SYDNEY GRAMMAR SCHOOL
MATHEMATICS DEPARTMENT
YEARLY EXAMINATIONS 2007

FORM V

MATHEMATICS EXTENSION 1

Examination date

Thursday 18th October 2007

Time allowed

3 hours

Instructions

All nine questions may be attempted.
All nine questions are of equal value.
All necessary working must be shown.
Marks may not be awarded for careless or badly arranged work.
Approved calculators and templates may be used.
A list of standard integrals is provided at the end of the examination paper.

Collection

Write your name, class and master clearly on each booklet.
Hand in the nine questions in a single well-ordered pile.
Hand in a booklet for each question, even if it has not been attempted.
If you use a second booklet for a question, place it inside the first.
Keep the printed examination paper and bring it to your next Mathematics lesson.

5A: WMP	5B: PKH	5C: REP
5D: BDD	5E: FMW	5F: GJ
5G: JNC	5H: DS	5I: KWM

Checklist

Folded A3 booklets: 9 per boy. A total of 1500 booklets should be sufficient.
Candidature: 134 boys.

Examiner

JNC

QUESTION ONE (15 marks) Use a separate writing booklet.

(a) Factorise $x^3 - y^3$.

(b) Express $\dfrac{1}{\sqrt{5}+1}$ with a rational denominator.

(c) Consider the arithmetic series $175 + 168 + 161 + \cdots$.

(i) Find the 300th term.

(ii) Find the sum of the first 300 terms.

(d) Differentiate $x^3 - 3x$.

(e) Find a primitive of $7 - 2x$.

(f) Express 140° in radians as a multiple of π.

(g) Find the exact value $\tan \frac{3\pi}{4}$.

(h) Write down the period and amplitude of $y = \sin 2x$.

(i) Solve $\sqrt{2}\cos\theta = -1$, for $0 \le \theta \le 2\pi$.

Exam continues next page ...

QUESTION TWO (15 marks) Use a separate writing booklet.

(a) Differentiate with respect to x:

(i) $\sqrt{4-x^2}$

(ii) $\dfrac{2x}{1-x^2}$

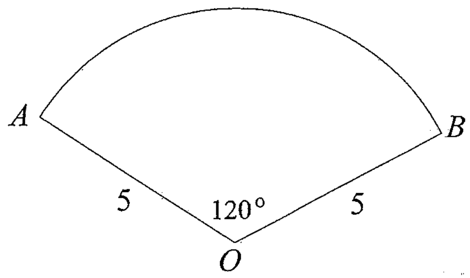
(b) Find a primitive function of each of the following:

(i) $\dfrac{3}{x^2}$

(ii) $x(2-x)$

(c) Evaluate $\displaystyle\int_{-1}^{3} 4x^3\,dx$.

(d)



In the diagram above, AB is an arc of a circle with centre O and radius 5 centimetres and $\angle AOB = 120°$.

(i) Find the length of the arc AB.

(ii) Find the area of the sector AOB

Exam continues overleaf ...

QUESTION THREE (15 marks) Use a separate writing booklet.

(a) Find a quadratic equation with roots $2+\sqrt{5}$ and $2-\sqrt{5}$. (Give your answer in the form $ax^2+bx+c=0$).

(b) For what values of k does the equation $2x^2-(k-3)x+8=0$ have equal roots?

(c) If α and β are the roots of the equation $2x^2+3x+5=0$, find:

(i) $\alpha+\beta$

(ii) $\alpha\beta$

(iii) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$

(iv) the quadratic equation with integer coefficients and with roots $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$

(d) Find the exact value of $\cos 75°$.

QUESTION FOUR (15 marks) Use a separate writing booklet.

(a) Find the equation of the curve passing through the point $(2,3)$ with gradient function

$$\frac{dy}{dx}=2x-1.$$

(b) Consider the function $f(x)=\dfrac{x^2+3}{x^2+1}$.

(i) Show that $f(x)$ is an even function.

(ii) Find any intercepts with the coordinate axes.

(iii) Show that the first derivative is $-\dfrac{4x}{(x^2+1)^2}$.

(iv) Find the co-ordinates of the stationary point.

(v) Given that $\dfrac{d^2y}{dx^2}=\dfrac{4(3x^2-1)}{(x^2+1)^3}$,

(α) determine the nature of the stationary point, and

(β) show that $(\frac{\sqrt{3}}{3},\frac{5}{2})$ is a point of inflexion.

(vi) Evaluate $\lim\limits_{x\to\infty} f(x)$.

(vii) On about one-third of a page, sketch the graph of $y=f(x)$, clearly indicating all features found in parts (i)-(vi).

Exam continues next page ...

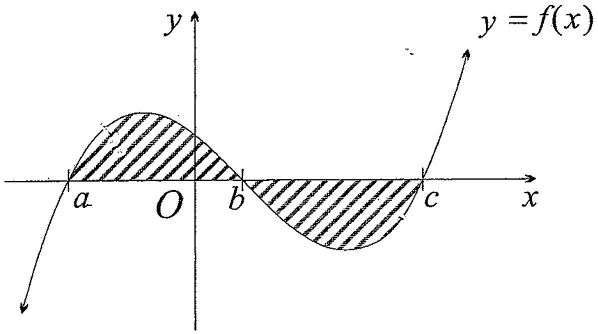
QUESTION FIVE (15 marks) Use a separate writing booklet.

(a) Find:

(i) $\displaystyle\int \frac{x^2+4}{\sqrt{x}}\,dx$

(ii) $\displaystyle\int (3-5x)^3\,dx$

(b)



In the diagram above, the graph of $y = f(x)$ intersects the x-axis at the points $(a,0)$, $(b,0)$ and $(c,0)$. The shaded area above the x-axis has magnitude 4 square units and the shaded area below the x-axis has magnitude 5 square units.
Write down the value of:

(i) $\displaystyle\int_b^c f(x)\,dx$

(ii) $\displaystyle\int_a^c f(x)\,dx$

(c) Find the value of k if $\displaystyle\int_0^k x^2\,dx = 9$.

