SYDNEY GRAMMAR SCHOOL

2010 Half-Yearly Examination

FORM VI
MATHEMATICS EXTENSION 1

Tuesday 16th March 2010

General Instructions
- Writing time — 2 Hours
- Write using black or blue pen.
- Board-approved calculators may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Structure of the paper
- Total marks — 84
- All seven questions may be attempted.
- All seven questions are of equal value.

Collection
- Write your candidate number clearly on each booklet.
- Hand in the seven questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place the question paper insisde your answer booklet for Question 1.

Checklist
- SGS booklets — 7 per boy
- Candidature — 133 boys

Examiner
RCF

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\quad x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\quad x > 0$

QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) Write down the exact value of:

(i) $\tan \frac{3\pi}{4}$ $\boxed{1}$

(ii) $2\log_e e^3$ $\boxed{1}$

(iii) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ $\boxed{1}$

(b) Express $160°$ in radian measure. Give your answer in terms of π. $\boxed{1}$

(c) Differentiate with respect to x:

(i) $\cos 4x$ $\boxed{1}$

(ii) $\log_e(2x+1)$ $\boxed{1}$

(iii) $e^x \tan x$ $\boxed{1}$

(iv) $\tan^{-1}(x^2)$ $\boxed{2}$

(d) Find:

(i) $\int \sin(4+2x)\,dx$ $\boxed{1}$

(ii) $\int e^{4x+1}\,dx$ $\boxed{1}$

(iii) $\int \frac{1}{\sqrt{4-x^2}}\,dx$ $\boxed{1}$

Exam continues next page ...

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) Solve $\dfrac{x}{x-1} \geq 3$. $\boxed{3}$

(b) Find, to the nearest minute, the acute angle between the lines $y = -\dfrac{x}{3} + 4$ and $y = x + 1$. $\boxed{2}$

(c) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the curve $x^2 = 4ay$.

(i) Show that the chord PQ has gradient $\frac{1}{2}(p+q)$. $\boxed{1}$

(ii) Hence find the equation of the chord PQ. $\boxed{2}$

(iii) If PQ is a focal chord, show that $p = -\dfrac{1}{q}$. $\boxed{1}$

(d) Suppose that α is obtuse and $\sin\alpha = \frac{\sqrt{7}}{3}$.
Find the exact value of:

(i) $\cos\alpha$ $\boxed{2}$

(ii) $\sin 2\alpha$ $\boxed{1}$

Exam continues overleaf ...

QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

(a) Find $\int \sin^4 x \cos x \, dx$. $\boxed{2}$

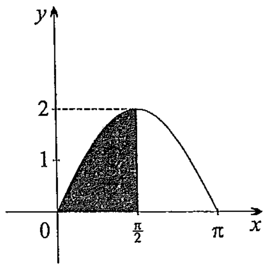
(b) (i) Sketch the graph of $f(x) = 2e^{-x}$. $\boxed{1}$

(ii) On the same number plane, sketch the graph of $f^{-1}(x)$. $\boxed{1}$

(iii) Find a simplified expression for $f^{-1}(x)$. $\boxed{2}$

(c) (i) Evaluate $\int_0^{\frac{\pi}{2}} \sin^2 x \, dx$. $\boxed{3}$

(ii)



The diagram above shows the region bounded by the curve $y = 2\sin x$, the x-axis and the line $x = \frac{\pi}{2}$. Using your answer from part (i), find the exact volume of the solid generated when the shaded region is rotated about the x-axis. $\boxed{1}$

(d) Sketch the graph of $f(x) = 5\cos^{-1}(x+1)$, indicating clearly the coordinates of the endpoints and any intercepts with the axes. $\boxed{2}$

Exam continues next page ...

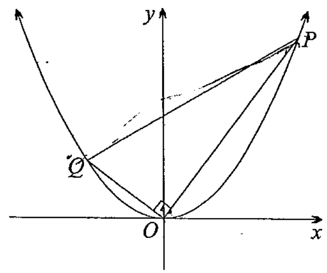
QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

(a) Find:

(i) $\int \frac{x^2+1}{x} \, dx$ $\boxed{1}$

(ii) $\frac{d}{dx}\left(\frac{-1}{\sqrt{1-x^2}}\right)$ $\boxed{2}$

(b)



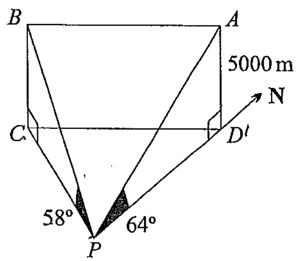
Two points $P(8p, 4p^2)$ and $Q(8q, 4q^2)$ lie on the parabola $x^2 = 16y$. The chord PQ subtends a right angle at the vertex of the parabola.

