Student Name: ____________________

Teacher: ____________________

2012
TRIAL HSC
EXAMINATION

Hurlstone Agricultural High School

Mathematics Extension 1

Examiners

Mr J. Dillon, Mrs S. Hackett, Mr G. Rawson and Mrs D. Crancher

General Instructions

- Reading time - 5 minutes.
- Working time - 2 hours.
- Write using black or blue pen. Diagrams may be drawn in pencil.
- Board-approved calculators and mathematics templates may be used.
- A table of standard integrals is provided at the back of this paper.
- Show all necessary working in Questions 11-14.
- Start each question in a separate answer booklet.
- Put your student number on each booklet.

Total marks - 70

Section I

10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II

60 marks

- Attempt Questions 11-14. Each of these four questions are worth 15 marks
- Allow about 1 hour 45 minutes for this section

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\; n\neq -1;\; x\neq 0,\text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\; x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\; a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\; a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\; a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\; a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\; a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\; a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\; a>0,\; -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\; x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x,\; x>0$

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

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1. What is another expression for $\cos(x+y)$?

(A) $\cos x\cos y-\sin x\sin y$ (B) $\sin x\cos y+\cos x\sin y$
(C) $\cos x\cos y+\sin x\sin y$ (D) $\sin x\cos y-\cos x\sin y$

2. Which of the following is an expression for $\int\cos^2 2x\,dx$?

(A) $x-\frac{1}{4}\sin 4x+c$ (B) $x+\frac{1}{4}\sin 4x+c$
(C) $\frac{x}{2}-\frac{1}{8}\sin 4x+c$ (D) $\frac{x}{2}+\frac{1}{8}\sin 4x+c$

3. What is the domain of the function $f(x)=2\sin^{-1}\left(\frac{x}{2}\right)$?

(A) $-\pi\le x\le\pi$ (B) $-2\pi\le x\le 2\pi$
(C) $-1\le x\le 1$ (D) $-2\le x\le 2$

4. Which of the following is the exact value of $\int_{\frac{3}{\sqrt{2}}}^{3}\frac{4}{\sqrt{9-x^2}}dx$?

(A) $-\pi$ (B) $-\frac{\pi}{4}$ (C) $\frac{\pi}{4}$ (D) π

5. Given $f(x)=\frac{3}{x}-4$, $f^{-1}(4)=?$

(A) $-\frac{13}{4}$ (B) $\frac{13}{4}$ (C) $\frac{3}{8}$ (D) $-\frac{3}{8}$

6 Mathematical induction is used to prove

$$1^2+3^2+5^2+...+(2n-1)^2=\frac{1}{3}n(2n+1)(2n-1) \text{ for all positive integers } n\ge 1.$$

Which of the following has an incorrect expression for part of the induction proof?

(A) Step 1: To prove the statement true for $n=1$

$\text{LHS}=1^2=1 \quad \text{RHS}=\frac{1}{3}\times 1\times(2\times 1+1)(2\times 1-1)=1$

Result is true for $n=1$

(B) Step 2: Assume the result true for $n=k$

$1^2+3^2+5^2+...+(2k-1)^2=\frac{1}{3}(k+1)(2k+1)(2k-1)$

(C) To prove the result is true for $n=k+1$

$1^2+3^2+5^2+...+(2k-1)^2+\big(2(k+1)-1\big)^2$

$=\frac{1}{3}(k+1)\big(2(k+1)+1\big)\big(2(k+1)-1\big)$

$=\frac{1}{3}(k+1)(2k+3)(2k+1)$

(D) $\text{LHS}=1^2+3^2+5^2+...+(2k-1)^2+\big(2(k+1)-1\big)^2$

$=\frac{1}{3}k(2k+1)(2k-1)+(2k+1)^2$

$=\frac{1}{3}(2k+1)\big(k(2k-1)+3(2k+1)\big)$

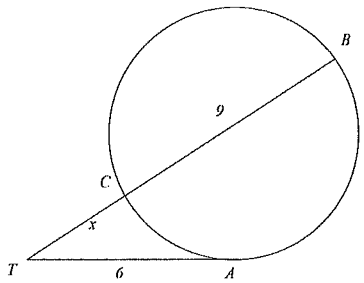
$=\frac{1}{3}(2k+1)\big(2k^2-k+6k+3\big)$

$=\frac{1}{3}(2k+1)\big(2k^2+5k+3\big)$

$=\frac{1}{3}(2k+1)(k+1)(2k+3)$

$=\text{RHS}$

7 Line TA is a tangent to the circle at A and TB is a secant meeting the circle at B and C.



Given that $TA=6$, $CB=9$ and $TC=x$, what is the value of x?

