EXTENSION 2 TEST
CONICS/ INTEGRATION 16-5-02

Name______________

INSTRUCTIONS:

- Begin each question on a new sheet of A4 paper
- All questions are of equal marks
- Show all necessary working
- Marks may be deducted for careless or badly arranged work
- Standard integrals are printed on the back page.

1. (a) Find the equation of the normal to the hyperbola $\frac{x^2}{6}-\frac{y^2}{8}=1$ at the point (3,2).

(3 m)

(b) $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. The tangent at P meets the tangents at the ends of the major axis at Q and R. Show that QR subtends a right angle at either focus.

(7 m)

(c) $P\left(ct,\frac{c}{t}\right)$, $t \neq 1$ lies on $xy=c^2$. The tangent and normal at P meet the line $y=x$ at T and N respectively. Show that $OT.ON=4c^2$.

(5 m)

[Begin a new page]

2. Find the following indefinite integrals: [3m each]

(i) $\int \frac{\sin^2 x}{1+\cos x}\,dx$

(ii) $\int \tan x \sec^3 x \, dx$

(iii) $\int \cos^3 x \sin^3 x \, dx$

(iv) $\int \sin^2 x \cos^2 x \, dx$

(v) $\int x^2 \sin x \, dx$

[Begin a new page]

3. (a) Find the following indefinite integrals:

(i) $\int x^2\sqrt{1-4x}\,dx$ [let $u=1-4x$] [3m]

(ii) $\int x^3 \tan^{-1} x\,dx$ [4m]

(iii) $\int \frac{4+3\cos x}{5+4\cos x}\,dx$ [4m]

(b) Show by direct integration, using partial fractions : [4m]

$$\int \frac{1-4x}{x(1-4x^2)}\,dx = \ln|x|+\frac{1}{2}\ln|2x-1|-\frac{3}{2}\ln|2x+1|+c$$

[Begin a new page]

4. (a) Prove that the following results are true:

(i) $\int_0^{\frac{\pi}{2}} \sin 4x\cos 6x\,dx = -\frac{2}{5}$ [5m]

(ii) $\int_0^1 \frac{4-x}{\sqrt{5-x-x^2}}\,dx = \sqrt{3}-\sqrt{5}+\frac{9}{2}\left[\sin^{-1}\left(\frac{3}{\sqrt{21}}\right)-\sin^{-1}\left(\frac{1}{\sqrt{21}}\right)\right]$ [5m]

(b) Prove that $\frac{d}{d\theta}(\cos\theta-\cos^3\theta+\frac{3}{5}\cos^5\theta-\frac{1}{7}\cos^7\theta)=-\sin^7\theta$ [3m]

Hence or otherwise show that $\int_0^{\pi}\sin^7\theta\,d\theta=\frac{32}{35}$ [2m]

(a) $\frac{x^2}{6} - \frac{y^2}{8} = 1$

$$\therefore \frac{2x}{6} - \frac{2y\frac{dy}{dx}}{8} = 0$$

$$\therefore \frac{x}{3} - \frac{y\frac{dy}{dx}}{4} = 0$$

$$\therefore y\frac{dy}{dx} = \frac{4x}{3} \qquad \therefore \frac{dy}{dx} = \frac{4x}{3y}$$

At $(3,2)$ $\frac{dy}{dx} = \frac{12}{6} = 2 = m_{tang}$.

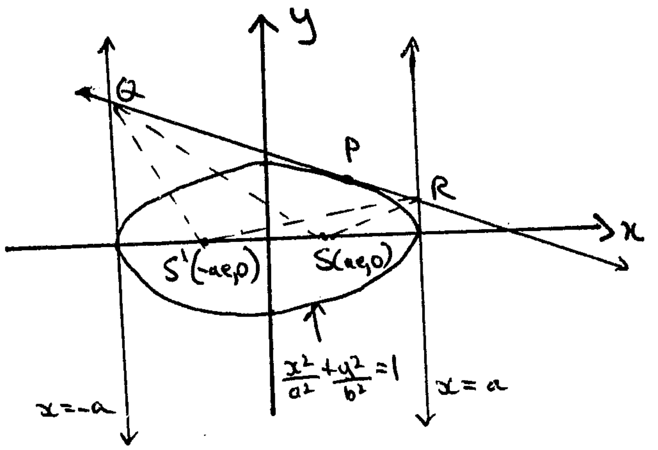
$$\therefore m_{normal} = -\frac{1}{2}$$

