GRAPHS 4 COMPLEX NOS – PRACTICE PAPER

Question 1 (24 marks)

(a) The graph of $y = f(x)$ is shown at right. (14 marks)

Sketch graphs of the following, clearly showing any axes intercepts and turning points.

(i) $y = f(x+3)$

(ii) $y = f(|x|)$

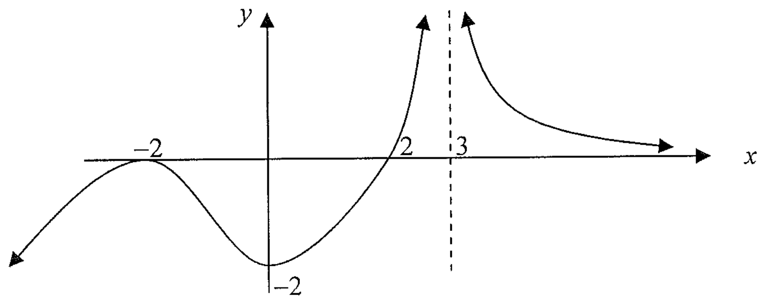
(iii) $y = \dfrac{1}{f(x)}$

(iv) $y^2 = -f(x)$

(v) $y = \ln(f(x))$

(vi) $y = e^{f(x)}$

(vii) $y = \tan^{-1}(f(x))$



(b) Given the graph of $y = f'(x)$ as above, sketch the graph of $y = f(x)$ given that $f(0) = 0$ and $f(x)$ is continuous. (2 marks)

(c) The polynomial $P(z)$ has the equation $P(z) = z^4 - 2z^3 + Az^2 + Bz + 10$, where A, B are real. Given that $2 + i$ is a zero of $P(z)$, find all the roots of $P(z)$ and write $P(z)$ as a product of two real quadratic factors. (2 marks)

(d) $z_1 = 4 + 3i, |z_2| = 2$ (6 marks)

(i) sketch and describe the locus of $z_1 + z_2$.

(ii) find the ranges of $|z_1 + z_2|$ and $|z_1 - z_2|$. Label A and D the points with maximum $|z_1 + z_2|$ and $|z_1 - z_2|$ and B and C the points with minimum $|z_1 + z_2|$ and $|z_1 - z_2|$ respectively.

(iii) find the range of $\arg \dfrac{z_1}{z_1 - z_2}$. Write your answers to the nearest degree.

Question 2 (16 marks)

(a) Sketch the following loci: (6 marks)

(i) $|z - i| = |z|$.

(ii) $\arg(z - \sqrt{3} + i) = \dfrac{\pi}{3}$.

(iii) $|z + 3 - 4i| \le 5$ and $0 \le \text{Re}(z) \le 1$.

(iv) $\arg\left(\dfrac{z - \sqrt{3} + i}{z + \sqrt{3} + i}\right) = \dfrac{\pi}{3}$, stating its centre and radius.

(b) (i) Express $z = 1 + i$ in modulus-argument form. (3 marks)

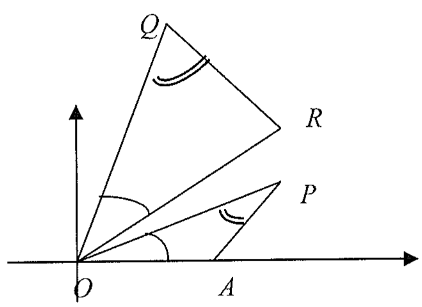
(ii) Hence write z^9 in the form $a + ib$ where a and b are real.

(c) In an Argand diagram, point A corresponds to the complex number $1 + bi$, b is real. (5 marks)

(i) If O is the origin, what complex numbers correspond to the vertices B and C if $OABC$ is a square, assuming O, A, B, C are in anticlockwise order?

(ii) Describe the locus of B if b varies.

(d) The points A, P and R correspond to the complex numbers 2, $3 + i$ and $3 + 2i$ respectively. Triangles OAP and ORQ are similar with corresponding angles as indicated. Find the complex number represented by Q. (2 marks)



Question 1 **16 marks**

(a) The graph of $y = f(x)$ is shown at right.