(i) Use gradients to show that $pq = -4$. $\boxed{1}$

(ii) Find the coordinates of the midpoint M of the chord PQ. $\boxed{1}$

(iii) Hence find the Cartesian equation of the locus of M. $\boxed{3}$

(c)



A helicopter's flightpath takes it from point A to point B at a constant altitude of 5000 metres. An observer at P on the ground follows its progress. Point A is due north of the observer at an angle of elevation of 64°. Two minutes later the helicopter reaches B which is on a bearing of 300°T from P and at an angle of elevation of 58°.

(i) Write down the size of $\angle CPD$, giving a clear explanation of your answer. $\boxed{1}$

(ii) Find the speed of the plane in metres per second, correct to one decimal place. $\boxed{3}$

Exam continues overleaf ...

QUESTION FIVE (12 marks) Use a separate writing booklet. **Marks**

(a) Find the exact value of $\sin\left(\tan^{-1}\left(-\frac{\sqrt{5}}{3}\right)\right)$. $\boxed{2}$

(b) (i) Find $\dfrac{d}{dx}(x\tan^{-1}x)$. $\boxed{1}$

(ii) Hence, or otherwise, find $\displaystyle\int \tan^{-1}x\,dx$. $\boxed{2}$

(c) By setting $t = \tan 22\frac{1}{2}^\circ$, use a t-formula to calculate the exact value of $\tan 22\frac{1}{2}^\circ$. $\boxed{3}$

(d) (i) Express $3\cos x + 4\sin x$ in the form $R\cos(x-\alpha)$ where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. $\boxed{2}$

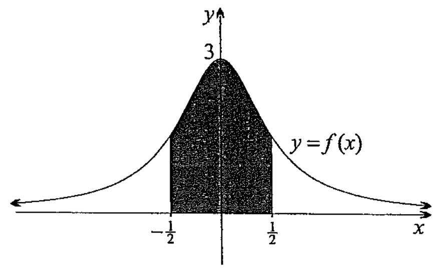
(ii) Hence, or otherwise, solve the equation $3\cos x + 4\sin x = \frac{5}{2}$, where $-\pi \leq x \leq \pi$. Give your solutions correct to four decimal places. $\boxed{2}$

Exam continues next page ...

QUESTION SIX (12 marks) Use a separate writing booklet. **Marks**

(a) Write down the general solution of the equation $\cos x = -\frac{\sqrt{3}}{2}$ in radian measure. $\boxed{1}$

(b)



The diagram above shows the graph of $f(x) = \dfrac{3}{1+4x^2}$. $\boxed{3}$

Calculate the area bounded by the curve, the x-axis and the lines $x = -\frac{1}{2}$ and $x = \frac{1}{2}$.

(c) Use mathematical induction to prove that $\boxed{4}$

$$2 \times 1! + 5 \times 2! + 10 \times 3! + \ldots + (n^2+1)n! = n(n+1)!$$

for all positive integers n, where $n! = 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$.

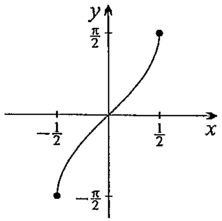
(d) (i) Show $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. $\boxed{2}$

(ii) Hence solve the equation $\cos 3\theta = -\cos\theta$ where $0 \leq \theta \leq 2\pi$. $\boxed{2}$

Exam continues overleaf ...

QUESTION SEVEN (12 marks) Use a separate writing booklet. **Marks**

(a)



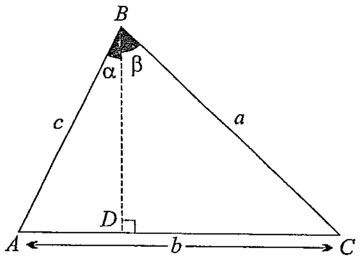
The diagram above shows the graph of $y = \sin^{-1} 2x$. Find the area of the region bounded by the curve, the x-axis and the line $x = \frac{1}{2}$. **3**

(b) (i) Find the constants A and B such that **2**

$$A(\sin x + 2\cos x) + B(\cos x - 2\sin x) \equiv \sin x + 12\cos x.$$

(ii) Hence find $\displaystyle\int \frac{\sin x + 12\cos x}{\sin x + 2\cos x}\, dx$. **2**

(c)



The diagram above shows $\triangle ABC$ with altitude BD. Let $\angle ABD = \alpha$ and $\angle CBD = \beta$.

