Ext.2 – Topic 11 – Integration I

1. Find $\int \sec^3 x . \tan x . dx$

2. Find $\int \frac{\sin 2x}{2+\sin^2 x} . dx$

3. Find $\int \frac{1}{x} . \sec^2(\ln x) . dx$

4. Find $\int \frac{e^x}{\sqrt{1-e^{2x}}} . dx$

5. Find $\int \frac{1}{e^{x}+e^{-x}}.dx$

6. Evaluate $\int_{0}^{\frac{1}{2}} \frac{1}{\sqrt{1-4x^{2}}}.dx$

7. Evaluate $\int_{0}^{\frac{\pi}{6}} \tan 2x.\sec 2x.dx$

8. Find $\int \frac{x^{2}}{(x+1)(x+2)}.dx$

9. Find $\int \frac{4x - x^2}{(x+1)(x^2+4)}.dx$

10. Find $\int \frac{3}{(x^2+1)(x^2+4)}.dx$

11. Evaluate $\int_1^2 \frac{2x-3}{x^2-2x+2}.dx$

12. Evaluate $\int_0^3 \frac{x^2+4x+5}{(x+1).(x+3)}.dx$

13. Evaluate $\int_0^{\sqrt{3}} \frac{8}{(x^2+1).(x^2+9)}.dx$

Topic 11 Integration - SOLUTIONS

Problem 1. Find $\int \sec^3 x \tan x dx$.

Answer: $\frac{\sec^3 x}{3} + c$.

Explanation: Using the pattern $\int \{f(x)\}^n f'(x)dx = \frac{\{f(x)\}^{n+1}}{n+1} + c$ with $f(x) = \cos x$ and $n = -4$,

we get $\int \sec^3 x \tan x dx = \int \frac{\sin x dx}{\cos^4 x} = -\int (\cos x)^{-4}(-\sin x)dx = \frac{(\cos x)^{-3}}{3} + c = \frac{\sec^3 x}{3} + c$.

Problem 2. Find $\int \frac{\sin 2x}{2 + \sin^2 x} dx$.

Answer: $\int \frac{\sin 2x}{2 + \sin^2 x} dx = \ln(\sin^2 x + 2) + c$.

Explanation: Using $\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$ with $f(x) = \sin^2 x + 2$, we get

$\int \frac{\sin 2x}{2 + \sin^2 x} dx = \int \frac{2 \sin x \cos x}{2 + \sin^2 x} dx = \ln|\sin^2 x + 2| + c = \ln(\sin^2 x + 2) + c$, since $\sin^2 x + 2 > 0$.

Problem 3. Find $\int \frac{1}{x} \sec^2(\ln x) dx$.

Answer: $\int \frac{1}{x} \sec^2(\ln x) dx = \tan(\ln x) + c$.

Explanation: Using the substitution $u = \ln x, du = \frac{1}{x} dx$, we get

$\int \frac{1}{x} \sec^2(\ln x) dx = \int \sec^2 u du = \int \frac{1}{\cos^2 u} du = \tan u + c = \tan(\ln x) + c$.

Problem 4. Find $\int \frac{e^x}{\sqrt{1 - e^{2x}}} dx$.

Answer: $\int \frac{e^x}{\sqrt{1 - e^{2x}}} dx = \sin^{-1}(e^x) + c$.

Explanation: Make the substitution $e^x = u; du = e^x dx$. Then

$\int \frac{e^x}{\sqrt{1 - e^{2x}}} dx = \int \frac{1}{\sqrt{1 - u^2}} du = \sin^{-1} u + c = \sin^{-1}(e^x) + c$.

Problem 5. Find $\int \frac{1}{e^x + e^{-x}} dx$.

Answer: $\int \frac{1}{e^x + e^{-x}} dx = \tan^{-1} e^x + c$.

Explanation: Using the substitution $e^x = u, du = e^x dx$, we get

$\int \frac{1}{e^x + e^{-x}} dx = \int \frac{e^x}{e^{2x} + 1} dx = \int \frac{1}{u^2 + 1} du = \tan^{-1} u + c = \tan^{-1} e^x + c$.

Problem 6. Evaluate $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{(1 - 4x^2)}} dx$.

Answer: $\int_0^{\frac{1}{2}} \frac{1}{\sqrt{(1 - 4x^2)}} dx = \frac{\pi}{4}$.