Sketch graphs of the following, clearly showing any axes intercepts and turning points.

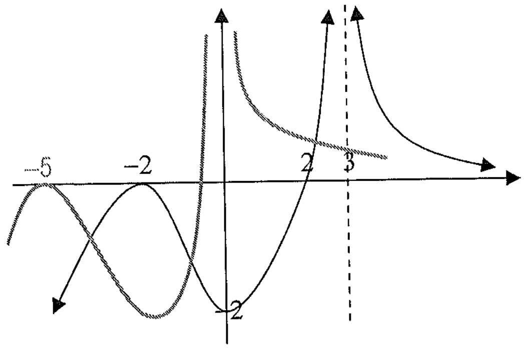
(i) $y = f(x+3)$ (ii) $y = f(|x|)$

(iii) $y = \dfrac{1}{f(x)}$ (iv) $y^2 = -f(x)$

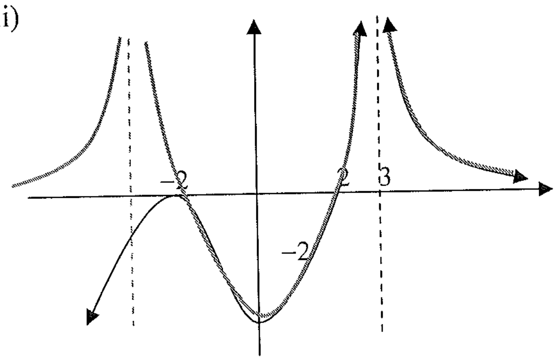
(v) $y = \ln(f(x))$ (vi) $y = e^{f(x)}$

(vii) $y = \tan^{-1}(f(x))$

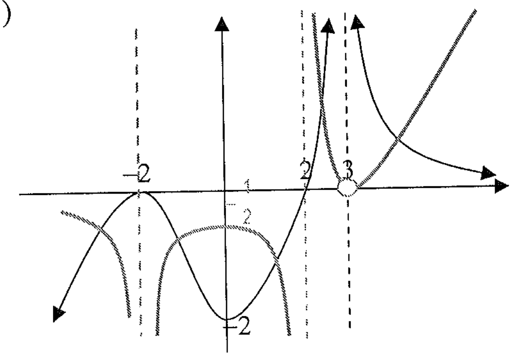
(i)



(ii)

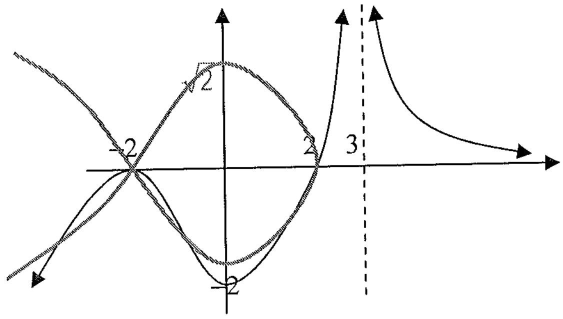


(iii)

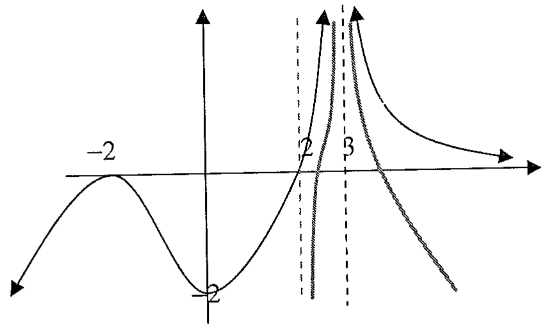


14m

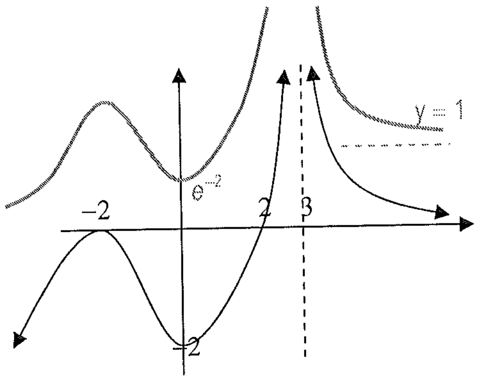
(iv)