(d) Find the area bounded by the graph of $y = x^2 - 2x$ and the x-axis, between $x = 1$ and $x = 3$.

QUESTION SIX (15 marks) Use a separate writing booklet.

(a) Find the equation of the parabola with focus $(1,-3)$ and vertex $(1,0)$.

(b) The parabola $\mathcal{P}$ has equation $x^2 = 6(y+4)$.

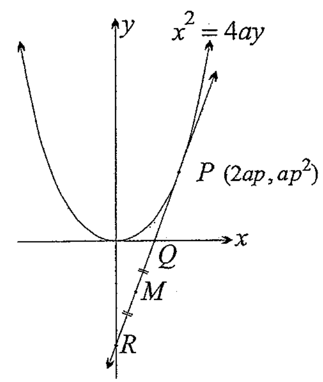
(i) Write down the coordinates of the vertex of $\mathcal{P}$.

(ii) Find the coordinates of the focus of $\mathcal{P}$.

(iii) Find the equation of the directrix of $\mathcal{P}$.

(iv) On about one-third of a page, sketch the graph of $\mathcal{P}$. On your diagram include and label all the features found in parts (i) to (iii).

(c)



In the diagram above, $P(2ap, ap^2)$ is a variable point on the parabola $x^2 = 4ay$. The tangent at P meets the x-axis at Q and the y-axis at R, and M is the midpoint of QR.

(i) Show that the gradient of the tangent at P is p.

(ii) Show that the equation of the tangent at P is $y = px - ap^2$.

(iii) Find the coordinates of Q and R.

(iv) Find the Cartesian equation of the locus of M.

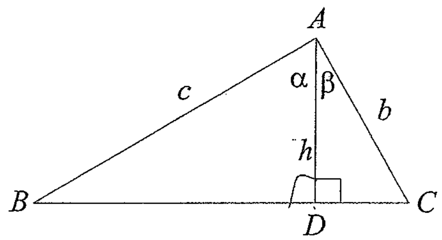
QUESTION SEVEN (15 marks) Use a separate writing booklet.

(a) (i) Copy and complete the table for the function $y = 2^{-x}$.

x	-2	-1	0	1	2
y					

(ii) Use Simpson's rule with five function values to approximate $\int_{-2}^{2} 2^{-x}\,dx$.

(b)



In the diagram above, $AD \perp BC$ and $AD = h$.

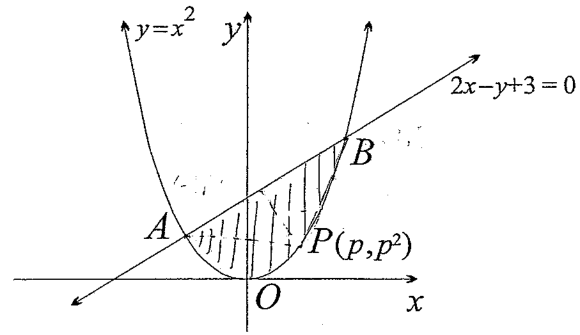
(i) Write down expressions for $\cos\alpha$ and $\cos\beta$.

(ii) Use the areas of triangles ABD, ADC and ABC to show that, for α and β acute,

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

QUESTION SEVEN CONTINUES ON THE NEXT PAGE

(c)



In the diagram above, $A(-1,1)$ and $B(3,9)$ are the points of intersection of the parabola $y = x^2$ and the line $2x - y + 3 = 0$. The point $P(p,p^2)$ is a variable point on the parabola between A and B.

(i) Find the area of the region enclosed between the parabola and the line.

(ii) Find the length of AB.

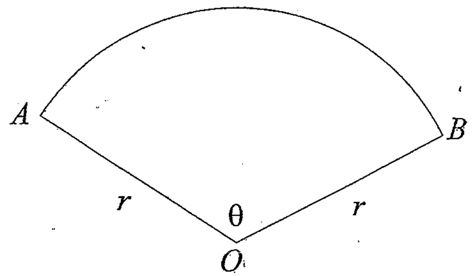
(iii) Find the perpendicular distance from P to the line $2x - y + 3 = 0$.

(iv) Show that the maximum area of the triangle APB is three-quarters of the area found in part (i).

QUESTION EIGHT IS ON THE NEXT PAGE

QUESTION EIGHT (15 marks) Use a separate writing booklet.

(a)

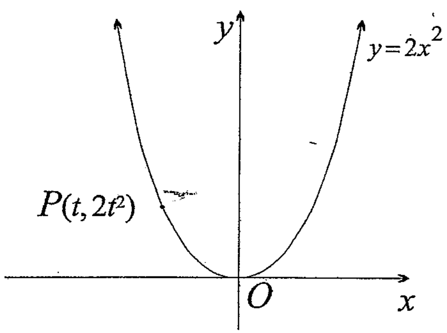


In the diagram above, AB is an arc of a circle with centre O and radius r centimetres. The angle AOB is θ radians. The radius is increasing at the rate of $\frac{1}{5}$ cm/s, while the perimeter of the sector AOB remains constant at 50 centimetres.

(i) Show that $\theta = \dfrac{50}{r} - 2$.

(ii) At what rate is θ decreasing when the radius is 20 centimetres?

(b)



In the diagram above, the point $P(t, 2t^2)$ lies on the parabola $y = 2x^2$.

(i) Show that the equation of the tangent at P is $y = 4tx - 2t^2$.

(ii) Find the equations of the two tangents to the parabola $y = 2x^2$ passing through the point $(-1, 0)$.