(i) Show that $a\cos\beta = c\cos\alpha$. **1**

(ii) Show that $b = c\sin\alpha + a\sin\beta$. **1**

(iii) Use the cosine rule to derive the compound angle formula for $\cos(\alpha + \beta)$. **3**

END OF EXAMINATION

2010 Extension One Half yearly solutions

Qu 1

a) (i) $\tan\frac{3\pi}{4} = -\tan\frac{\pi}{4} = (-1)$ ✓

(ii) $2\log_e e^3 = 2\times 3 = 6$ ✓

(iii) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ ✓

b) $160 \times \frac{\pi}{180} = \frac{8\pi}{9}$ radians ✓

c) (i) $y = \cos 4x$

$\frac{dy}{dx} = -4\sin 4x$ ✓

(ii) $y = \log_e(2x+1)$

$\frac{dy}{dx} = \frac{2}{2x+1}$ ✓

(iii) $y = e^x \tan x$

$\frac{dy}{dx} = e^x \sec^2 x + e^x \tan x$
$= e^x(\sec^2 x + \tan x)$ ✓

(iv) $y = \tan^{-1}(x^2)$

$\frac{dy}{dx} = \frac{1}{1+(x^2)^2} \times 2x$

$= \frac{2x}{1+x^4}$ ✓ Numerator ✓ Denominator

d) (i) $\int \sin(4+2x)\,dx$

$= \frac{-\cos(4+2x)}{2} + c$ ✓

(ii) $\int e^{4x+1}\,dx$

$= \frac{1}{4}e^{4x+1} + c$ ✓

(iii) $\int \frac{1}{\sqrt{4-x^2}}\,dx$

$= \sin^{-1}\frac{x}{2} + c$ ✓

Question 2

a) $\frac{x}{x-1} \geq 3 \quad (x \neq 1)$

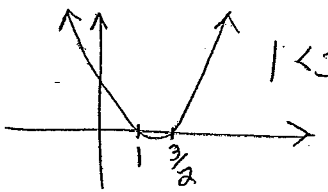
$\times (x-1)^2$

$x(x-1) \geq 3(x-1)^2$ ✓

$x^2 - x \geq 3x^2 - 6x + 3$

$0 \geq 2x^2 - 5x + 3$

$0 \geq (2x-3)(x-1)$ ✓



$1 < x \leq \frac{3}{2}$ ✓

($1 \leq x \leq \frac{3}{2}$ worth 2/3 mks)

b) $y = -\frac{x}{3} + 4 \quad m_1 = \left(-\frac{1}{3}\right)$

$y = x + 1 \quad m_2 = 1$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{-\frac{1}{3} - 1}{1 - \frac{1}{3}}\right|$ ✓

$= \left|\frac{-\frac{4}{3}}{\frac{2}{3}}\right|$

$= 2$

$\theta \doteqdot 63°26'$ ✓ (Do not penalise rounding)

2 c) (i) $m = \frac{ap^2 - aq^2}{2ap - 2aq}$

$= \frac{a(p+q)(p-q)}{2a(p-q)}$

$= \frac{p+q}{2}$ "SHOW" ✓

(ii) $y - y_1 = m(x - x_1)$ ✓

$y - ap^2 = \left(\frac{p+q}{2}\right)(x - 2ap)$ ✓

$y = \left(\frac{p+q}{2}\right)x - ap^2 - apq + ap^2$

$y = \left(\frac{p+q}{2}\right)x - apq$ ✓

(iii) If a focal chord passes through $S(0, a)$

$\therefore a = \left(\frac{p+q}{2}\right) \times 0 - apq$

$\therefore a = -apq$

$(-1) = pq$

$\left(-\frac{1}{q}\right) = p$

✓ "SHOW"