$\therefore$ Eq'n of req'd normal is: $y-2 = -\frac{1}{2}(x-3)$

$$\therefore 2y-4 = -x+3$$

$$\therefore x+2y-7=0$$

(b)



At $P(a\cos\theta, b\sin\theta)$

$x = a\cos\theta$, $y = b\sin\theta$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta}\cdot\frac{d\theta}{dx}$$

$$= b\cos\theta \cdot \frac{1}{-a\sin\theta}$$

$$= \frac{-b\cos\theta}{a\sin\theta}$$

$\therefore$ Eq'n of tangent at P is:

$$y - b\sin\theta = \frac{-b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

$$\therefore ay\sin\theta - ab\sin^2\theta = -xb\cos\theta + ab\cos^2\theta$$

$$\therefore bx\cos\theta + ay\sin\theta = ab$$

$$\therefore \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

At Q $x=-a$ $\therefore y = \frac{b(1+\cos\theta)}{\sin\theta}$

At R $x=a$ $\therefore y = \frac{b(1-\cos\theta)}{\sin\theta}$

$$\Rightarrow Q = \left(-a, \frac{b(1+\cos\theta)}{\sin\theta}\right), R = \left(a, \frac{b(1-\cos\theta)}{\sin\theta}\right)$$

Now $m_{SQ}\cdot m_{SR} = \left[\frac{\frac{b(1+\cos\theta)}{\sin\theta}}{-a-ae} \cdot \frac{\frac{b(1-\cos\theta)}{\sin\theta}}{a-ae}\right]$

$$= \frac{\frac{b^2(1-\cos^2\theta)}{\sin^2\theta}}{-a^2 + \cancel{a^2e} - \cancel{a^2e} + a^2e^2}$$

$$= \frac{b^2}{-a^2(1-e^2)}$$

$$= \frac{-b^2}{b^2} \quad \left(\text{as } b^2 = a^2(1-e^2) \text{ for this ellipse.}\right)$$

$$= -1$$

Similarly $m_{S'Q}\cdot m_{S'R} = \left[\frac{\frac{b(1+\cos\theta)}{\sin\theta}}{-a+ae} \cdot \frac{\frac{b(1-\cos\theta)}{\sin\theta}}{a+ae}\right]$

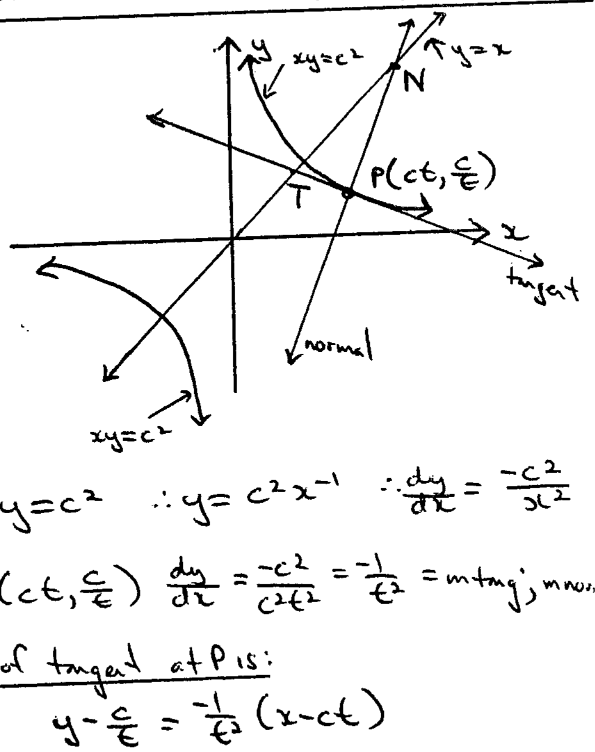
$$= \frac{b^2}{-a^2 - \cancel{a^2e} + \cancel{a^2e} + a^2e^2}$$

$$= \frac{b^2}{-a^2(1-e^2)}$$

$$= -1$$

$\Rightarrow$ QR subtends a right angle at either f

(c)



$$xy = c^2 \quad \therefore y = c^2x^{-1} \quad \therefore \frac{dy}{dx} = \frac{-c^2}{x^2}$$

At $P(ct, \frac{c}{t})$ $\frac{dy}{dx} = \frac{-c^2}{c^2t^2} = \frac{-1}{t^2} = m_{tang}$; m_{nor}