QUESTION EIGHT CONTINUES ON THE NEXT PAGE

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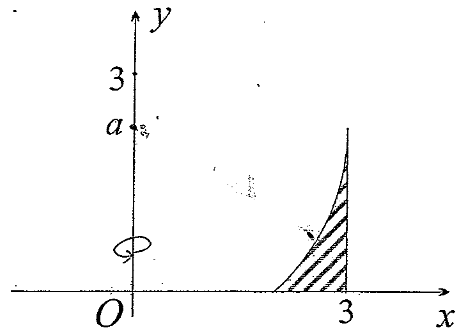
(c)



In the diagram above, $0 < a \leq 3$ and the shaded region is bounded by the x-axis, the line $x = 3$ and an arc of the circle with centre $(0, a)$ and radius 3. Find the volume of the solid formed when the shaded region is rotated about the y-axis. Give your answer exactly in terms of π.
(Note: The equation of the circle is $x^2 + (y - a)^2 = 9$.)

QUESTION NINE (15 marks) Use a separate writing booklet.

(a) Consider the equation

$$\frac{x}{x-a} + \frac{x}{x-b} = 1 + c,$$

where a,b and c are constants, a and b are non-zero and $|a| \neq |b|$. Suppose that the equation has exactly one real solution.

(i) Rewrite the equation in the form $\alpha x^2 + \beta x + \gamma = 0$.

(ii) Find the discriminant of the quadratic.

(iii) Explain why $c \neq 0$.

(iv) Find the maximum possible value of c. (You must fully justify your answer).

(b) (i) Find, and fully simplify,

$$\int_{-1}^{1} \frac{dx}{\sqrt{1 + 2ax + a^2}},$$

where a is a constant.

(ii) For what values of a is $\displaystyle\int_{-1}^{1} \frac{dx}{\sqrt{1 + 2ax + a^2}}$ defined?

END OF EXAMINATION

(1)

(a) $x^3 - y^3 = (x-y)(x^2+y^2+xy)$

(b) $\dfrac{1}{\sqrt5+1} \times \dfrac{\sqrt5-1}{\sqrt5-1} = \dfrac{\sqrt5-1}{5-1} = \dfrac{\sqrt5-1}{4}$

c) A.P $175+168+161+\dots$

(i) $a=175 \quad d=-7$

$T_n = a+(n-1)d$

$\therefore T_n = 175-(n-1)-7$

$T_n = 175+7(n-1)$

so $T_{300} = 175+7(299)$

$= 2268$

(ii) S_{300} $\quad 175+168+161+\dots+2268$

$a=175 \quad \ell=2268$

$S_n = \frac{1}{2}n(a+\ell)$

$= 150(175+2268)$

$S_{300} = 366\,450$

error carried

d) x^3-3x

$\therefore \dfrac{dy}{dx} = 3x^2-3 = 3(x^2-1)$

(e) $7-2x$

$F(x) = 7x - \dfrac{2x^2}{2} + C$

$= 7x - x^2 + C$

(f) 140° in radians

π radians $= 180^\circ \qquad 180^\circ = \pi$

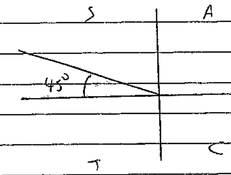
$\therefore 1^\circ = \dfrac{\pi}{180}$

$140^\circ = \dfrac{\pi \times 140}{180^\circ} = \dfrac{140\pi}{180}$

$= \dfrac{7\pi}{9}$ radians.

(g) $\tan\dfrac{3\pi}{4} = \tan\dfrac{3\times180}{4} = \tan\dfrac{540}{4}$

$= \tan 135^\circ$



$= -\tan 45^\circ$

$= -1$

hence $\tan\frac{3\pi}{4} = -1$

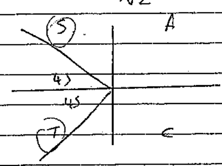
(h) $y = \sin 2x \qquad (n=2)$

amplitude is 1

period is $\dfrac{2\pi}{n} = \dfrac{2\pi}{2} = \pi$

(i) $\sqrt2\cos\theta = -1 \quad 0 \le \theta \le 360$

$\cos\theta = \dfrac{-1}{\sqrt2}$ related $\angle$ is 45°



θ is 135° or 225°

(a) (i) $\sqrt{4-x^2} = (4-x^2)^{\frac12}$ $\qquad$ 15

$\therefore \dfrac{dy}{dx} = \frac12(4-x^2)^{-\frac12} \times -2x$

$= -x(4-x^2)^{-\frac12}$

(ii) $\dfrac{2x}{1-x^2} \quad \begin{matrix}-(u)\\-(v)\end{matrix} \qquad \therefore u=2x \quad u'=2$

$v = 1-x^2 \quad v' = -2x$

$\therefore \dfrac{dy}{dx} = \dfrac{vu'-uv'}{v^2} = \dfrac{2(1-x^2)-(2x \times -2x)}{(1-x^2)^2}$

$= \dfrac{2(1-x^2)+4x^2}{(1-x^2)^2}$

$= \dfrac{2-2x^2+4x^2}{(1-x^2)^2}$

$\dfrac{2+2x^2}{(1-x^2)^2} = \dfrac{2(1+x^2)}{(1-x^2)^2}$

b) (i) $\dfrac{3}{x^2} = 3x^{-2}$

$F(x) = \dfrac{3x^{-1}}{-1} = -3x^{-1}$

$= -3 \times \dfrac{1}{x}$

$= \dfrac{-3}{x}$

(ii) $x(2-x) = 2x-x^2$

$F(x) = \dfrac{2x^2}{2} - \dfrac{x^3}{3}$

$= x^2 - \frac13 x^3$

(c) $\displaystyle\int_{-1}^{3} 4x^3\,dx$

$= \left[\dfrac{4x^4}{4}\right]_{-1}^{3} = \left[x^4\right]_{-1}^{3}$

$= 3^4 - (-1)^4$

$= 81 - 1$

$= 80$

(d) (i) $\ell = r\theta$

$r=5 \quad \theta = 120^\circ = \dfrac{2\pi}{3}$

$\therefore \ell = 5 \times \dfrac{2\pi}{3} = \dfrac{10\pi}{3}$ units

(ii) $A = \frac12 r^2\theta$

$r=5 \quad r^2=25 \quad \theta=\frac{2\pi}{3}$

$\therefore A = \frac12 \times 25 \times \frac{2\pi}{3}$

$= \dfrac{50\pi}{6}$

$= \dfrac{25\pi}{3}$ units^2

(3)