d) $\sin\alpha = \frac{\sqrt{7}}{3}$ α obtuse ie 2^{nd} quad

$\sqrt{7}$ 3 x

$x^2 = 9 - 7 = 2$

$x = \sqrt{2}$ ✓

(i) $\cos\alpha = \left(-\frac{\sqrt{2}}{3}\right)$ ✓

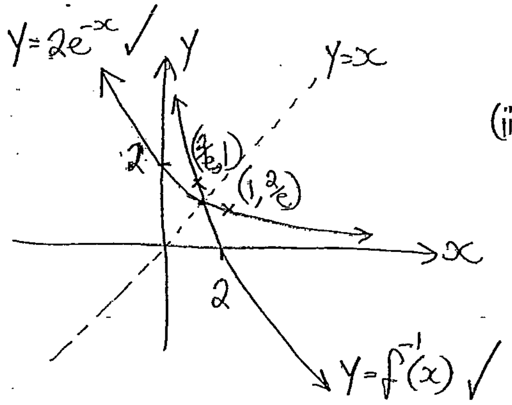
(ii) $\sin 2\alpha = 2\sin\alpha\cos\alpha = 2 \times \frac{\sqrt{7}}{3} \times \left(-\frac{\sqrt{2}}{3}\right)$

$= \left(\frac{-2\sqrt{14}}{9}\right)$ ✓

Qu 3

a) $\int \sin^4 x \cos x\,dx = \frac{1}{5}\sin^5 x + c$ ✓✓ $\sin^5 x$, co-eff (Do not penalise arbitrary constant)

b)



(iii) $x = 2e^{-y}$ ✓

$\frac{x}{2} = e^{-y}$

$\frac{x}{2} = \frac{1}{e^y}$

$e^y = \frac{2}{x}$

$y = \log_e\left(\frac{2}{x}\right)$ ✓ $x > 0$

c) (i) $\int_0^{\pi/2} \sin^2 x\,dx$

$= \int_0^{\pi/2} \frac{1}{2} - \frac{1}{2}\cos 2x\ dx$ ✓

$= \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\pi/2}$ ✓

$= \left(\frac{\pi}{4} - \frac{1}{4}\sin\pi\right) - 0$

$= \frac{\pi}{4}$ ✓

(ii) $V = \pi\int_0^{\pi/2} y^2\,dx$

$= \pi\int_0^{\pi/2} (2\sin x)^2\,dx$

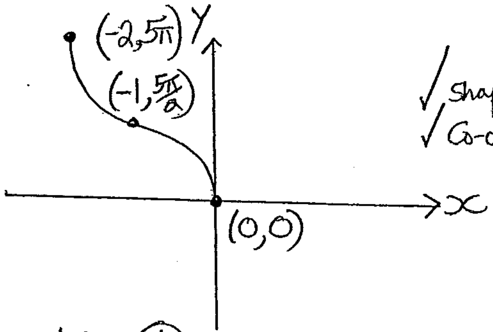
$= 4\pi \times \int_0^{\pi/2} \sin^2 x\,dx$

$= 4\pi \times \frac{\pi}{4}$ ✓

$= \pi^2$ units cubed.

(Award mark for $4\pi \times$ (i) ans) - NSE

d)



✓ Shape
✓ Co-ordinates of endpts

Question 4

a)(i) $\int \frac{x^2+1}{x}\,dx = \int x + \frac{1}{x}\,dx$

$= \frac{x^2}{2} + \ln x + c$ ✓

(ii) $\frac{d}{dx}\left(-\frac{1}{\sqrt{1-x^2}}\right) = \frac{d}{dx}\left(-(1-x^2)^{-\frac{1}{2}}\right)$

$= \frac{1}{2}(1-x^2)^{-\frac{3}{2}} \times (-2x)$ ✓

$= \frac{-x}{\sqrt{(1-x^2)^3}}$ ✓

b.) (i) Vertex @ origin

$\left.\begin{array}{l} m_{OP} \times m_{OQ} = -1 \\ \frac{4p^2}{8p} \times \frac{4q^2}{8q} = -1 \end{array}\right\}$ ✓

$\frac{p}{2} \times \frac{q}{2} = -1$

$pq = -4$

(ii) $M\left(\frac{8p+8q}{2}, \frac{4p^2+4q^2}{2}\right) = \left(4(p+q), 2(p^2+q^2)\right)$ ✓