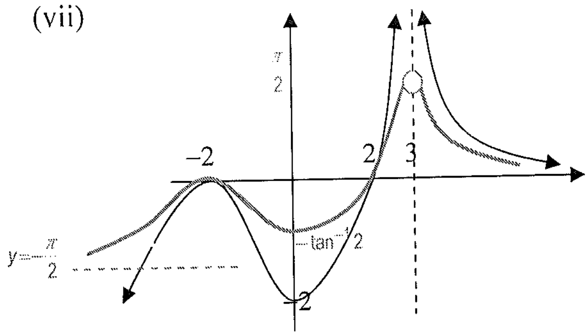
(v)



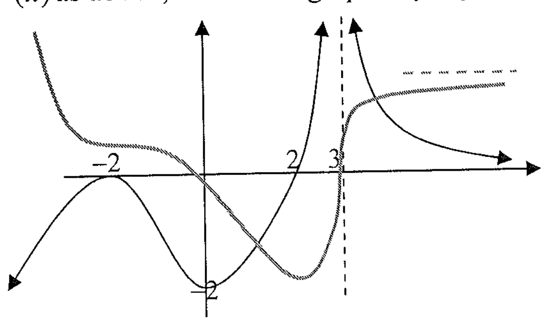
(vi)



(vii)



(b) Given the graph of $y = f'(x)$ as above, sketch the graph of $y = f(x)$ given that $f(0) = 0$ and $f(x)$ is continuous.



2m

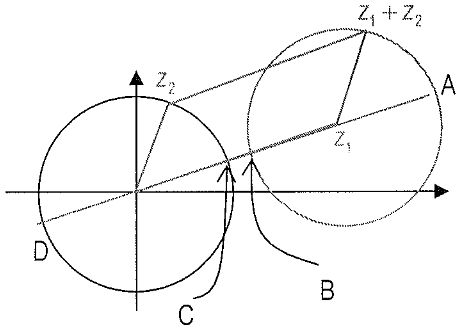
(c) The polynomial $P(z)$ has the equation $P(z) = z^4 - 2z^3 + Az^2 + Bz + 10$, where A, B are real. **24 marks**

Given that $2 + i$ is a zero of $P(z)$, find all the roots of $P(z)$ and write $P(z)$ as a product of two real quadratic factors.

$2 - i$ is also a root because all coefficients are real. $\therefore (z-2+i)(z-2+i) = z^2 - 4z + 5$ is a factor.

$z^4 - 2z^3 + Az^2 + Bz + 10 = (z^2 - 4z + 5)(z^2 + 2z + 2)$. 2m

Other roots are $z = \dfrac{-2 \pm \sqrt{-4}}{2} = -1 \pm i$.



(d) $z_1 = 4 + 3i, |z_2| = 2$ find

(i) the locus of $z_1 + z_2$.

As z_1 is a fixed point and the distance from z_1 to $(z_1 + z_2)$ is a constant, so the locus is a circle of centre z_1, radius 2. 1m

(ii) the ranges of $|z_1 + z_2|$ and $|z_1 - z_2|$.

$z_1 + z_2$ is the distance from the origin to $z_1 + z_2$ so, it's max at A $(= 5 + 2 = 7)$and min at B$(= 5 - 2 = 3)$. $\therefore 3 \le |z_1 + z_2| \le 7$. 3m

$z_1 - z_2$ is the distance from z_1 to z_2 so, it's max at D and min at C. $\therefore 3 \le |z_1 - z_2| \le 7$.

(iii) the range of $\arg \dfrac{z_1}{z_1 - z_2}$.