(a) $\alpha = 2+\sqrt{5}$ $\beta = 2-\sqrt{5}$

$\therefore$ we want $(x-\alpha)(x-\beta) = (x-(2+\sqrt{5}))(x-(2-\sqrt{5}))$ ✓

$= (x-2-\sqrt{5})(x-2+\sqrt{5})$

$= x^2 - 2x + \cancel{x\sqrt{5}} - 2x + 4 - \cancel{2\sqrt{5}} - \cancel{x\sqrt{5}} + \cancel{2\sqrt{5}} - 5$

$= x^2 - 4x - 1$ ✓

hence $x^2-4x-1=0$ ✓

(b) $2x^2-(k-3)x+8=0$

will have EQUAL roots, $\Longleftrightarrow$ when $\Delta=0$

$\Delta = b^2-4ac$ $\quad -k+3$

$a=2$ $b=-(k-3)$ $c=8$

$\therefore \Delta = (3-k)^2-(4\times2\times8) = 0$ ✓

$= 9+k^2-6k-(64)=0$

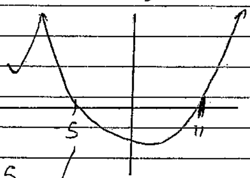
$= 9+k^2-6k-64=0$ $\qquad$ $5 \quad -11$

$k^2-6k-55=0$ $\qquad$ $+(-6)\ \times(-55)$

$k^2+5k-11k-55=0$

$k(k+5)-11(k+5)=0$

$(k-11)(k+5)=0$ ✓



$\therefore$ when $k=11, -5$ ✓

(c) $2x^2+3x+5=0$ $\quad \therefore a=2 \quad b=3 \quad c=5$

(i) $\alpha+\beta = \frac{-b}{a} = \frac{-3}{2}$ ✓

(ii) $\alpha\beta = \frac{c}{a} = \frac{5}{2}$ ✓

(iii) $\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\beta+\alpha}{\alpha\beta} = \frac{\alpha+\beta}{\alpha\beta}$

$= \frac{-\frac{3}{2}}{\frac{5}{2}}$

$= -3/5$ ✓✓

(iv) $(x-\frac{1}{\alpha})(x-\frac{1}{\beta}) = x^2 - \frac{x}{\beta} - \frac{x}{\alpha} + \frac{1}{\alpha\beta}$

$\cancel{= x^2 - \left(\frac{\alpha x - \beta x}{\alpha\beta}\right) + \frac{1}{\alpha\beta}}$

$-\frac{x}{\beta} - \frac{x}{\alpha} = \frac{-x\alpha - x\beta}{\alpha\beta} = \frac{-x(\alpha+\beta)}{\alpha\beta}$

$\therefore$ quadratic is $x^2 - \frac{x(\alpha+\beta)}{\alpha\beta} + \frac{1}{\alpha\beta}$

$= x^2 - \frac{x\left(\frac{-3}{2}\right)}{\frac{5}{2}} + \frac{1}{\frac{5}{2}}$

$= x^2 - \frac{2}{5}x\left(\frac{-3}{2}\right) + \frac{2}{5}$

$= x^2 + \frac{6}{10}x + \frac{2}{5}$

$= x^2 + \frac{3}{5}x + \frac{2}{5}$ ✓✓

$= 5x^2+3x+2$

$\therefore$ quadratic is $5x^2+3x+2=0$

(d) $\cos 75^\circ = \cos(30^\circ+45^\circ)$

where $\cos(A+B) = CC - SS$

$\cos(A+B) = \cos A\cos B - \sin A \sin B$

hence $\cos(30^\circ+45^\circ) = \cos 30^\circ \cos 45 - \sin 30^\circ \sin 45$

$= \left(\frac{\sqrt{3}}{2} \times \frac{1}{\sqrt{2}}\right) - \left(\frac{1}{2}\times\frac{1}{\sqrt{2}}\right)$

$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{3}-1}{2\sqrt{2}} \times \frac{2\sqrt{2}}{2\sqrt{2}}$

$= \frac{2\sqrt{2}(\sqrt{3}-1)}{8}$

$= \frac{\sqrt{2}(\sqrt{3}-1)}{4}$

$= \frac{\sqrt{6}-\sqrt{2}}{4}$ ✓✓✓

(4)

(a) $P(2,3)$ $\quad y' = 2x-1$ $\quad (x=2)$

$\therefore$ gradient $m = 2(2)-1$

$= 4-1 = 3$ $\qquad y = \int 2x-1\,dx$

$= x^2-x+c$

$\cancel{\therefore y-y_1 = m(x-x_1)}$ $\qquad$ At $(2,3)$

$\cancel{y-3 = 3(x-2)}$ $\qquad 3 = 2^2-2+c$

$\cancel{y-3 = 3x-6}$ $\qquad \therefore c=1$

$\cancel{y-3x-3+6=0}$ $\qquad \therefore y = x^2-x+1$

$\cancel{y-3x+3=0}$

(b) $f(x) = \frac{x^2+3}{x^2+1}$

(i) $f(-x) = \frac{(-x)^2+3}{(-x)^2+1} = \frac{x^2+3}{x^2+1}$

$= f(x)$ ✓

since $f(-x)=f(x)$, function is EVEN.