(A) -12 (B) 2 (C) 3 (D) 4

8 At a dinner party, the host, hostess and their six guests sit at a round table. In how many ways can they be arranged if the host and hostess are separated?

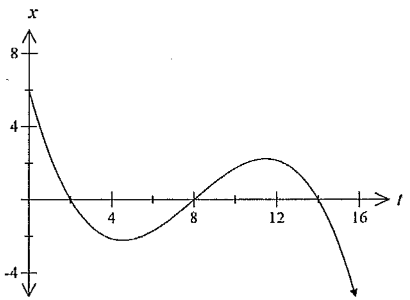
(A) 720 (B) 1440 (C) 3600 (D) 5040

9 A curve has parametric equations $x=\frac{2}{t}$ and $y=2t^2$.

What is the Cartesian equation of this curve?

(A) $y=\frac{4}{x}$ (B) $y=\frac{8}{x}$ (C) $y=\frac{4}{x^2}$ (D) $y=\frac{8}{x^2}$

10 The displacement, x metres, from the origin of a particle moving in a straight line at any time (t seconds) is shown in the graph.



When was the particle at rest?

(A) $t=4\cdot5$ and $t=11\cdot5$ (B) $t=0$

(C) $t=2$, $t=8$ and $t=14$ (D) $t=8$

Section II

60 marks
Attempt Questions 11 – 14
Allow about 1 hour and 45 minutes for this section

Answer each question in a new answer booklet.

All necessary working should be shown in every question.

Question 11 (15 marks) **Start a new answer booklet** **Marks**

(a) Solve $\frac{3}{x+2}<1$ **2**

(b) At any point on the curve $y=f(x)$, the gradient function is given by $\frac{dy}{dx}=\sin^2 x$. Show that the value of $f\left(\frac{3\pi}{4}\right)-f\left(\frac{\pi}{4}\right)$ is equal to $\frac{\pi}{4}+\frac{1}{2}$. **3**

(c) Find the general solutions of $2\sin^3 x-\sin x-2\sin^2 x+1=0$ **3**

(d) Show that $\frac{1+\cos x}{\sin x}=\frac{1}{t}$ for $t=\tan\frac{x}{2}$. **2**

(e) Show that $\sin A\cos B=\frac{1}{2}[\sin(A+B)+\sin(A-B)]$ **2**

(f) A vertical tower of height h metres stands on horizontal ground.

From a point P on the ground due east of the tower the angle of elevation of the top of the tower is 45 degrees.

From a point Q on the ground due south of the tower the angle of elevation of the top of the tower is 30 degrees.

(i) Draw a neat diagram in your answer booklet to represent this situation. **1**

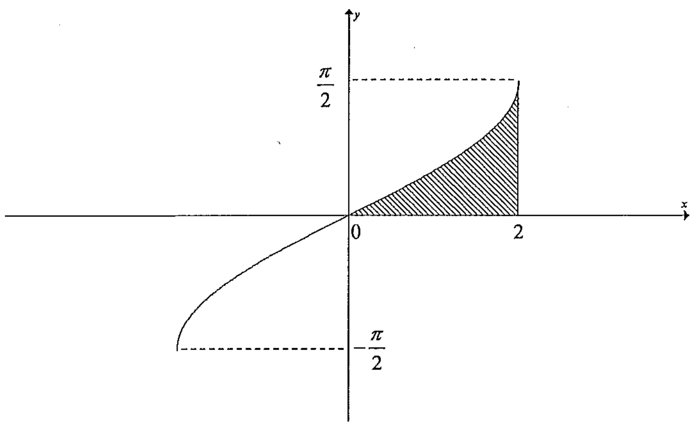
(ii) If the distance PQ is 40 metres, find the exact height of the tower. **2**

Question 12 (15 marks) **Start a new answer booklet** **Marks**

(a) What is the inverse function of $y=\dfrac{1}{4-x}$? **1**

(b) Show that $\dfrac{d}{dx}\left(x\cos^{-1}x-\sqrt{1-x^2}\right)=\cos^{-1}x$ **2**

(c)



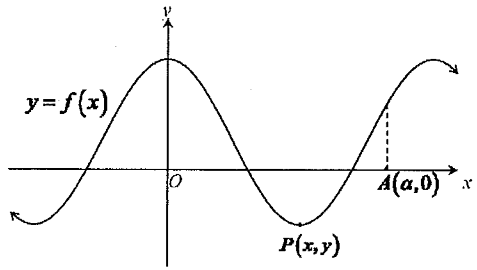
The shaded area is represented by $\displaystyle\int_0^2 \sin^{-1}\frac{x}{2}\,dx$.