Eq'n of tangent at P is:

$$y - \frac{c}{t} = \frac{-1}{t^2}(x - ct)$$

$$\therefore t^2y - ct = -x + ct$$

$$\therefore x + t^2y = 2ct$$

At T: $y=x$ $\therefore x + t^2x = 2ct$ $\therefore x = \frac{2ct}{1+t^2}$

$$\Rightarrow T = \left(\frac{2ct}{1+t^2}, \frac{2ct}{1+t^2}\right)$$

② (i) $I = \int \frac{\sin^2 x}{1+\cos x}\, dx$

$= \int \frac{1-\cos^2 x}{1+\cos x}\, dx$

$= \int \frac{(1-\cos x)\cancel{(1+\cos x)}}{\cancel{(1+\cos x)}}\, dx$

$= x - \sin x + c$

(ii) $I = \int \tan x \sec^3 x \, dx$

$= \int \tan x \sec x \sec^2 x \, dx$

let $u = \sec x \quad \therefore \frac{du}{dx} = \sec x \tan x$

$\therefore I = \int u^2 \, du$

$= \frac{u^3}{3} + c$

$= \frac{1}{3}\sec^3 x + c$

(iii) $I = \int \cos^3 x \sin^3 x \, dx$

$= \int \cos^3 x \sin x (1-\cos^2 x)\, dx$

let $u = \cos x \quad \therefore \frac{du}{dx} = -\sin x$

$\therefore I = -\int u^3(1-u^2)\, du$

$= -\int u^3 - u^5 \, du$

$= -\frac{u^4}{4} + \frac{u^6}{6} + c$

$= -\frac{1}{4}\cos^4 x + \frac{1}{6}\cos^6 x + c$

(iv) $I = \int \sin^2 x \cos^2 x \, dx$

$= \int (\sin x \cos x)^2 \, dx$

$= \int (\tfrac{1}{2}\sin 2x)^2 \, dx$

$= \frac{1}{4}\int \sin^2 2x \, dx$

$\cos 2x = 1 - 2\sin^2 x$

$\therefore \cos 4x = 1 - 2\sin^2 2x$

$\therefore \sin^2 2x = \frac{1}{2}[1-\cos 4x]$

$\therefore I = \frac{1}{4}\int \frac{1}{2}[1-\cos 4x]\, dx$

$= \frac{1}{8}\left[x - \frac{\sin 4x}{4}\right] + c$

(v) $I = \int x^2 \sin x \, dx$

let $u = x^2 \qquad dv = \sin x \, dx$

$\frac{du}{dx} = 2x \qquad v = -\cos x$

$\therefore I = -x^2\cos x + \underbrace{\int \cos x . 2x \, dx}$

let $u = 2x \qquad dv = \cos x \, dx$

$\frac{du}{dx} = 2 \qquad v = \sin x$

$\therefore I = -x^2\cos x + 2x\sin x - 2\int \sin x$

$= -x^2\cos x + 2x\sin x + 2\cos x + c$

[1(c) ctd] Eqn of normal at P is:

$y - \frac{c}{t} = t^2(x - ct)$

$\therefore t^3 x - ty = c(t^4 - 1)$

At N: $y = x \quad \therefore t^3 x - tx = c(t^4-1)$

$\therefore x = \frac{c(t^4-1)}{t(t^2-1)} = \frac{c(t^2+1)}{t}$

$\therefore N = \left(\frac{c(t^2+1)}{t}, \frac{c(t^2+1)}{t}\right)$

Now $OT . ON = \sqrt{\left(\frac{2ct}{1+t^2}\right)^2 + \left(\frac{2ct}{1+t^2}\right)^2} . \sqrt{\left(\frac{c(t^2+1)}{t}\right)^2 + \left(\frac{c(t^2+1)}{t}\right)^2}$

$= \sqrt{2}\left(\frac{2ct}{1+t^2}\right)\sqrt{2}\left(\frac{c(t^2+1)}{t}\right)$

$= 2\left(\frac{2c\cancel{t}}{\cancel{1+t^2}} . \frac{c\cancel{(t^2+1)}}{\cancel{t}}\right)$

$= 4c^2$

(a) (i) $I = \int x^2 \sqrt{1-4x}\, dx$

let $u = 1-4x$ $\quad \therefore x = \frac{1-u}{4}$

$\therefore \frac{du}{dx} = -4 \quad \therefore x^2 = \left(\frac{1-u}{4}\right)^2$

$$\therefore I = \int \left(\frac{1-u}{4}\right)^2 \sqrt{u} \cdot -\frac{du}{4}$$

$$= -\frac{1}{4}\int \frac{1-2u+u^2}{16} \cdot u^{\frac{1}{2}}\, du$$

$$= -\frac{1}{64}\int u^{\frac{1}{2}} - 2u^{3/2} + u^{5/2}\, du$$

$$= -\frac{1}{64}\left[\frac{2u^{3/2}}{3} - \frac{4u^{5/2}}{5} + \frac{2u^{7/2}}{7}\right] + c$$

$$= -\frac{1}{64}\left[\frac{2}{3}(1-4x)^{3/2} - \frac{4}{5}(1-4x)^{5/2} + \frac{2}{7}(1-4x)^{7/2}\right] + c$$