(ii) $y = \frac{x^2+3}{x^2+1}$

x-int when $y=0$ $\qquad$ y-int when $x=0$

$\therefore x^2+3=0$ $\qquad y = \frac{3}{1} = 3$ ✓

$x^2 = -3$

no x-intercept exists $\qquad$ $(0,3)$ is the y-int.

(iii) $f(x) = \frac{x^2+3}{x^2+1}$ $\quad u = x^2+3 \quad u' = 2x$

$v = x^2+1 \quad v' = 2x$

$f'(x) = \frac{vu'-uv'}{v^2} = \frac{2x(x^2+1)-2x(x^2+3)}{(x^2+1)^2} = \frac{\cancel{2x^3}+2x-\cancel{2x^3}-6x}{(x^2+1)^2}$

$f'(x) = \frac{-4x}{(x^2+1)^2}$ ✓

(iv) stat pt when $f'(x) = 0$

$\therefore \frac{-4x}{(x^2+1)^2} = 0$

$\therefore -4x = 0$

$x = 0$

Stat pt when $x = 0$

when $x = 0$ $y = 3$

hence Stat pt exists at $(0,3)$

(v) $y'' = \frac{4(3x^2-1)}{(x^2+1)^3}$

(α) when $x = 0$

$y'' = \frac{4(0-1)}{(0+1)^3}$

$= \frac{4(-1)}{1} = -4$

since $f''(x) < 0$, it is $\cap$, maximum point.

(β) point of inflexion when $f''(x) = 0$

$\therefore \frac{4(3x^2-1)}{(x^2+1)^3} = 0$

$4(3x^2-1) = 0$

so $3x^2 = 1$

$x^2 = \frac{1}{3}$ $\quad x = \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{\sqrt{3}}{3}$

when $x = \frac{\sqrt{3}}{3}$, $y = \frac{\left(\frac{\sqrt{3}}{3}\right)^2+3}{\left(\frac{\sqrt{3}}{3}\right)^2+1} = \frac{\frac{3}{9}+3}{\frac{3}{9}+1}$

$= \frac{3\frac{1}{3}}{1\frac{1}{3}}$

You need to test for change in concavity X

$y = 2\frac{1}{2} = \frac{5}{2}$

hence point of inf is $\left(\frac{\sqrt{3}}{3}, \frac{5}{2}\right)$

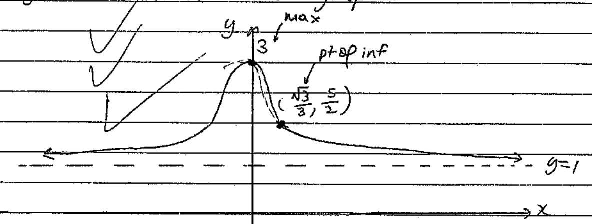
(vi) $\lim_{x \to \infty} \frac{x^2+3}{x^2+1}$

$\therefore \frac{\frac{x^2}{x^2} + \frac{3}{x^2}}{\frac{x^2}{x^2} + \frac{1}{x^2}} = \frac{1 + \frac{3}{x^2}}{1 + \frac{1}{x^2}}$

as $x \to \infty$ $y \to \frac{1+0}{1+0} = 1$

so $y = 1$ is a horizontal asymptote.

$y = \frac{x^2+3}{x^2+1}$



(5)

a) (i) $\int \frac{x^2+4}{\sqrt{x}}\,dx = \int \frac{x^2+4}{x^{\frac{1}{2}}}\,dx$

$= \cancel{\int (x^2+4)(x^2)^{-\frac{1}{2}}\,dx}$

$= \int \frac{x^2}{x^{\frac{1}{2}}} + \frac{4}{x^{\frac{1}{2}}}$

~~let $u = x^2+4$~~

~~$\frac{du}{dx} = 2x$~~

$= \int x^{3/2} + 4x^{-\frac{1}{2}}\,dx$

$= \frac{x^{2\frac{1}{2}}}{2\frac{1}{2}} + \frac{4x^{\frac{1}{2}}}{\frac{1}{2}} + c$

$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + 8x^{\frac{1}{2}} + c$

$= \frac{2}{5}x^{\frac{5}{2}} + 8x^{\frac{1}{2}} + c$

(ii) $\int (3-5x)^3\,dx = \frac{(3-5x)^4}{4 \times -5} + c$

$= \frac{(3-5x)^4}{-20} + c$

b) (i) $\int_6^c f(x)\,dx = -5$

(ii) $\int_a^c f(x)\,dx = -1$

(c) $\int_0^k x^2\,dx = 9$

$\therefore \left[\frac{x^3}{3}\right]_0^k = 9$

$\left[\frac{1}{3}x^3\right]_0^k = 9$

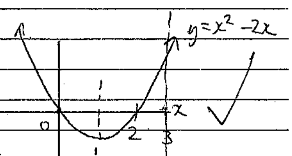
so $\frac{1}{3}k^3 = 9$

$\frac{k^3}{3} = 9$

$k^3 = 27$

$k = 3$

(d) $y = x^2 - 2x = x(x-2)$



ie, this is $\left|\int_1^2 x^2-2x\,dx\right| + \int_2^3 x^2-2x$

$= \left|\left[\frac{x^3}{3} - \frac{2x^2}{2}\right]_1^2\right| + \left[\frac{x^3}{3} - \frac{2x^2}{2}\right]_2^3$ *

$\cancel{\left[\frac{1}{3}x^2 - x^2\right]_1^2 + \left[\frac{1}{3}x^2 - x^2\right]_2^3}$??

