Ext 2 Mathematics
Easter Holiday Assignment

Complex Numbers, Polynomials, Volumes & Some Harder Ext 1 Questions

Date Due: Wednesday 28th April, 2004

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1) Obtain the solutions of the quadratic equation $(1 - 4i)z^2 - 4z + 1 = 0$ in the form $a + ib$.

2) The complex number z has modulus r and argument θ where $0 < \theta < 2\pi$. Find in terms of r and θ the modulus and argument of:

a. z^2 b. $\frac{1}{z}$ c. iz

3) Find the locus in the Argand diagram of the point P which represents the complex number z where $z\bar{z} - 4(z + \bar{z}) = 9$.

4) On an Argand diagram the points P, Q, R represent the complex numbers p, q, r respectively. If $p - q + iq - ir = 0$, what type of triangle is ΔPQR? Give a reason for your answer.

5) The complex number z is represented by the point P on an Argand Diagram. Indicate clearly on a single diagram the locus of P in each of the following cases:

i. $|z - 4| = |z + 2i|$ ii. $\arg(z + 3) = \frac{\pi}{4}$.

Show that there is a point representing a complex number of the form ib, where b is real, which lies on both loci.

6) $P(x) = x^6 + x^3 + 1$

i. Show that the roots of $P(x) = 0$ are amongst the roots of $x^9 - 1 = 0$.

ii. Hence show the roots of $P(x) = 0$ on the unit circle, centre the origin, on an Argand Diagram.

iii. Show that $P(x) = (x^2 - 2x\cos\frac{2\pi}{9} + 1)(x^2 - 2x\cos\frac{4\pi}{9} + 1)(x^2 - 2x\cos\frac{8\pi}{9} + 1)$.

iv. Evaluate $\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} + \cos\frac{4\pi}{9}\cos\frac{8\pi}{9} + \cos\frac{8\pi}{9}\cos\frac{2\pi}{9}$.

7) i. If α is a double zero of the polynomial P(x), show that α is a zero of P′(x).

ii. $(x - 1)^2$ is a factor of $x^5 + 2x^4 + ax^3 + bx^2$. Find the values of a and b.

8) The equation $x^3 + 2x - 1 = 0$ has roots α, β, γ. In each of the following cases find an equation with numerical coefficients having the roots stated.

a. $-\alpha, -\beta, -\gamma$ b. $\alpha, -\alpha, \beta, -\beta, \gamma, -\gamma$ c. $\alpha^2, \beta^2, \gamma^2$.

9) A polynomial P(x) is divided by $x^2 - a^2$ where $a \neq 0$ and the remainder is $px + q$. Show that $p = \frac{1}{2a}\{P(a) - P(-a)\}$ and $q = \frac{1}{2}\{P(a) + P(-a)\}$.

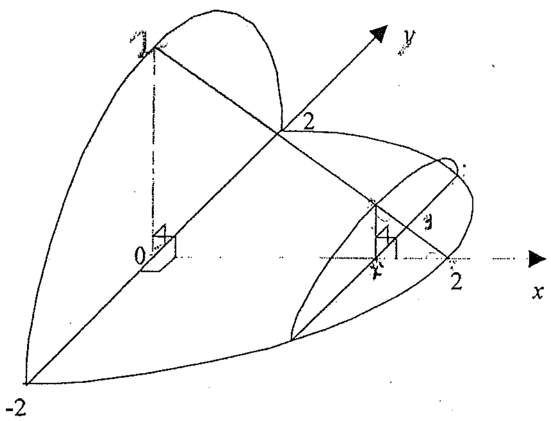
Find the remainder when the polynomial $P(x) = x^n - a^n$ is divided by $x^2 - a^2$ for the cases

a. n even. b. n odd.

10) i. On the same axes sketch the graphs of the functions $y = \dfrac{e^{x} + e^{-x}}{2}$ and $y = \dfrac{e^{x} - e^{-x}}{2}$ showing clearly the coordinates of any points of intersection with the x axis and the y axis.

ii. The region between the two curves bounded by the y axis and the line $x = 1$ is rotated through one complete revolution about the y axis. Use cylindrical shells to show that the volume V of the solid of revolution so formed is given by $V = 2\pi \int_0^1 xe^{-x}\,dx$ and hence find this volume. Hint : $\int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C$

11)

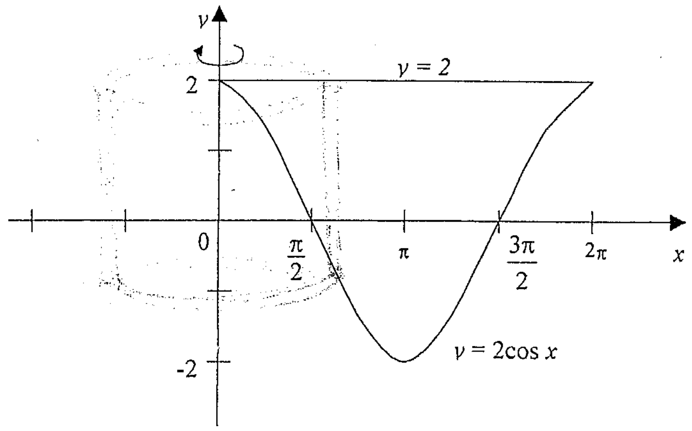


A solid figure has a semicircular base of radius 2. Cross sections taken at right angles to the semi-diameter of this base are semi-ellipses. The vertical cross section containing the semi-diameter of the base is a right isosceles triangle as shown.

a. Given that the area of an ellipse with semi-axes a and b is πab, show that the volume of the solid is given by $V = \pi \int_0^2 \sqrt{4 - x^2}\,dx - \dfrac{\pi}{2} \int_0^2 x\sqrt{4 - x^2}\,dx$.

b. Find the volume of the solid using the substitution $u = 4 - x^2$ where necessary.

12)



A mathematically inclined microwave cooking enthusiast decided to design his own cake pan. The shape of the interior of the cake pan is obtained by rotating the region bounded by the curve $y = 2\cos x$, $0 \le x \le 2\pi$ and the line $y = 2$ through 360° about the y-axis. Use the method of cylindrical shells to show that the volume of the cake pan is given by

$4\pi \int_0^{2\pi} x(1 - \cos x)\,dx$ and hence calculate this volume.

Hint: $\int x\cos x\,dx = x\sin x + \cos x + C$

13) A sequence of numbers T_n, n = 1 , 2, 3, ... is given by $T_1 = 1$, $T_2 = 3$ and $T_n = T_{n-1} + T_{n-2}$, n = 3, 4, 5, ... Use the method of mathematical induction to show that

$$T_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n, \quad n = 1, 2, 3, \ldots .$$

14) Let $f(x) = \cos x - (1 - \frac{x^2}{2})$

i. Show that f(x) is an even function.

ii. Find expressions for $f'(x)$ and $f''(x)$.

iii. Deduce that $f'(x) \geq 0$ for $x \geq 0$.

iv. Hence show that $\cos x \geq 1 - \frac{x^2}{2}$ for all x.

15) The numbers a, b, c are said to be in harmonic progression if their reciprocals $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in arithmetic progression, and b is then said to be the harmonic mean of a and c.

i. Show that the numbers 6, 8, 12 are in harmonic progression.

ii. Show that the harmonic mean of a and c is equal to $\frac{2ac}{a+c}$.

iii. If $a > 0$, $c > 0$ show that the geometric mean $\sqrt{ac}$ is greater than or equal to the harmonic mean $\frac{2ac}{a+c}$.

The End ...