SYDNEY GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT
HALF-YEARLY EXAMINATIONS 2008

FORM IV

MATHEMATICS

Examination date

Friday 16th May 2008

Time allowed

Two hours

Instructions

All eight questions may be attempted.
All eight questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.

Collection

Write your name, class and master clearly on the front.
Hand in all the writing paper in a single well-stapled bundle.
Keep the printed examination paper and bring it to your next Mathematics lesson.

4A: REP	4B: LYL	4C: JMR	4D: MLS
4E: RCF	4F: MW	4G: KWM	4H: SJE
4I: AMD	4J: GJ		

Checklist

Writing paper required.
Candidature: 188 boys.

Examiner

MLS

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

QUESTION ONE Start a new page.

(a) What is the radius of the circle with equation $x^2 + y^2 = 16$?

(b) What is the surface area of a cube with edge length 5 cm?

(c) Solve $2x(x-4) = 0$.

(d) Find the midpoint of the interval joining $A(2,8)$ and $B(4,4)$.

(e) Write down the gradient of a line parallel to the line $5x - 2y - 3 = 0$.

(f) Write 9% p.a. as a monthly rate.

(g) Find the simple interest earned if \$5000 is invested for 5 years at 6·5% p.a.

(h) Fully factorise $8x^2 - 4x$.

(i) Expand and simplify $(2x-3)^2$.

(j) What is the probability of rolling a five on a six-sided die?

(k) Find the area of a rhombus with diagonal lengths 16 cm and 10 cm.

(l) Write down the equation of the vertical line passing through the point $(5, -3)$.

QUESTION TWO Start a new page.

(a) Find the equation of the line with gradient $\frac{1}{3}$ and y-intercept 2.

(b) (i) Solve $x^2 - 2x - 15 = 0$ by factoring.

(ii) Solve $x^2 - x - 15 = 0$ by the quadratic formula.

(c) (i) Sketch $x + y = 4$. Show clearly the intercepts with the axes.

(ii) Sketch the parabola $y = x^2 + 4$.

(iii) Sketch the circle $(x-3)^2 + y^2 = 9$. Show clearly the intercepts with the axes.

Exam continues next page ...

QUESTION THREE Start a new page.

(a) Let $P = (-2, 1)$ and $Q = (6, 5)$.

(i) Find the gradient of PQ.

(ii) A line ℓ is perpendicular to PQ and passes through Q. Find its equation.

(iii) Hence find the y-intercept of ℓ.

(b) A parabola has the equation $y = x^2 - 2x - 3$.

(i) Find the y-intercept.

(ii) Find the x-intercepts.

(iii) Find the equation of the axis of symmetry.

(iv) Find the coordinates of the vertex.

(v) Sketch a neat graph of $y = x^2 - 2x - 3$ showing all the above information.

QUESTION FOUR Start a new page.

(a) Solve $x^2 - 10x + 10 = 0$ by completing the square. Give your answer in exact form.

(b) Find the total value of an investment of \$3000 earning 18% per annum compounded monthly for two years. Give your answer correct to the nearest cent.

(c) Draw a tree diagram to show the possible outcomes when three coins are tossed. Use this to determine the probability of the following outcomes:

(i) 3 heads,

(ii) 2 heads and 1 tail,

(iii) at least two heads.

(d) Consider the function $y = 2^x$.

(i) Copy and complete the table

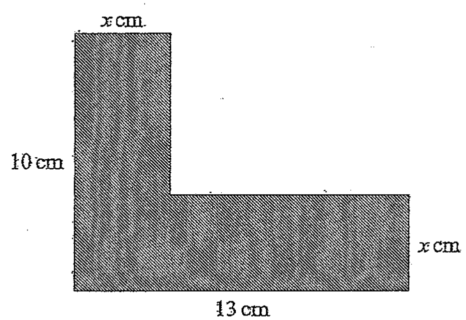
x	-2	-1	0	1	2	3
y						

(ii) Sketch the graph of $y = 2^x$. Show clearly where the graph cuts a coordinate axis and mark the coodinates of any other point on the graph.