$= \left[\left(\frac{4}{3} - 4\right) - \left(\frac{1}{3} - 1\right)\right] + \left[(3-9) - \left(\frac{4}{3} - 4\right)\right]$

$= \left[-2\frac{2}{3} - \left(-\frac{2}{3}\right)\right] + \left[(-6) - \left(-2\frac{2}{3}\right)\right]$

$= (-2) + \left(-3\frac{1}{3}\right)$

$= 5\frac{1}{3}$ units^2

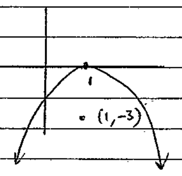
* $\left|\left(\frac{8}{3} - 4\right) - \left(\frac{1}{3} - 1\right)\right| + (9-9) - \left(\frac{8}{3} - 4\right)$

$= \left|-\frac{2}{3}\right| + 1\frac{1}{3}$

$= 2$

(6)

(a) $F(1,-3)$ vertex $(1,0)$



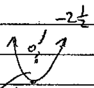
$a = 3$

so, $(x-h)^2 = -4a(y-k)$ $\quad (h,k) = (1,0)$

$(x-1)^2 = -4\times 3(y-0)$

$(x-1)^2 = -12y$

(b) $x^2 = 6(y+4)$



(i) vertex is $(0,-4)$

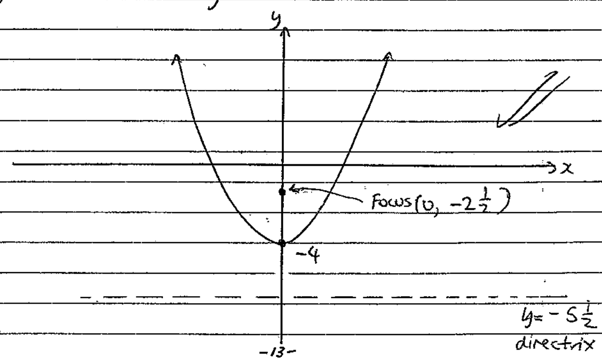
(ii) $4a = 6$

$a = \frac{6}{4} = \frac{3}{2}$

$\therefore$ so Focus is $(0, -2\frac{1}{2})$

(iii) $\therefore$ directrix is $y = -5\frac{1}{2}$

(iv)



(c) (i) $x^2 = 4ay$

$4ay = x^2$

$y = \frac{x^2}{4a} = \frac{1}{4a}\times x^2$

$y' = \frac{2}{4a}\times x$

$= \frac{1}{2a}\times x = \frac{x}{2a}$

so at P, tangent $x = 2ap$

gradient $= \frac{2ap}{2a} = p$ as required.

(ii) Point $(2ap, ap^2)$ $\quad m = p$

$y - y_1 = m(x - x_1)$

$y - ap^2 = p(x - 2ap)$

$y - ap^2 = px - 2ap^2$

$y = px - 2ap^2 + ap^2$

$y = px - ap^2$

(iii) Q is int of $y = px - ap^2$ with x-axis

ie, $y = 0$

$0 = px - ap^2$

$ap^2 = px$

$ap = x$ $\quad \therefore Q$ is $(ap, 0)$

R is int with y-axis

ie, $x = 0$

$y = 0 - ap^2$

$y = -ap^2$ $\quad \therefore R$ is $(0, -ap^2)$

(iv) midpoint $Q(ap,0)$ $R(0,-ap^2)$ is M

$x = \frac{x_1+x_2}{2}$ $\quad y = \frac{y_1+y_2}{2}$

$x = \frac{ap}{2}$ ✓ $\quad y = \frac{-ap^2}{2}$ ✓

if $\frac{ap}{2} = x$ $\quad p = \frac{2x}{a} \Rightarrow y = -\frac{a}{2}\left(\frac{4x^2}{a^2}\right)$

$\therefore ap = 2x$ $\quad = \frac{-2x^2}{a}$

$a = \frac{2x}{p}$ $\quad x^2 = -\frac{ay}{2}$

sub into $y = \frac{-ap^2}{2}$

$y = \frac{\left(\frac{-2x}{p}\right)(p^2)}{2}$

$= \frac{\frac{-2xp^2}{p}}{2}$ $\quad y = \frac{-2xp}{2}$

so $2y = -2xp$

$y = -xp$

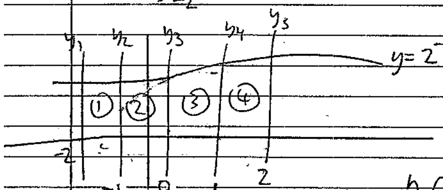
XX

(7)

(a) (i) $y = 2^{-x}$

x	-2	-1	0	1	2
y	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$

(ii) $\int_{-2}^{2} 2^{-x}\,dx$



$n = 4$

$\therefore h = \frac{a-b}{n} = 1$

$A = \frac{h}{3}\left[(y_1+y_n) + 2(\text{odd}) + 4(\text{even})\right]$

when $x = -2$ $\quad y_1 = 4$ $\quad = \frac{1}{3}\left[(4+\frac{1}{4}) + 2(1) + 4(2+\frac{1}{2})\right]$

$x = -1$ $\quad y_2 = 2$ $\quad = \frac{1}{3}\left(4\frac{1}{4} + 2 + 10\right)$

$x = 0$ $\quad y_3 = 1$ $\quad = \frac{1}{3}\times 16\frac{1}{4}$

$x = 1$ $\quad y_4 = \frac{1}{2}$ $\quad = \frac{1}{3}\times\frac{65}{4}$

$x = 2$ $\quad y_5 = \frac{1}{4}$

area $= \frac{65}{12}$ units^2

(b) (i) $\cos\alpha = \frac{h}{c}$ $\quad \cos\beta = \frac{h}{b}$

(ii) for $\triangle ABC$, $\text{area} = \frac{1}{2}ab\sin C$

$= \frac{1}{2} \times b \times c \times \sin(\alpha+\beta)$

$= \frac{bc\sin(\alpha+\beta)}{2}$

area for $\triangle ABD = \frac{1}{2}ab\sin C$ $\quad$ area for $\triangle ADC = \frac{1}{2}ab\sin C$