$\arg \dfrac{z_1 - 0}{z_1 - z_2} = \alpha =$ angle Oz_1z_2, so it's max when z_1z_2 is tangent to the circle $|z_2| = 2$. $\therefore \sin\alpha = \dfrac{2}{5}$. $\therefore \alpha = 24°$. 2m

$\therefore -24° \le \arg \dfrac{z_1 - 0}{z_1 - z_2} \le 24°$.

Question 2

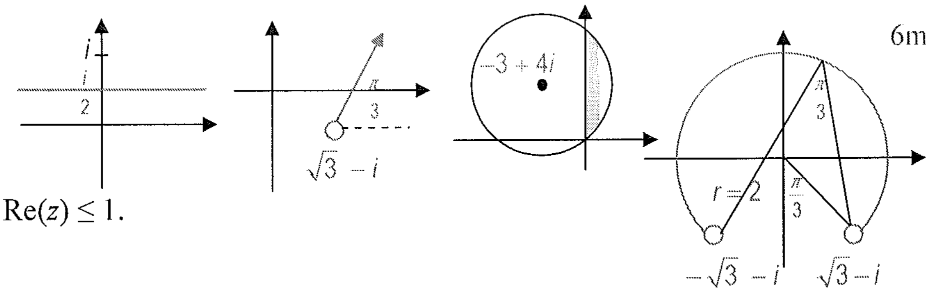
(a) Sketch the following loci: 6m

(i) $|z - i| = |z|$.

(ii) $\arg(z - \sqrt{3} + i) = \dfrac{\pi}{3}$.

(iii) $|z + 3 - 4i| \le 5$ and $0 \le \mathrm{Re}(z) \le 1$.

(iv) $\arg\left(\dfrac{z - \sqrt{3} + i}{z + \sqrt{3} + i}\right) = \dfrac{\pi}{3}$.



(b) (i) Express $z = 1 + i$ in modulus-argument form. $\sqrt{2}\,\mathrm{cis}\dfrac{\pi}{4}$ 1m

(ii) Hence write z^9 in the form $a + ib$ where a and b are real.

$\left(\sqrt{2}\,\mathrm{cis}\dfrac{\pi}{4}\right)^9 = 16\sqrt{2}\,\mathrm{cis}\dfrac{9\pi}{4} = 16\sqrt{2}\,\mathrm{cis}\dfrac{\pi}{4} = 16\sqrt{2}\left(\dfrac{1}{\sqrt{2}} + i\dfrac{1}{\sqrt{2}}\right) = 16 + 16i.$ 2m

(c) In an Argand diagram, point A corresponds to the complex number $1 + bi$, where b is real.

(i) If O is the origin, what complex numbers correspond to the vertices B and C if $OABC$ is a square, assuming O, A, B, C are in anticlockwise order?

$\overrightarrow{OC} = \overrightarrow{OA}$ rot. 90°, $\therefore C = iA = i(1 + bi) = -b + i$ 3m

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC}$, $\therefore B = 1 + bi + (-b + i) = (1 - b) + i(1 + b)$

(ii) Describe the locus of B if b varies. Let $x = 1 - b, y = 1 + b, \therefore x + y = 2$. The locus of B is the line $x + y = 2$. 2m

(d) The points A, P and R correspond to the complex numbers 2, $3 + i$ and $3 + 2i$ respectively.

Triangles OAP and ORQ are similar with corresponding angles as indicated.

Find the complex number represented by Q.

$\dfrac{OR}{OA} = \dfrac{OQ}{OP}, \therefore OQ = \dfrac{OR.OP}{OA} = \dfrac{|r||p|}{2}$.

$\angle QOR = \angle POA, \therefore \arg(q) - \arg(r) = \arg(p), \therefore \arg(q) = \arg(p) + \arg(r) = \arg(p) + \arg(r) - \arg(2) = \arg(\dfrac{pr}{2})$. 2m

$\therefore q = \dfrac{pr}{2} = \dfrac{(3+i)(3+2i)}{2} = \dfrac{7 + 9i}{2}$.