Exam continues overleaf ...

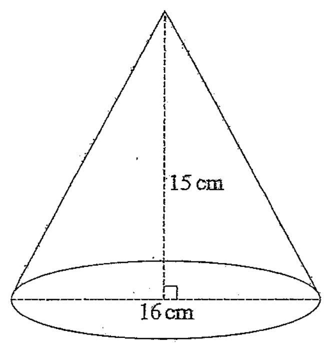
QUESTION FIVE Start a new page.

(a)



The area of the above figure is $51{\cdot}25\text{ cm}^2$. By forming a quadratic equation, find the value of x.

(b)



The cone shown above has a diameter of 16 cm and a height of 15 cm.

(i) Calculate the slant height of this cone.

(ii) Calculate the exact area of the curved surface of this cone.

(iii) Calculate the exact volume of this cone.

(iv) I create a sphere with the same volume as this cone. What is the radius of this sphere? Give your answer correct to two decimal places.

(c) The present value of a company's asset is \$350 000. If it has been depreciating at 17·5% per annum for the past six years, what was the original value of the asset? Give your answer to the nearest dollar.

Exam continues next page ...

QUESTION SIX Start a new page.

(a) A square pyramid of base length x centimetres has a volume of 110·25 cubic centimetres and a vertical height of 12 centimetres. Find x in simplest form.

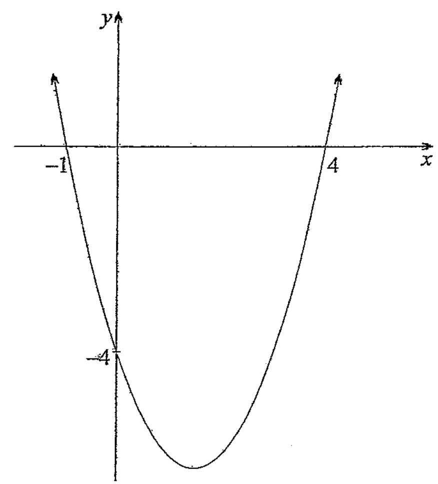
(b) (i) Factorise fully $x^3 + 6x^2 - 4x - 24$.

(ii) The graph of $y = x^3 + 6x^2 - 4x - 24$ is a cubic curve. Find the coordinates of the points where:

(α) it cuts the y-axis,

(β) it cuts the x-axis.

(c)



The equation of the parabola graphed above is $y = x^2 + bx + c$.

(i) Find the values of b and c.

(ii) Find the coordinates of the vertex.

Exam continues overleaf ...

QUESTION SEVEN Start a new page.

(a) Find the centre and radius of the circle $x^2+y^2+4x-2y+1=0$.

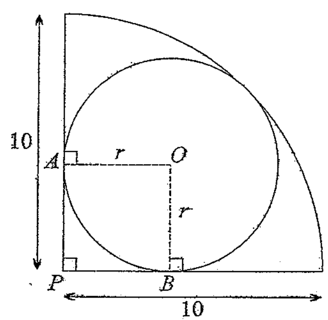
(b) The hypotenuse of a right-angled triangle is 25 cm and the sum of the other two sides is 31 cm. Let x be the length in centimetres of one of the other two sides.

(i) Show that $x+\sqrt{25^2-x^2}=31$.

(ii) Find the length of the shortest side.

(c) Fifty two tagged fish were released into a dam known to contain fish. Later a sample of thirty fish was netted from this dam, of which eight were found to be tagged. Estimate the total number of fish in the dam just before the sample of thirty were removed.

(d)



The figure above represents a circle of radius r units inscribed in a quadrant of a circle of radius 10 units. You may assume that $\angle OAP = \angle OBP = 90°$. Calculate the exact value of r.

Exam continues next page ...

QUESTION EIGHT Start a new page.