Explain why the area is given by $\pi-2\displaystyle\int_0^{\frac{\pi}{2}}\sin y\,dy$. **2**

Question 12 continued next page.

Question 12 continued this page.

(d) Let $f(x)=1+2\cos\dfrac{x}{2}$. The diagram shows the graph $y=f(x)$.



(i) State the period and the amplitude of the curve given that x is expressed in radians. **2**

(ii) The point $P(x,y)$ is a turning point on the curve. Find its coordinates. **1**

(iii) What is the largest possible positive domain, containing $x=0$, for which $y=f(x)$ has an inverse function? **1**

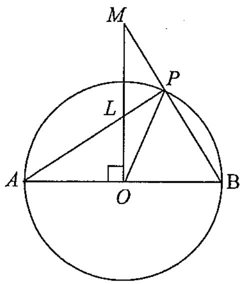
(iv) Find the equation of $f^{-1}(x)$ for this restricted domain of $f(x)$. **2**

(v) Sketch the curve $y=f^{-1}(x)$. **2**

(vi) $A(\alpha,0)$ lies to the right of P as indicated in the diagram above.
Find a simplified expression for the exact value of $y=f^{-1}(f(\alpha))$. **2**

Question 13 (15 marks) **Start a new answer booklet** **Marks**

(a) (i)



NOT TO SCALE

O is the centre of the circle ABP. $MO \perp AB$. M, P and B are collinear. MO intersects AP at L.

(α) Prove that A, O, P and M are concyclic. **2**

(β) Prove that $\angle OPA = \angle OMB$. **2**

(ii) Use Mathematical Induction to prove that

$$4 \times 2^n + 3^{3n}$$

is divisible by 5 for all integers n, $n \geq 0$. **3**

(b) (i) The mass M of a male silverback gorilla is modelled by $M = 200 - 198e^{-kt}$, where M is measured in kilograms, t is the age of the gorilla in years and k is a positive constant.

(α) Show that the rate of growth of the mass of the gorilla is given by the differential equation $\frac{dM}{dt} = k(200 - M)$ **1**

(β) When the gorilla is 6 years old its mass is 68 kilograms. Find the value of k, correct to three decimal places. **2**

(γ) According to this model, what is the limiting mass of the gorilla? **1**

(ii) (α) Show that $\frac{u^3}{u+1} = u^2 - u + 1 - \frac{1}{u+1}$ **1**

(β) Hence, by using the substitution $u = \sqrt{x}$, show that

$$\int_0^4 \frac{x}{1+\sqrt{x}}\,dx = \frac{16}{3} - \ln 9$$ **3**

Question 14 (15 marks) **Start a new answer booklet** **Marks**

(a) How many distinct eight letter arrangements can be made using the letters of the word PARALLEL? **2**

(b) Alex's playlist consists of 40 different songs that can be arranged in any order.

(i) How many arrangements are there for the 40 songs? **1**

(ii) Alex decides that she wants to play her three favourite songs first, in any order.

How many arrangements of the 40 songs are now possible? **1**

(c) Two particles moving in a straight line are initially at the origin. The velocity of one particle is $\frac{2}{\pi}$ m/s and the velocity of the other particle at time t seconds is given by $v = -2\cos t$ m/s.

(i) Determine equations that give the displacements, x_1 and x_2 metres, of the particles from the origin at time t seconds. **2**

(ii) Hence, or otherwise, show that the particles will never meet again. **2**

(d) $P(2p, p^2)$ and $Q(2q, q^2)$ are two points on the parabola $x^2 = 4y$.

(i) Show that the equation of the tangent to $x^2 = 4y$ at P is $y = px - p^2$. **2**

(ii) Show that the coordinates of T, the point of intersection of the tangents from P and Q is given by $(p+q,\ pq)$. **1**

(iii) Given that $\frac{1}{p} + \frac{1}{q} = 2$, find the equation of the locus of T. **1**

(e) A resting adult's breathing cycle is 5 seconds long.

For time t seconds, $0 \leq t \leq 2\frac{1}{2}$, air is taken into the lungs.

For $2\frac{1}{2} < t < 5$ air is expelled from the lungs.

The rate, R litres/second, at which air is taken in or expelled from the lungs can be modelled on the equation $R = \frac{1}{2}\sin\left(\frac{2\pi}{5}t\right)$.