$= \frac{1}{2} \times c \times h \times \sin\alpha$ $\quad = \frac{1}{2} \times h \times b \times \sin\beta$

$= \frac{ch\sin\alpha}{2}$ $\quad = \frac{hb\sin\beta}{2}$

$\therefore$ $\triangle ABC$ area = areas of $\triangle ABD + \triangle ADC$

$$\frac{bc\sin(\alpha+\beta)}{2} = \frac{ch\sin\alpha}{2} + \frac{hb\sin\beta}{2}$$

$$bc\sin(\alpha+\beta) = ch\sin\alpha + hb\sin\beta$$

from part (i), $\cos\alpha = \frac{h}{c}$ $\quad \cos\beta = \frac{h}{b}$

$\therefore h = \cos\alpha\, c$ $\quad \therefore h = b\cos\beta$

sub into equation,

thus, $bc\sin(\alpha+\beta) = cb\cos\beta\sin\alpha + \cos\alpha\, cb\sin\beta$

$\therefore bc\sin(\alpha+\beta) = bc(\sin\alpha\cos\beta) + bc(\cos\alpha\sin\beta)$

$\therefore \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$

as required

(c) (i) area $= \int_{-1}^{3}$ line − parabola dx

line is $2x - y + 3 = 0$ $\quad$ | parabola is $y = x^2$

$\therefore y = 2x + 3$

$$\therefore \int_{-1}^{3} 2x + 3 - x^2 \, dx = \left[\frac{2x^2}{2} + 3x - \frac{x^3}{3}\right]_{-1}^{3}$$

$$= \left[x^2 + 3x - \tfrac{1}{3}x^3\right]_{-1}^{3}$$

$$= \left[9 + 9 - (\tfrac{1}{3} \times 27)\right] - \left(1 - 3 - (\tfrac{1}{3} \times -1)\right)$$

$$= (18 - 9) - (-2 + \tfrac{1}{3}) = 9 - (-1\tfrac{2}{3})$$

$$= 9 + 1\tfrac{2}{3} = 10\tfrac{2}{3} \text{ units}^2$$

(ii) length $A(-1, 1)$ $B(3, 9)$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(3+1)^2 + (9-1)^2}$$

$$= \sqrt{(4)^2 + (8)^2}$$

$$= \sqrt{16 + 64} = \sqrt{80}$$

$$= 4\sqrt{5} \text{ units}$$

(iii) $P(p, p^2)$ to $2x - y + 3 = 0$

$\therefore$ where $a = 2$ $b = -1$ $c = 3$

$$pd = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

$$= \frac{|2p - p^2 + 3|}{\sqrt{4 + 1}} = \frac{|2p - p^2 + 3|}{\sqrt{5}} \text{ units}$$

(iv) A of $\triangle$

$\perp$ distance

$$A = \frac{1}{2} \times AB \times pd = \frac{1}{2} \times 4\sqrt{5} \times \frac{|2p - p^2 + 3|}{\sqrt{5}}$$

why?

$$= \frac{4(2p - p^2 + 3)}{2}$$

$$= 2(2p - p^2 + 3) = 4p - 2p^2 + 6$$

as $-1 \le p \le 3$

$\frac{dA}{dp} = 4 - 6p^2$

stat pt when $\frac{dA}{dp} = 0$, $4 = 6p^2$

$6p^2 = 4$

$p^2 = \frac{2}{3}$ $\quad p = \sqrt{\frac{2}{3}}$

$\therefore$ maximum area when $p = \sqrt{\frac{2}{3}}$

(test) p	0.8	$\sqrt{\frac{2}{3}}$	0.9
y'	+	0	−

hence is maximum.

$\therefore$ max area is $4p - 2p^3 + 6$ (from before) sub in $\sqrt{\frac{2}{3}}$

$4 \times \frac{\sqrt{2}}{\sqrt{3}} - 2\left(\sqrt{\frac{2}{3}}\right)^3 + 6$

$4\sqrt{2}$

see further!

area of $\triangle = 4p - 2p^2 + 6$

$\frac{dA}{dp} = 4 - 4p = 4(1 - p)$

max area stat point when $\frac{dA}{dp} = 0$

$\therefore 4(1 - p) = 0$. $\quad p = 1$

test p	0	1	2
y'	+	0	−

hence MAXIMUM.

$\therefore$ when $p = 1$, we get a max area for $\triangle$

this is $= 4p - 2p^2 + 6$ sub in 1

$\therefore 4 - 2 + 6 = 8$

this and 8 is $\frac{3}{4}$ of $10\frac{2}{3}$ units^2

since $\frac{32}{3} \times \frac{3}{4} = \frac{32}{4} = 8$ units.

(8)

(a) $\frac{dr}{dt} = \frac{1}{5}$ cm/s perimeter = 50cm

(i) arc length $l = r\theta$

perimeter of diagram is $r + r + l = 2r + l$

$= 2r + r\theta = 50$cm

$\therefore r\theta = 50 - 2r$

$\theta = \frac{50}{r} - 2$ as required.

(ii) $\frac{d\theta}{dt} = ?$ $r = 20$cm.

$\frac{dr}{dt}$ ie, $\frac{d\theta}{dt} = \frac{d\theta}{dr} \times \frac{dr}{dt}$

$= \frac{d\theta}{dr} \times \frac{1}{5} = \frac{-50}{r^2} \times \frac{1}{5}$

$\theta = 50r^{-1} - 2$

$= \frac{-10}{r^2}$

$\frac{d\theta}{dr} = -50r^{-2}$

$= -50 \times \frac{1}{r^2}$ and $r = 20$cm

$= \frac{-50}{r^2}$ hence $\frac{d\theta}{dt} = \frac{-10}{(20)^2}$

$= \frac{-10}{400}$

$= -\frac{1}{40}$ radians/s.

so θ is decreasing at $\frac{1}{40}$ radians/s.