(ii) $I = \int x^3 \tan^{-1} x\, dx$

let $u = \tan^{-1} x \quad dv = x^3\, dx$

$\therefore \frac{du}{dx} = \frac{1}{1+x^2} \quad v = \frac{x^4}{4}$

$$\therefore I = \frac{1}{4}x^4 \tan^{-1} x - \frac{1}{4}\int \frac{x^4}{1+x^2}\, dx$$

$$\begin{array}{r|l} & x^2 - 1 \\ x^2+1 & x^4 \\ & -(x^4 + x^2) \\ & -x^2 \\ & -(-x^2 - 1) \\ & 1 \end{array}$$

$$\therefore I = \frac{1}{4}x^4 \tan^{-1} x - \frac{1}{4}\int x^2 - 1 + \frac{1}{x^2+1}\, dx$$

$$= \frac{1}{4}x^4 \tan^{-1} x - \frac{1}{4}\left[\frac{x^3}{3} - x + \tan^{-1} x\right] + c$$

$$= \frac{1}{4}\left[x^4 \tan^{-1} x - \frac{x^3}{3} + x - \tan^{-1} x\right] + c$$

(iii) $I = \int \frac{4 + 3\cos x}{5 + 4\cos x}\, dx$

$$= \int \frac{\frac{3}{4}(5+4\cos x) + \frac{1}{4}}{5+4\cos x}\, dx$$

$$= \int \frac{3}{4}\, dx + \frac{1}{4}\int \frac{1}{5+4\cos x}\, dx$$

let $t = \tan\frac{x}{2}$, $\cos x = \frac{1-t^2}{1+t^2}$, $dx = \frac{2d}{1+}$

$$\therefore I = \frac{3x}{4} + \frac{1}{4}\int \frac{1}{5+4\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2dt}{1+t^2}$$

$$= \frac{3x}{4} + \frac{1}{4}\int \frac{1}{\frac{5+5t^2+4-4t^2}{\cancel{1+t^2}}} \cdot \frac{2dt}{\cancel{1+t^2}}$$

$$= \frac{3x}{4} + \frac{1}{2}\int \frac{dt}{9+t^2}$$

$$= \frac{3x}{4} + \frac{1}{2}\cdot\frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) + c$$

$$= \frac{3x}{4} + \frac{1}{6}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{3}\right) + c$$

(b) $I = \int \frac{1-4x}{x(1-4x^2)}\, dx$

By partial fractions: $\frac{1-4x}{x(1-4x^2)} = \frac{A}{x} + \frac{Bx}{1-4}$

$$\therefore 1-4x = A(1-4x^2) + (Bx+C)x$$
$$= x^2(B-4A) + Cx + A$$

Equating corr. coeffs of powers of x:

x^2: $0 = B - 4A$ — (1)

x : $-4 = C$ — (2)

x^0: $1 = A$ — (3)

$\therefore A = 1, B = 4, C = -4.$

$$\therefore I = \int \frac{1}{x} + \frac{4(x-1)}{1-4x^2}\, dx$$

$$= \ln|x| + \int \frac{-\frac{1}{2}(-8x) - 4}{1-4x^2}\, dx$$

$$= \ln|x| - \frac{1}{2}\ln|1-4x^2| - 4\int \frac{dx}{-4(x^2}$$

$$= \ln|x| - \frac{1}{2}\ln|(1-2x)(1+2x)| + \frac{1}{2\cdot\frac{1}{2}}\ln\left|\frac{x}{x}\right.$$

$$= \ln|x| - \frac{1}{2}\ln|1-2x| - \frac{1}{2}\ln|1+2x| + \ln\left|\frac{2}{2}\right.$$

$$= \ln|x| - \frac{1}{2}\ln|2x-1| - \frac{1}{2}\ln|1+2x| + \ln|2x-1| - \ln|2x+1| + c$$

$$= \ln|x| + \frac{1}{2}\ln|2x-1| - \frac{3}{2}\ln|2x+1| + c$$

4(a)(i) $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$\therefore \frac{1}{2}(\sin A + \sin B) = \sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