How many litres of air does a resting adult take **into** their lungs during one breathing cycle? **3**

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Year 12 Questions 1 to 10 Multiple Choice Guidelines	Extension 1 Mathematics Solutions and Marking	Task 4-Trial HSC 2012

Outcome Addressed in this Question

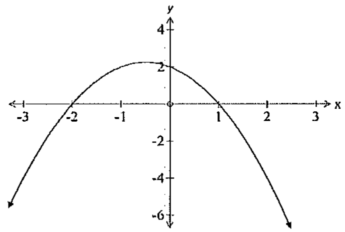
PE5 determines derivatives which require the application of more than one rule of differentiation
HE4 uses the relationship between functions, inverse functions and their derivatives
HE2 uses inductive reasoning in the construction of proofs
PE2 uses multi-step deductive reasoning in a variety of contexts
PE3 solves problems involving permutations and combinations, inequalities, polynomials, circle geometry and parametric representations
PE4 uses the parametric representation together with differentiation to identify geometric properties of parabolas
HE3 uses a variety of strategies to investigate mathematical models of situations involving binomial probability, projectiles, simple harmonic motion, or exponential growth and decay
HE7 evaluates mathematical solutions to problems and communicates them in an appropriate form

Outcome	Solutions	Marking Guidelines
HE7	**Question 1.** $\cos(x+y)=\cos x\cos y-\sin x\sin y$ **Answer is A**	**1 mark for correct answer**
PE5	**Question 2.** $\int\cos^2 2x\,dx=\int\left(\frac{1}{2}+\frac{1}{2}\cos 4x\right)dx$ $=\frac{1}{2}x+\frac{1}{8}\sin 4x+C$ $=\frac{x}{2}+\frac{1}{8}\sin 4x+C$ **Answer is D**	**1 mark for correct answer**
HE4	**Question 3.** Now, $-1\le\frac{x}{2}\le 1$ $-2\le x\le 2$ Therefore domain is $-2\le x\le 2$. **Answer is D**	**1 mark for correct answer**
HE4	**Question 4.** $\int_{\frac{3}{\sqrt{2}}}^{3}\frac{4}{\sqrt{9-x^2}}dx$ $=\left[4\sin^{-1}\left(\frac{x}{3}\right)\right]_{\frac{3}{\sqrt{2}}}^{3}$ $=4\sin^{-1}1-4\sin^{-1}\frac{1}{\sqrt{2}}$ $=4\left(\frac{\pi}{2}\right)-4\left(\frac{\pi}{4}\right)$ $=2\pi-\pi$ $=\pi$ **Answer is D**	**1 mark for correct answer**
HE4	**Question 5.** Function is: $f(x)=\frac{3}{x}-4$ *i.e.* $y=\frac{3}{x}-4$ Inverse function is: $x=\frac{3}{y}-4$ $x+4=\frac{3}{y}$ $y=\frac{3}{x+4}$ *i.e.* $f^{-1}(x)=\frac{3}{x+4}$ Therefore, $f^{-1}(4)=\frac{3}{4+4}=\frac{3}{8}$ **Answer is C**	**1 mark for correct answer**
HE2	**Question 6.** The incorrect expression for part of the induction proof is: (B) Step 2: Assume the result true for $n=k$ $1^2+3^2+5^2+...+(2k-1)^2=\frac{1}{3}(k+1)(2k+1)(2k-1)$ **Answer is B**	**1 mark for correct answer**

Outcome	Solutions	Marking Guidelines
	Question 7.	
PE2 PE3	$6^2 = x(x+9)$ [The square of the intercept on tangent to a circle equals the product of the intercepts on the secant] $36 = x^2 + 9x$ $x^2 + 9x - 36 = 0$ $(x+12)(x-3) = 0$ $x = -12,\ 3$ The value of x is 3. **Answer is C**	**1 mark for correct answer**
	Question 8.	
PE3	$5 \times 6! = 3600$ **Answer is C**	**1 mark for correct answer**
	Question 9.	
PE4	Now, $x = \frac{2}{t}$ ....A Rearrange A $t = \frac{2}{x}$ $y = 2t^2$ ....B *substitute* $t = \frac{2}{x}$ *into B* $\therefore y = 2\left(\frac{2}{x}\right)^2$ $\therefore y = \frac{8}{x^2}$ **Answer is D**	**1 mark for correct answer**
	Question 10.	
HE3	The particle is at rest when $\frac{dy}{dx} = 0$ which occurs at the maximum and minimum turning points. *i.e.* $t = 4\cdot5$ and $t = 11\cdot5$ **Answer is A**	**1 mark for correct answer**