(a) (i) Write $\dfrac{1}{k}+\dfrac{1}{k+1}$ as one fraction.

(ii) Hence or otherwise simplify

$$\frac{5}{2\times 3}-\frac{7}{3\times 4}+\frac{9}{4\times 5}-\frac{11}{5\times 6}+\cdots+\frac{4n+1}{2n(2n+1)}-\frac{4n+3}{(2n+1)(2n+2)}$$

(b) If $\dfrac{a}{b}=\dfrac{c}{d}$, prove that $\dfrac{a-b}{a+b}=\dfrac{c-d}{c+d}$.

(c) The points P, Q and T are collinear, where $P=(2p,p^2)$, $Q=(-\frac{2}{p},\frac{1}{p^2})$ and $T=(x,-1)$. Find x in terms of p. Give x in simplest form.

END OF EXAMINATION

Solutions Form IV AY 2008.

(96)

Q1

a) 4 ✓

b) $6\times5\times5\ cm^2 = 150\ cm^2$ ✓

c) $x = 0$ or 4 ✓

d) $\left(\frac{4+2}{2}, \frac{8+4}{2}\right) = (3, 6)$ ✓

e) $2y = 5x - 3$

$y = \frac{5}{2}x - \frac{3}{2}$ so gradient is $\frac{5}{2}$ or $2\frac{1}{2}$ ✓

f) $\frac{9}{12}$ or $\frac{3}{4}$ ✓ (either)

g) $I = 5000 \times \frac{6.5}{100} \times 5$

$= \$1625$ ✓

h) $8x^2 - 4x = 4x(2x-1)$ ✓

i) $(2x-3)^2 = 4x^2 - 12x + 9$ ✓

j) $\frac{1}{6}$ ✓

k) $A = \frac{1}{2}xy$

$= \frac{1}{2}\times16\times10\ cm^2$

$= 80\ cm^2$ ✓ (units count in Q5)

l) $x = 5$ ✓

Q2

(a) $y = \frac{1}{3}x + 2$ ✓

(b) (i) $x^2 - 2x - 15 = 0$

$(x+3)(x-5) = 0$ ✓✓

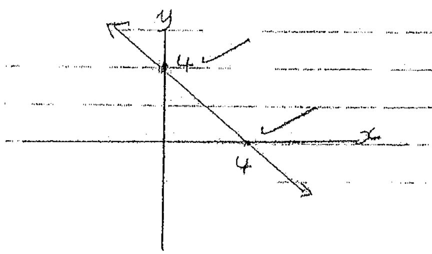
$x = -3$ or 5 ✓

(ii) $x^2 - x - 15 = 0$

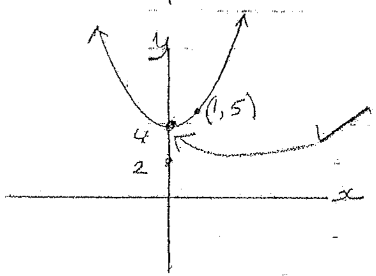
$x = \frac{1 \pm \sqrt{1+60}}{2}$ ✓

$= \frac{1+\sqrt{61}}{2}$ or $\frac{1-\sqrt{61}}{2}$ ✓

(c) (i)

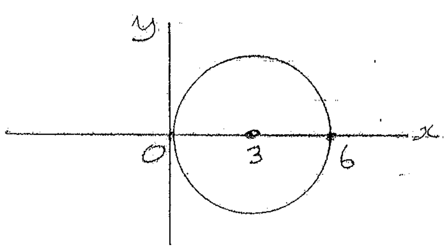


(ii)



✓ shape

(iii)



✓ for 0 and 6 intercepts

✓ centre (3, 0).