Explanation: Using the pattern $\int \frac{1}{\sqrt{(a^2 - x^2)}} dx = \sin^{-1} \frac{x}{a}$ with $a = 1/2$, we obtain

$\int_0^{\frac{1}{2}} \frac{1}{\sqrt{(1 - 4x^2)}} dx = \frac{1}{2} \int_0^{\frac{1}{2}} \frac{1}{\sqrt{\frac{1}{4} - x^2}} dx = \frac{1}{2} \sin^{-1}(2x) \Big|_0^{\frac{1}{2}} = \frac{1}{2} \sin^{-1}(1) = \frac{\pi}{4}$.

Problem 7. Evaluate $\int_0^{\frac{\pi}{6}} \tan 2x \sec 2x dx$.

Answer: $\int_0^{\frac{\pi}{6}} \tan 2x \sec 2x dx = \frac{1}{2}$.

Explanation: Using the pattern $\int \{f(x)\}^{-2} f'(x) dx = -\{f(x)\}^{-1} + c$ with $f(x) = \cos 2x$, we get $\int_0^{\frac{\pi}{6}} \tan 2x \sec 2x dx = \int_0^{\frac{\pi}{6}} \frac{\sin 2x}{\cos^2 2x} dx = -\frac{1}{2}\int \frac{(\cos 2x)'}{\cos^2 2x} dx = \frac{1}{2}\left[\frac{1}{\cos 2x}\right]_0^{\frac{\pi}{6}} = \frac{1}{2}\left(\frac{1}{\cos(\frac{\pi}{3})} - \frac{1}{\cos 0}\right) = \frac{1}{2}$.

Problem 8. Find $\int \frac{x^2}{(x+1)(x+2)} dx$.

Answer: $\int \frac{x^2}{(x+1)(x+2)} dx = x + \ln\left(\frac{|x+1|}{(x+2)^4}\right) + c$.

Explanation: By division $\frac{x^2}{(x+1)(x+2)} = 1 - \frac{3x+2}{(x+1)(x+2)}$.

Let $\frac{3x+2}{(x+1)(x+2)} \equiv \frac{a}{x+1} + \frac{b}{x+2}$.

Then we get $3x+2 \equiv a(x+2) + b(x+1)$.

Put $x=-1$: $a=-1$.

Put $x=-2$: $b=4$.

Hence

$$\int \frac{x^2}{(x+1)(x+2)} dx = \int 1 dx - \int \frac{3x+2}{(x+1)(x+2)} dx = x - \int\left(-\frac{1}{x+1} + \frac{4}{x+2}\right) dx$$

$$= x + \int \frac{1}{x+1} dx - 4\int \frac{1}{x+2} dx = x + \ln|x+1| - 4\ln|x+2| + c = x + \ln\left(\frac{|x+1|}{(x+2)^4}\right) + c.$$

Problem 9. Find $\int \frac{4x - x^2}{(x+1)(x^2+4)} dx$.

Answer: $\int \frac{4x-x^2}{(x+1)(x^2+4)} dx = -\ln|x+1| + 2\tan^{-1}\left(\frac{x}{2}\right) + c$.

Explanation: Let $\frac{4x-x^2}{(x+1)(x^2+4)} \equiv \frac{a}{x+1} + \frac{bx+c}{x^2+4}$.

Then $4x - x^2 \equiv a(x^2+4) + (bx+c)(x+1)$.

Put $x=-1$: $-5 = 5a \Rightarrow a = -1$.

Equate coefficients of x^2: $-1 = a + b \Rightarrow b = 0$.

Equate coefficients of x^1: $4 = b + c \Rightarrow c = 4$.

Hence

$$\int \frac{4x-x^2}{(x+1)(x^2+4)} dx = \int\left(\frac{-1}{x+1} + \frac{4}{x^2+4}\right) dx = \int \frac{-1}{x+1} dx + \int \frac{4}{x^2+4} dx = -\ln|x+1| + 2\tan^{-1}(\frac{x}{2}) + c.$$

Problem 10. Find $\int \frac{3}{(x^2+1)(x^2+4)} dx$.

Answer: $\int \frac{3}{(x^2+1)(x^2+4)} dx = \tan^{-1} x - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c$.

Explanation: Let $\frac{3}{(x^2+1)(x^2+4)} \equiv \frac{ax+b}{x^2+1} + \frac{cx+d}{x^2+4}$.

Then $3 \equiv (ax+b)(x^2+4) + (cx+d)(x^2+1)$.