$= \sin 4x \cos 6x$

$\therefore \frac{A+B}{2} = 4x \quad \therefore A+B = 8x$ — ①

$\frac{A-B}{2} = 6x \quad \therefore A-B = 12x$ — ②

①+②: $2A = 20x \quad \therefore A = 10x$

①−②: $2B = -4x \quad \therefore B = -2x$

$\therefore \frac{1}{2}(\sin 10x + \sin(-2x)) = \sin 4x \cos 6x$

Now $LHS = \int_0^{\frac{\pi}{2}} \sin 4x \cos 6x \, dx$

$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 10x - \sin 2x \, dx$

$= \frac{1}{2}\left[-\frac{\cos 10x}{10} + \frac{\cos 2x}{2}\right]_0^{\frac{\pi}{2}}$

$= \frac{1}{2}\left[\left(\frac{1}{10} + \frac{(-1)}{2}\right) - \left(\frac{-1}{10} + \frac{1}{2}\right)\right]$

$= \frac{1}{2}\left[\frac{1}{5} - 1\right]$

$= -\frac{2}{5}$

$= RHS.$

(ii) $LHS = \int_0^1 \frac{4-x}{\sqrt{5-x-x^2}} dx$

$= \int_0^1 \frac{\frac{1}{2}(-2x-1) + \frac{9}{2}}{\sqrt{5-x-x^2}} dx$

$= \frac{1}{2}\int_0^1 \frac{-2x-1}{\sqrt{5-x-x^2}} dx + \frac{9}{2}\int_0^1 \frac{dx}{\sqrt{-[x^2+x-5]}}$

let $u = 5-x-x^2$

$\therefore \frac{du}{dx} = -1-2x$

when $x=0$ $u=5$

$x=1$ $u=3$

$\therefore LHS = \frac{1}{2}\int_5^3 \frac{du}{\sqrt{u}} + \frac{9}{2}\int_0^1 \frac{dx}{\sqrt{-[(x+\frac{1}{2})^2 - \frac{21}{4}]}}$

$= \frac{1}{2}\left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}}\right]_5^3 + \frac{9}{2}\int_0^1 \frac{dx}{\sqrt{(\frac{\sqrt{21}}{2})^2 - (x+\frac{1}{2})^2}}$

$\therefore LHS = \sqrt{3} - \sqrt{5} + \frac{9}{2}\left[\sin^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{21}}{2}}\right)\right]_0^1$

$= \sqrt{3} - \sqrt{5} + \frac{9}{2}\left[\sin^{-1}\left(\frac{2x+1}{\sqrt{21}}\right)\right]_0^1$

$= \sqrt{3} - \sqrt{5} + \frac{9}{2}\left[\sin^{-1}\frac{3}{\sqrt{21}} - \sin^{-1}\frac{1}{\sqrt{21}}\right.$

$= RHS$

(b) $LHS = \frac{d}{d\theta}\left(\cos\theta - \cos^3\theta + \frac{3}{5}\cos^5\theta - \frac{1}{7}\right.$

$= -\sin\theta - 3\cos^2\theta \cdot -\sin\theta$
$+ \frac{3}{5} \cdot 5\cos^4\theta \cdot -\sin\theta - \frac{1}{7} \cdot 7\cos^6\theta \cdot$

$= -\sin\theta + 3(1-\sin^2\theta)\sin\theta$
$+ 3(1-\sin^2\theta)^2 - \sin\theta + (1-\sin^2\theta)^3 \sin$

$= -\sin\theta + 3\sin\theta - 3\sin^3\theta$
$- 3\sin\theta(1 - 2\sin^2\theta + \sin^4\theta)$
$+ \sin\theta(1 - 3\sin^2\theta + 3\sin^4\theta - \sin$

$= 2\sin\theta - 3\sin^3\theta - 3\sin\theta + 6\sin^3$
$- 3\sin^5\theta + \sin\theta - 3\sin^3\theta + 3\sin^5$
$- \sin^7\theta$

$= -\sin^7\theta$

$= RHS$

Now $\int_0^{\pi} \sin^7\theta \, d\theta = -\left[\cos\theta - \cos^3\theta + \frac{3}{5}\cos^5\theta - \frac{1}{7}\cos^7\theta\right]_0^{\pi}$

$= -\left[\left((-1) + 1 - \frac{3}{5} + \frac{1}{7}\right) - \left[1 - \right.\right.$

$= -\left[-\frac{6}{5} + \frac{2}{7}\right]$

$= \frac{32}{35}$