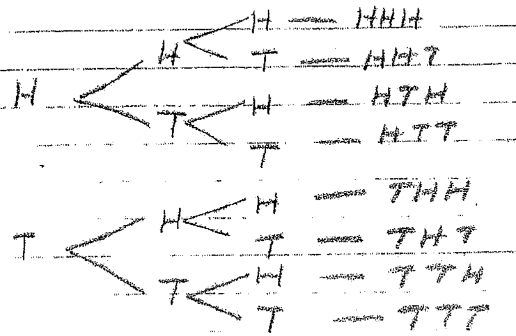
Q4.

(a)
$$x^2 - 10x + 10 = 0$$
$$x^2 - 10x + 25 = -10 + 25$$
$$(x-5)^2 = 15$$
$$x - 5 = \sqrt{15} \text{ or } -\sqrt{15}$$
$$x = 5+\sqrt{15} \text{ or } 5-\sqrt{15}$$

(b) Amount $= 3000\left(1 + \frac{18}{1200}\right)^{24}$ for 24; for 12

$= \$4288.51$

(c)



(i) $P(HHH) = \frac{1}{8}$

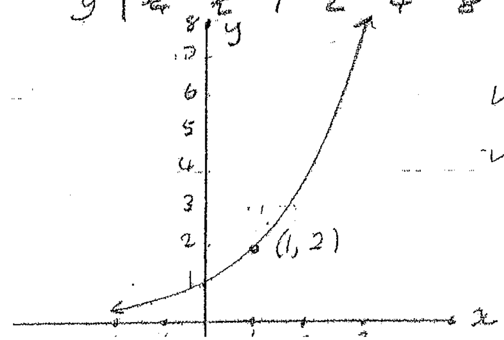
(ii) $P(2H, 1T) = \frac{3}{8}$

(iii) $P(\text{at least } 2H) = \frac{4}{8} = \frac{1}{2}$

(d)

x	-2	-1	0	1	2	3
y	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

$y = 2^x$



for general shape

for y int, $y=1$ and another point

Q3

a) (i) $m_{PQ} = \frac{5-1}{6+2}$
$$= \frac{4}{8} = \frac{1}{2}$$

(ii) $m = -2$, $Q = (6, 5)$
$$y - y_1 = m(x - x_1)$$
$$y - 5 = -2(x-6) \quad \text{for } m=-2$$
$$y = -2x + 17 \quad \text{or} \quad 2x + y - 17 = 0$$

(iii) $(0, 17)$ or just 17

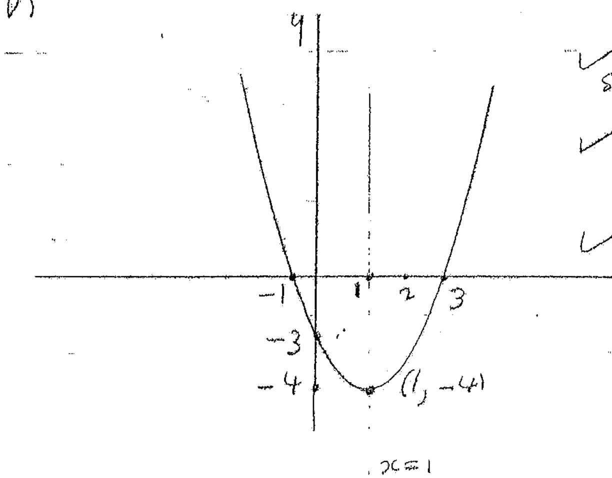
b) (i) $(0, -3)$ or just -3

(ii) $y = x^2 - 2x - 3$
$$= (x-3)(x+1)$$
so x intercepts are 3 and -1 or $(3,0)$ and $(-1,0)$

(iii) $x = 1$

(iv) $x=1$, $y = 1-2-3 = -4$
vertex is $(1, -4)$

(v)



for concave up smooth parabola

for 3 intercepts marked

for vertex marked

$A = 350\,000 = P\left(1-\frac{17.5}{100}\right)^6$

$P = 350\,000 \div \left(1-\frac{17.5}{100}\right)^6$

$= \$1{,}110{,}054.63$

$\doteq \$1{,}110{,}055$

Q5.