Equate coefficients of x^3: $0 = a + c$.

Equate coefficients of x^2: $0 = b + d$.

Equate coefficients of x^1: $0 = 4a + c$.

Equate constant terms: $3 = 4b + d$.

Thus we obtain $a = c = 0$, $b = 1$, $d = -1$.

Hence

$$\int \frac{3}{(x^2+1)(x^2+4)} dx = \int\left(\frac{1}{x^2+1} - \frac{1}{x^2+4}\right) dx = \int \frac{1}{x^2+1} dx - \int \frac{1}{x^2+4} dx = \tan^{-1} x - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + c.$$

Problem 11. Evaluate $\int_1^2 \frac{2x-3}{x^2-2x+2} dx$.

Answer: $\int_1^2 \frac{2x-3}{x^2-2x+2} dx = \ln 2 - \frac{\pi}{4}$.

Explanation: Make the substitution $x-1 = u, dx = du, x = 1 \Rightarrow u = 0, x = 2 \Rightarrow u = 1, x = u+1$.

Hence

$$\int_1^2 \frac{2x-3}{x^2-2x+2} dx = \int_1^2 \frac{2x-3}{(x-1)^2+1} dx = \int_0^1 \frac{2u-1}{u^2+1} du = \int_0^1 \frac{2u}{u^2+1} du - \int_0^1 \frac{du}{u^2+1} = \left[\ln(u^2+1)\right]_0^1 - \left[\tan^{-1} u\right]_0^1$$

$\ln 2 - \ln 1 - (\tan^{-1} 1 - \tan^{-1} 0) = \ln 2 - \frac{\pi}{4}$.

Problem 12. Evaluate $\int_0^3 \frac{x^2+4x+5}{(x+1)(x+3)}dx$.

Answer: $\int_0^3 \frac{x^2+4x+5}{(x+1)(x+3)}dx = 3+\ln 2$.

Explanation: By division $\frac{x^2+4x+5}{(x+1)(x+3)} \equiv 1 + \frac{2}{(x+1)(x+3)}$.

Let $\frac{2}{(x+1)(x+3)} \equiv \frac{a}{(x+1)} + \frac{b}{(x+3)}$, a, b constants.

Then $2 \equiv a(x+3)+b(x+1)$.

Put $x=-1$: $2=2a \Rightarrow a=1$.

Put $x=-3$: $2=-2b \Rightarrow b=-1$.

Thus we get

$$\int_0^3 \frac{x^2+4x+5}{(x+1)(x+3)}dx = \int_0^3 1dx + \int_0^1 \frac{2}{(x+1)(x+3)}dx = [x]_0^3 + \int_0^3 \frac{1}{x+1}dx - \int_0^3 \frac{1}{x+3}dx$$

$$= 3 + \left[\ln|x+1|\right]_0^3 - \left[\ln|x+3|\right]_0^3 = 3+\ln 4 - \ln 1 - (\ln 6 - \ln 3) = 3+\ln 2.$$

Problem 13. Evaluate $\int_0^{\sqrt{3}} \frac{8}{(x^2+1)(x^2+9)}dx$.

Answer: $\int_0^{\sqrt{3}} \frac{8}{(x^2+1)(x^2+9)}dx = \frac{5\pi}{8}$.

Explanation: Let $\frac{8}{(x^2+1)(x^2+9)} \equiv \frac{ax+b}{x^2+1} + \frac{cx+d}{x^2+9}$.

Then $8 \equiv (ax+b)(x^2+9)+(cx+d)(x^2+1)$.

Equate coefficients of x^3: $0=a+c$.

Equate coefficients of x^2: $0=b+d$.

Equate coefficients of x^1: $0=9a+c$.

Equate constant terms: $8=9b+d$.

Thus we get $a=0, c=0, b=1, d=-1$.

Hence $\int_0^{\sqrt{3}} \frac{8}{(x^2+1)(x^2+9)}dx = \int_0^{\sqrt{3}} \frac{1}{x^2+1}dx - \int_0^{\sqrt{3}} \frac{1}{x^2+9}dx = \left[\tan^{-1} x\right]_0^{\sqrt{3}} - \frac{1}{3}\left[\tan^{-1}(\frac{x}{3})\right]_0^{\sqrt{3}}$

$= \frac{\pi}{3} - \frac{\pi}{18} = \frac{5\pi}{18}$.