(b) (i) $y = 2x^2$ $y' = 4x$

so at $P(t, 2t^2)$ gradient m of tangent is $M = 4(t) = 4t$

$y - y_1 = m(x - x_1)$

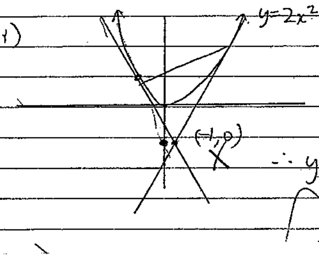
$y - 2t^2 = 4t(x - t)$

$y - 2t^2 = 4xt - 4t^2$

$y = 4tx - 4t^2 + 2t^2$

$y = 4tx - 2t^2$ is equ for tangent at P.

(ii)



$xx_0 = 2a(y - y_0)$

will give you equation of chord of contact

$\therefore y = 2x^2$

$2x^2 = y$

$x^2 =$

$x =$

You have

(ii) Since the tangent passes through $(-1, 0)$

$\therefore 0 = 4t(-1) - 2t^2$

$= -4t - 2t^2$

$= -2t(2 + t)$

$\therefore$ ~~both~~ $t = -2$ or 0

$\therefore$ Eq^{ns}. of tangents are

$y = 0$ or $y = -8x - 8$

(c) volume $= \pi \int x^2 \, dy$

for circle $x^2 + (y - a)^2 = 9$

$x^2 = 9 - (y - a)^2$

for line $x = 3$ $x^2 = 9$

volume $= \pi \int_0^a 9 - (9 - (y - a)^2) \, dy$

int of circle with x-axis is when $y = 0$

ie, $x^2 + (-a)^2 = 9$

$x^2 + a^2 = 9$

$x^2 = 9 - a^2$

$x = \sqrt{9 - a^2}$

Volume $= \int_0^a 9 - 9 + (y - a)^2 \, dy$

$= \int_0^a (y - a)^2 \, dy$

$= \left[\frac{(y - a)^3}{3} \right]_0^a$

$= \frac{1}{3} \left[0 + a^3 \right]$

$= \frac{a^3}{3}$ units^3.

(9)

(a) (i) $\frac{x}{x - a} + \frac{x}{x - b} = 1 + c$

$\frac{x(x - b) + x(x - a)}{(x - a)(x - b)} = 1 + c$

$x^2 - xb + x^2 - xa = (1 + c)(x^2 - xb - ax + ab)$

$x^2 - \cancel{xb} + \cancel{x^2} - \cancel{xa} = \cancel{x^2} - \cancel{xb} - \cancel{ax} + ab + cx^2 - xbc - acx + abc$

$\therefore x^2 = ab + cx^2 - bcx - acx + abc$

$\therefore x^2 - cx^2 + bcx + acx - ab - abc = 0$

$x^2(1 - c) + x(bc + ac) - (ab + abc) = 0$

thus $(1 - c)x^2 + (bc + ac)x - (ab + abc) = 0$

(ii) $\Delta = b^2 - 4ac$ where $a = 1 - c$ $b = bc + ac$ $c = -(ab + abc)$

$\Delta = (bc + ac)^2 - 4((1 - c)(-(ab + abc)))$

$= c^2(b + a)^2 + 4ab(1 - c^2)$.

(iii) If $c = 0$, $\Delta = 4ab$. $b \neq a \neq 0$.

> 0 since a, b are non-zero,
(means 2 roots)

but the equation has only ONE real solution

$\therefore c \neq 0$.

(iv) For max value $\Delta = 0$

$\therefore c^2(b + a)^2 + 4ab - 4ac^2b = 0$

$c^2[(a + b)^2 - 4ab] = 0 - 4ab$

$c^2 = \frac{-4ab}{(a - b)^2}$

Note: $a^2 + 2ab + b^2 - 4ab$
$= a^2 - 2ab + b^2$
$= (a - b)^2$

$c^2 > 0$ if $ab < 0$

i.e. $c = \sqrt{\frac{4ab}{(a - b)^2}}$

$c_{max} = \frac{2\sqrt{ab}}{a - b}$

(9) (i) $\displaystyle\int_{-1}^{1} \frac{dx}{(1+2ax+a^2)^{1/2}}$

$$= \frac{1}{2a}\int_{-1}^{1} 2a\,(1+2ax+a^2)^{-1/2}\,dx$$

$$= \frac{1}{\cancel{2}a}\left[\frac{(1+2ax+a^2)^{1/2}}{\frac{1}{\cancel{2}}}\right]_{-1}^{1} \quad \text{by reverse chain rule}$$

$$= \frac{1}{a}\left[\sqrt{1+2a+a^2} - \sqrt{1-2a+a^2}\right]$$

$$= \frac{1}{a}\left[\sqrt{(a+1)^2} - \sqrt{(a-1)^2}\right]$$

$$= \frac{1}{a}\left[(a+1) - (a-1)\right]$$

$$= \frac{1}{a}\left[2\right]$$

$$= \frac{2}{a}$$

(ii) For $\displaystyle\int_{-1}^{1} \frac{dx}{\sqrt{1+2ax+a^2}}$ to be defined

$1+2ax+a^2 > 0$ (Positive definite)

i.e. $\Delta < 0$

$\therefore (2x)^2 - 4 < 0$

$4x^2 - 4 < 0$

$(x+1)(x-1) < 0$

$\therefore -1 < x < 1$ which is true for all a.