(a) $A = 10x + x(13-x) = 51.25$

$10x + 13x - x^2 = 51.25$

$\therefore 23x - x^2 - 51.25 = 0$

$4x^2 - 92x + 205 = 0$

$(2x-41)(2x-5) = 0$

$x = \frac{5}{2}$ or $\frac{41}{2}$

$= 2\frac{1}{2}$ or $20\frac{1}{2}$

However, $x \neq 20\frac{1}{2}$ as the sides are only 10 and 13

So $x = 2\frac{1}{2}$ cm

(b) (i) $s^2 = 15^2 + 8^2$

$= 289$

$s = 17$ cm

(ii) $SA = \pi r s$

$= \pi \times 8 \times 17$

$= 136\pi$ cm^2

(iii) $V = \frac{1}{3}Ah$

$= \frac{1}{3} \times \pi \times 8^2 \times 15$

$= 320\pi$ cm^3

(iv) $V = 320\pi = \frac{4}{3}\pi r^3$

$320 = \frac{4}{3}r^3$

$r^3 = \frac{320 \times 3}{4}$

$= 240$

$r \doteq 6.21$ cm

Q2.

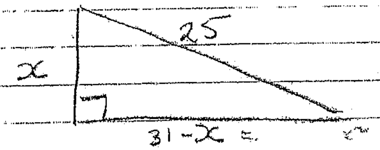
(a) $x^2+y^2+4x-2y+1=0$

$x^2+4x+4 \quad + y^2-2y+1 = 4$

$(x+2)^2+(y-1)^2 = 2^2$

centre is $(-2,1)$, radius is 2

(b)



(i) By Pythagoras, $x^2+(31-x)^2 = 25^2$

so $25^2-x^2 = (31-x)^2$

$\sqrt{25^2-x^2} = 31-x$

$x+\sqrt{25^2-x^2} = 31$

(ii) $(31-x)^2 = 25^2-x^2$

$961-62x+x^2 = 625-x^2$

$2x^2-62x+336 = 0$

$x^2-31x+168 = 0$

$(x-7)(x-24) = 0$

$x = 7$ or 24

So the shortest side is 7.

c) Let x be the total number of fish in the dam before the 30 are removed

Then P(getting a tagged fish) $= \frac{52}{x}$

So $\frac{52}{x} = \frac{8}{30}$

$8x = 30 \times 52$

$x = 195$

Q6.

(a) $V = 110{\cdot}25 = \frac{1}{3}Ah$

$\frac{1}{3} \times x^2 \times 12 = 110{\cdot}25$

$x^2 = 110{\cdot}25 \times \frac{3}{12}$

$= \frac{441}{16}$

$x = \frac{21}{4}$ or $5\frac{1}{4}$ cm

(b) (i) $x^3+6x^2-4x-24$

$= x^2(x+6)-4(x+6)$

$= (x^2-4)(x+6)$

$= (x-2)(x+2)(x+6)$

(ii) (α) on the y axis $x=0$, $y=-24$

$(0,-24)$

β on the x axis, $y=0$

$(x-2)(x+2)(x+6) = 0$

$x = 2, -2$ or -6

$(2,0)$ $(-2,0)$, $(-6,0)$

(c) (i) $c=-4$

at $(4,0)$, $0 = 4^2+4b-4$

$0 = 12+4b$

so $b=-3$

(ii) axis of symmetry is $x=1\frac{1}{2}$

at vertex $y=(1\frac{1}{2})^2-3(1\frac{1}{2})-4$

$= -6\frac{1}{4}$

vertex is $(1\frac{1}{2}, -6\frac{1}{4})$

Q8.

a) (i) $\frac{1}{k} + \frac{1}{k+1} = \frac{k+1+k}{k(k+1)}$

$= \frac{2k+1}{k(k+1)}$

(ii) $\frac{5}{2\times 3} - \frac{7}{3\times 4} + \frac{9}{4\times 5} - \frac{11}{5\times 6} + \dots + \frac{4n-1}{2n(2n+1)} - \frac{4n+3}{(2n+1)(2n+2)}$