(iii) $x = 4(p+q)$ $\quad y = 2(p^2+q^2)$ ✓

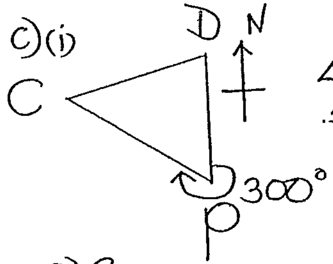
$\therefore \frac{x}{4} = p+q$ $\quad \frac{y}{2} = (p+q)^2 - 2pq$ ✓

$\frac{y}{2} = (p+q)^2 + 8$ ✓

$\therefore \frac{y}{2} = \left(\frac{x}{4}\right)^2 + 8$

$y = \frac{x^2}{8} + 16.$ ✓

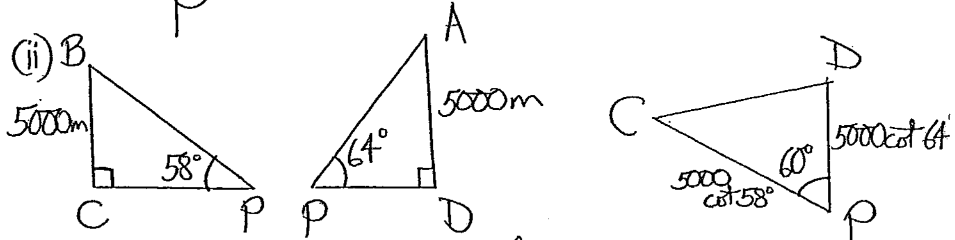
c)(i)



$\angle CPD$ (Reflex) $= 300$ (Given bearing)

$\therefore \angle CPD = 60°$ (Angles in full revolution) ✓

(ii)



$\tan 58° = \frac{5000}{PC}$ $\quad \tan 64° = \frac{5000}{PD}$

$\therefore PC = 5000\cot 58°$ $\quad PD = 5000\cot 64°$ ✓

using cosine rule in $\triangle CDP$

$CD^2 = 5000^2(\cot^2 58 + \cot^2 64) - 2 \times 5000\cot 58 \times 5000\cot 64 \times \cos 60$

$CD^2 = 5000^2\left[\cot^2 58 + \cot^2 64 - \cot 58\cot 64\right]$ ✓

$CD^2 \doteqdot 8\,089\,390.96\ldots$

$CD \doteqdot 2844$ m.

Speed $= \frac{\text{distance}}{\text{time}}$

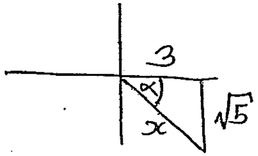
$= \frac{2844}{120}$

$\doteqdot 23.7$ m/s ✓

Question (5)

a) $\sin\left(\tan^{-1}\left(-\frac{\sqrt5}{3}\right)\right)$

$= -\frac{\sqrt5}{\sqrt{14}} = \left(-\sqrt{\frac{5}{14}}\right)$ ✓



$3^2+(\sqrt5)^2 = x^2$

$x = \sqrt{14}$ ✓

b)

(i) $\frac{d}{dx}(x\tan^{-1}x)$

$= x\cdot\frac{1}{1+x^2} + \tan^{-1}x$

$= \frac{x}{1+x^2} + \tan^{-1}x$ ✓

(ii) $\int \frac{x}{1+x^2} + \tan^{-1} dx = x\tan^{-1}x$

$\therefore \int \tan^{-1} dx = x\tan^{-1}x - \int\frac{x}{1+x^2}dx$ ✓

$= x\tan^{-1}x - \frac{1}{2}\ln(1+x^2) + C$ ✓

c) $\tan 45° = 1$ let $t = \tan\frac{45°}{2} = \tan 22\frac{1}{2}°$

$\tan 45° = \frac{2t}{1-t^2}$

$\therefore 1 = \frac{2t}{1-t^2}$ ✓ $\therefore$ $1-t^2 = 2t$

$t^2+2t-1 = 0$

$t = \frac{-2\pm\sqrt8}{2} = -1\pm\sqrt2$ ✓

(since $\Delta = b^2-4ac = (2)^2-4\times1\times(-1) = 8$.)

but $\tan 22\frac{1}{2}°$ (in first quadrant) is positive hence positive root.

