x \, dx$

$\frac{du}{dx} = 1 \quad v = -\cos x$

$$\therefore I_1 = \left[-x \cos x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x \, dx$$

$$= \left[0 - 0\right] + \left[\sin x\right]_0^{\frac{\pi}{2}}$$

$$= 1 - 0 = 1$$

Now $I_3 = \int_0^{\frac{\pi}{2}} x^3 \sin x \, dx$

$$= 3\left(\frac{\pi}{2}\right)^2 - 3(2) I_1$$

$$= \frac{3\pi^2}{4} - 6.1$$

$$= \frac{3\pi^2}{4} - 6$$

(c) $I = \int_0^{\frac{\pi}{4}} \frac{dx}{2 \sin 2x + \cos x}$

$$= \int_0^{\frac{\pi}{4}} \frac{dx}{4 \sin x \cos x + \cos x}$$

$$= \int_0^{\frac{\pi}{4}} \frac{dx}{\cos x (4 \sin x + 1)}$$

let $t = \tan \frac{x}{2}$, $x \neq \pi$ $\quad$ when $x = 0$ $t = 0$; $x = \frac{\pi}{4}$ $t = \tan \frac{\pi}{8}$

$\cos x = \frac{1 - t^2}{1 + t^2}$, $\sin x = \frac{2t}{1 + t^2}$, $dx = \frac{2dt}{1 + t^2}$

$$\therefore I = \int_0^{\tan \frac{\pi}{8}} \frac{\frac{2dt}{1+t^2}}{\left(\frac{1-t^2}{1+t^2}\right)\left(\frac{8t}{1+t^2} + 1\right)}$$

$$= \int_0^{\tan \frac{\pi}{8}} \frac{2(1+t^2) \, dt}{(1-t^2)(t^2 + 8t + 1)}$$

By partial fractions $\dfrac{2(1+t^2)}{(1-t^2)(t^2+8t+1)} = \dfrac{A}{1-t} + \dfrac{B}{1+t} + \dfrac{Ct+D}{t^2+8t+1}$

$\therefore 2+2t^2 = A(1+t)(t^2+8t+1) + B(1-t)(t^2+8t+1) + (Ct+D)(1-t^2)$

Let $t=-1 \quad \therefore 4 = -12B \quad \therefore B = -\frac{1}{3}$

$t=1 \quad \therefore 4 = 12A \quad \therefore A = \frac{1}{3}$

$t=0 \quad \therefore 2 = A+B+D \quad \therefore D=2$

$t=2 \quad \therefore 10 = 63A - 21B - 6C - 3D \quad \therefore C=2$

$$\therefore I = \int_0^{\tan\frac{\pi}{8}} \frac{\frac{1}{3}}{1-t} - \frac{\frac{1}{3}}{1+t} + \frac{2t+2}{t^2+8t+1}\,dt$$

$$= \int_0^{\tan\frac{\pi}{8}} \frac{\frac{1}{3}}{1-t} \; \frac{-\frac{1}{3}}{1+t} + \frac{(2t+8)-6}{t^2+8t+1}\,dt$$

$$= \int_0^{\tan\frac{\pi}{8}} \frac{\frac{1}{3}}{1-t} - \frac{\frac{1}{3}}{1+t} \; \frac{+2t+8}{t^2+8t+1} - \frac{6}{(t+4)^2-15}\,dt$$

$$= \left[-\frac{1}{3}\ln|1-t| - \frac{1}{3}\ln|1+t| + \ln|t^2+8t+1| - \frac{6}{2\sqrt{15}}\ln\left|\frac{t+4-\sqrt{15}}{t+4+\sqrt{15}}\right|\right]_0^{\tan\frac{\pi}{8}}$$

$$= \Bigg[-\frac{1}{3}\ln\left|1-\tan\frac{\pi}{8}\right| - \frac{1}{3}\ln\left|1+\tan\frac{\pi}{8}\right| + \ln\left|\tan^2\frac{\pi}{8} + 8\tan\frac{\pi}{8} + 1\right|$$
$$-\frac{3}{\sqrt{15}}\ln\left|\frac{\tan\frac{\pi}{8}+4-\sqrt{15}}{\tan\frac{\pi}{8}+4+\sqrt{15}}\right| + \frac{3}{\sqrt{15}}\ln\left|\frac{4-\sqrt{15}}{4+\sqrt{15}}\right|\Bigg]$$

$$= \ln\left|\tan^2\frac{\pi}{8} + 8\tan\frac{\pi}{8} + 1\right| + \frac{3}{\sqrt{15}}\ln\left|\frac{4-\sqrt{15}}{4+\sqrt{15}}\right| - \frac{1}{3}\ln\left|1-\tan^2\frac{\pi}{8}\right|.$$

5(a) alternative:

$$I = \int_{5\frac{1}{2}}^{6\frac{1}{2}} \frac{dx}{\sqrt{(x-5)(7-x)}}$$

$$= \int_{5\frac{1}{2}}^{6\frac{1}{2}} \frac{dx}{\sqrt{-x^2+12x-35}}$$

$$= \int_{5\frac{1}{2}}^{6\frac{1}{2}} \frac{dx}{\sqrt{-[(x-6)^2-1]}}$$

$$= \int^{6\frac{1}{2}} \frac{dx}{\quad}$$

$$\therefore I = \left[\sin^{-1}\left(\frac{x-6}{1}\right)\right]_{5\frac{1}{2}}^{6\frac{1}{2}}$$

$$= \sin^{-1}\frac{1}{2} - \sin^{-1}\left(-\frac{1}{2}\right)$$

$$= \frac{\pi}{6} + \frac{\pi}{6}$$

$$= \frac{\pi}{3}$$

$\therefore$ LHS = RHS.

3(c) Using $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-\ }{2}\right.$

$\frac{A+B}{2} = 2x, \; \frac{A-B}{2} = x$

$\therefore A = 3x, \; B = x$

$$\therefore I = \int \cos 2x \cos x\,dx$$

$$= \frac{1}{2}\int \cos 3x + \cos x\,dx$$

$$= \frac{1}{2}\left(\frac{\sin 3x}{3} + \sin x\right) + C$$