$= \left(\frac{1}{2} + \cancel{\frac{1}{3}}\right) - \left(\cancel{\frac{1}{3}} + \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} + \cancel{\frac{1}{5}}\right) - \left(\cancel{\frac{1}{5}} + \cancel{\frac{1}{6}}\right) + \dots + \left(\cancel{\frac{1}{2n}} + \cancel{\frac{1}{2n+1}}\right)$

$- \left(\cancel{\frac{1}{(2n+1)}} + \frac{1}{(2n+2)}\right)$

$= \frac{1}{2} - \frac{1}{2n+2}$

$= \frac{n+1-1}{2(n+1)}$

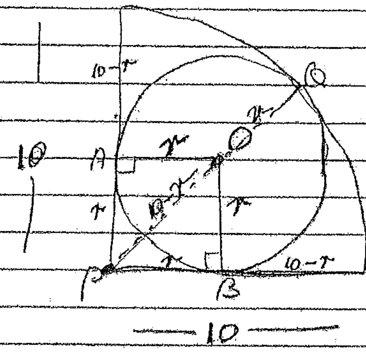
$= \frac{n}{2(n+1)}$

b) $\frac{a-b}{a+b} = \frac{\frac{a}{b}-1}{\frac{a}{b}+1}$

$= \frac{\frac{c}{d}-1}{\frac{c}{d}+1}$

$= \frac{c-d}{c+d}$

(d)



① $PO = \sqrt{2}\, r$

so $PQ = 10 = (\sqrt{2})r + r$

$r(\sqrt{2}+1) = 10$

$r = \frac{10}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}$

$= 10(\sqrt{2}-1)$.

② $PO^2 = r^2 + r^2 = 2r^2$, using Pythagoras

$= (10-r)^2$

So $(10-r)^2 = 2r^2$

$100 - 20r + r^2 = 2r^2$

$r^2 + 20r - 100 = 0$

$r = \frac{-20 \pm \sqrt{400+400}}{2}$

$= \frac{-20 \pm \sqrt{800}}{2}$

$= \frac{-20 \pm 20\sqrt{2}}{2}$

$= -10 + 10\sqrt{2}$ or $-10 - 10\sqrt{2}$

But $r > 0$, so $r = -10 + 10\sqrt{2}$

$= 10(\sqrt{2} - 1)$ units

C.

$P(2p, p^2)$ $\quad Q\left(-\frac{2}{p}, \frac{1}{p^2}\right)$ $\quad T(x, -1)$

Equating gradients ✓ for an attempt to do this

$$M_{PQ} = \frac{\frac{1}{p^2} - p^2}{-\frac{2}{p} - 2p} = \frac{1-p^4}{-2p-2p^3} = \frac{(1-p^2)(1+p^2)}{-2p(1+p^2)}$$

$$= \frac{(1-p)(1+p)}{-2p}$$

$$M_{QT} = \frac{\frac{1}{p^2}+1}{-\frac{2}{p}-x} = \frac{1+p^2}{-2p-p^2x} = \frac{1+p^2}{-p(2+px)}$$

So, $\frac{(1-p)(1+p)}{-2p} = \frac{1+p^2}{-p(2+px)}$ ✓

$$\frac{(1-p)(1+p)}{2} = \frac{1+p^2}{2+px}$$

$$2+px = \frac{2(1+p^2)}{(1+p)(1-p)}$$

✓ for a sensible attempt to find x

$$x = \left(\frac{2+2p^2}{1-p^2} - 2\right) \times \frac{1}{p}$$

$$= \frac{2+2p^2-2+2p^2}{p(1-p^2)}$$

$$= \frac{4p^2}{p(1-p^2)}$$

$$= \frac{4p}{1-p^2}$$ ✓