$\therefore \tan 22\frac{1}{2} = \sqrt2 - 1$ ✓

d) (i) $3\cos x + 4\sin x \equiv R\cos(x-\alpha)$

$= R\cos x\cos\alpha + R\sin x\sin\alpha$

Equating coeffs $\sin x$: $4 = R\sin\alpha$ ①

$\cos x$: $3 = R\cos\alpha$ ②

① ÷ ②: $\tan\alpha = \frac{4}{3}$

$\alpha = \tan^{-1}\left(\frac{4}{3}\right)$ ✓

①² + ②²: $16+9 = R^2(\sin^2\alpha+\cos^2\alpha)$

$25 = R^2$

$5 = R$ $(R>0)$ ✓

(ii) $3\cos x + 4\sin x = \frac{5}{2}$

$5\cos(x-\alpha) = \frac{5}{2}$ $-\pi \le x \le \pi$

$\cos(x-\alpha) = \frac{1}{2}$ $-\pi-\alpha \le x-\alpha \le \pi-\alpha$ ✓

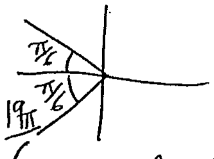
$(x-\alpha) = -\frac{\pi}{3}$ or $\frac{\pi}{3}$ ✓

$x = -\frac{\pi}{3}+\alpha$ or $\frac{\pi}{3}+\alpha$

(since $\alpha \doteqdot 0.927295\ldots$) $x \doteqdot -0.1199$ or 1.9745 ✓

Question (6)

a) $\cos x = -\frac{\sqrt3}{2}$



$x = \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{17\pi}{6}, \frac{19\pi}{6} \ldots$ or $x = 2n\pi \pm \frac{5\pi}{6}$ ✓

$-\frac{5\pi}{6}, -\frac{7\pi}{6}, -\frac{17\pi}{6} \ldots$ where $n \in \mathbb{Z}$

b) $A = \int_{-\frac12}^{\frac12} \frac{3}{1+4x^2}dx = 2\int_0^{\frac12}\frac{3}{1+4x^2}dx$ (by even symmetry)

$= \frac{6}{4}\int_0^{\frac12}\frac{1}{\frac14+x^2}dx$ ✓

$= \frac{3}{2}\left[\frac{1}{(\frac12)}\tan^{-1}\frac{x}{(\frac12)}\right]_0^{\frac12}$ ✓

$= \left[3\tan^{-1}2x\right]_0^{\frac12}$

$= 3(\tan^{-1}1 - \tan^{-1}0)$

$= \frac{3\pi}{4}u^2$ ✓

(6c) Prove $2\times1!+5\times2!+10\times3!+\dots+(n^2+1)n! = n(n+1)!$

Step A : Show for $n=1$.

LHS $= 2\times1! = 2$ RHS $= 1\times2! = 2$ ∴ Conjecture is true for $n=1$. ✓

Step B : Assume true for $n=k$ ie

$$2\times1!+5\times2!+\dots+(k^2+1)k! = k(k+1)!$$ ✓

Result to prove true for $n=k+1$ ie

$$2\times1!+5\times2!+\dots+(k^2+1)k!+[(k+1)^2+1](k+1)! = (k+1)(k+2)$$

LHS $= k(k+1)!+[k^2+2k+2](k+1)!$ Using assumption ✓

$= (k+1)![k+k^2+2k+2]$

$= (k+1)![k^2+3k+2]$

$= (k+1)!(k+2)(k+1)$

$= (k+2)!\times(k+1)$

$=$ RHS ✓ If conjecture true for $n=k$, also true for $n=k+1$.

Step C By the principle of mathematical induction the conjecture is proven true for all positive integer n.

d) (i) $\cos 3\theta = \cos(\theta+2\theta)$ ✓

$= \cos\theta\cos 2\theta - \sin\theta\sin 2\theta$ ✓

$= \cos\theta(2\cos^2\theta-1) - \sin\theta(2\sin\theta\cos\theta)$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta(1-\cos^2\theta)$ ✓SHOW

$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$

(6) d) (ii) $\cos 3\theta = -\cos\theta$

$\cos 3\theta + \cos\theta = 0$

$4\cos^3\theta - 2\cos\theta = 0$

$\cos\theta(4\cos^2\theta - 2) = 0$

$2\cos\theta(2\cos^2\theta - 1) = 0$ ✓

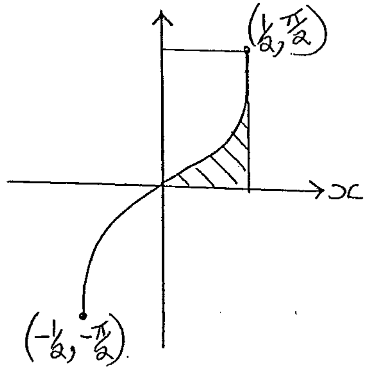
$\cos\theta = 0$ OR $\cos^2\theta = \frac{1}{2}$

$\cos\theta = \pm\frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ OR $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$ ✓

Question 7

a)



Shaded area req'd. − By subtraction

Area of rectangle $= \frac{1}{2}\times\frac{\pi}{2}$

$= \frac{\pi}{4}\ u^2$

Area under curve to y axis

$\int_{y=0}^{\frac{\pi}{2}} x\,dy = \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin y\,dy$ ✓

$= \left[-\frac{1}{2}\cos y\right]_0^{\frac{\pi}{2}}$

$= -\frac{1}{2}(\cos\frac{\pi}{2} - \cos 0)$

$= \frac{1}{2}\ u^2$ ✓

Area of required region

$= \left(\frac{\pi}{4} - \frac{1}{2}\right) u^2$ ✓

b) (i) $A\sin x + 2A\cos x + B\cos x - 2B\sin x \equiv \sin x + 12\cos x$

Equating Coeffs. $A - 2B = 1$ ①
$2A + B = 12$ ② ✓

①$+2\times$② $5A = 25$

$A = 5$

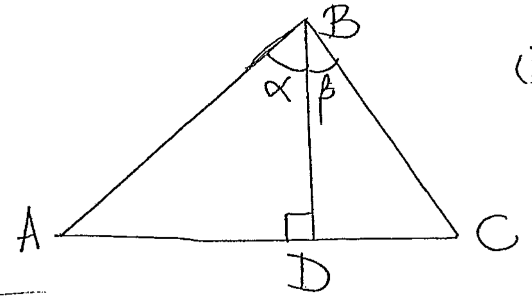
$B = 2$ ✓

(ii) $\displaystyle\int \frac{\sin x + 12\cos x}{\sin x + 2\cos x}\,dx = \int \frac{5(\sin x + 2\cos x)}{\sin x + 2\cos x} + \frac{2(\cos x - 2\sin x)}{\sin x + 2\cos x}\,dx$

$\displaystyle = \int 5 + \frac{2(\cos x - 2\sin x)}{\sin x + 2\cos x}\,dx$

$= 5x + 2\ln(\sin x + 2\cos x) + c$

b)



(i) In $\triangle ABD$ $\cos\alpha = \frac{BD}{AB}$

$BD = c\cos\alpha$

In $\triangle BCD$ $\cos\beta = \frac{BD}{BC}$

$BD = a\cos\beta$

$\therefore c\cos\alpha = a\cos\beta$

(ii) In $\triangle ABD$ $\sin\alpha = \frac{AD}{AB}$

In $\triangle BCD$ $\sin\beta = \frac{CD}{BC}$

$\therefore AD = c\sin\alpha$

$CD = a\sin\beta$

$AC = AD + DC$

$b = a\sin\beta + c\sin\alpha$

(iii) In $\triangle ABC$ $b^2 = a^2 + c^2 - 2ac\cos(\alpha+\beta)$

$$\therefore \cos(\alpha+\beta) = \frac{a^2+c^2-b^2}{2ac}$$

$$= \frac{a^2+c^2-(a\sin\beta + c\sin\alpha)^2}{2ac}$$

$$= \frac{a^2+c^2-a^2\sin^2\beta - 2ac\sin\alpha\sin\beta - c^2\sin^2\alpha}{2ac}$$

$$= \frac{a^2(1-\sin^2\beta) + c^2(1-\sin^2\alpha) - 2ac\sin\alpha\sin\beta}{2ac}$$

$$= \frac{a^2\cos^2\beta + c^2\cos^2\alpha - 2ac\sin\alpha\sin\beta}{2ac}$$

$$= \frac{a\cos\beta\, c\cos\alpha + a\cos\beta\, c\cos\alpha - 2ac\sin\alpha\sin\beta}{2ac}$$

$$= \frac{2ac(\cos\beta\cos\alpha - \sin\alpha\sin\beta)}{2ac}$$

$= \cos\alpha\cos\beta - \sin\alpha\sin\beta.$