Outcome Addressed in this Question

PE3	solves problems involving permutations and combinations, inequalities, polynomials, circle geometry and parametric representations
HE7	evaluates mathematical solutions to problems and communicates them in an appropriate form
PE6	makes comprehensive use of mathematical language, diagrams and notation for communicating in a wide variety of situations
H4	expresses practical problems in mathematical terms based on simple given models

Outcome	Solutions	Marking Guidelines
	Question 11.	
PE3	(a) $\frac{3}{x+2} < 1 \qquad x \neq -2$ $(x+2)^2 \times \frac{3}{(x+2)} < 1 \times (x+2)^2$ $3(x+2) < 1(x+2)^2$ $(x+2)(3-x-2) < 0$ $(x+2)(1-x) < 0$  $x < -2$ or $x > 1$	**2 Marks for complete correct solution.** **1 Mark for finding one correct solution or multiplying both sides of the inequality by $(x+2)^2$.**
HE7	(b) $\frac{dy}{dx} = \sin^2 x$ $\frac{dy}{dx} = \frac{1}{2} - \frac{1}{2}\cos 2x$ $y = \frac{1}{2}x - \frac{1}{2} \times \frac{\sin 2x}{2} + C$ $y = \frac{x}{2} - \frac{1}{4}\sin 2x + C$ $f\left(\frac{3\pi}{4}\right) - f\left(\frac{\pi}{4}\right) = \left(\frac{3\pi}{8} - \frac{1}{4}\sin\frac{3\pi}{2} + C\right) - \left(\frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2} + C\right)$ $= \left(\frac{3\pi}{8} + \frac{1}{4}\right) - \left(\frac{\pi}{8} - \frac{1}{4}\right)$ $= \frac{\pi}{4} + \frac{1}{2}$	**3 Marks for complete correct solution** **2 Marks for finding correct equation for y and correctly substituting into y for both values** **1 Mark for finding correct equation for y**

HE7

(c)

$$2\sin^3 x-\sin x-2\sin^2 x+1=0$$
$$2\sin^2 x(\sin x-1)-(\sin x-1)=0$$
$$(\sin x-1)(2\sin^2 x-1)=0$$
$$\therefore \sin x=1,\ \sin x=\frac{1}{\sqrt{2}},\ \sin x=-\frac{1}{\sqrt{2}}$$
$$\therefore x=n\pi+(-1)^n\frac{\pi}{2},\ x=n\pi+(-1)^n\frac{\pi}{4},\ x=n\pi-(-1)^n\frac{\pi}{4}$$

where n is any integer.

3 marks for correct solution

2 mark for factorising the equation and then solving the resulting equation correctly

1 mark for factorising correctly

HE7

(d)

$$\sin x=\frac{2t}{1+t^2}\ and\ \cos x=\frac{1-t^2}{1+t^2}\ for\ t=\tan\frac{x}{2}$$

Now,

$$LHS=\frac{1+\cos x}{\sin x}$$
$$=\frac{1+\frac{1-t^2}{1+t^2}}{\frac{2t}{1+t^2}}$$
$$=\frac{1+t^2+1-t^2}{2t}$$
$$=\frac{2}{2t}$$
$$=\frac{1}{t}$$
$$=RHS$$

$$\therefore \frac{1+\cos x}{\sin x}=\frac{1}{t}\quad for\quad t=\tan\frac{x}{2}.$$

2 marks for complete correct solution

1 mark for correctly substituting the t results

HE7

(e)

$$RHS=\frac{1}{2}[\sin(A+B)+\sin(A-B)]$$
$$=\frac{1}{2}(\sin A\cos B+\cos A\sin B+\sin A\cos B-\cos A\sin B)$$
$$=\frac{1}{2}(2\sin A\cos B)$$
$$=\sin A\cos B$$
$$=LHS$$

$$\therefore \sin A\cos B=\frac{1}{2}[\sin(A+B)+\sin(A-B)]$$

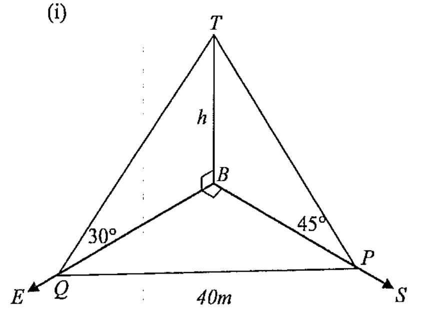
2 marks for complete correct solution

1 mark for correct expansions

PE6

(f)

(i)



1 mark for correct diagram

H4, HE7

(ii)

$$\tan 45°=\frac{h}{PB}\qquad \tan 30°=\frac{h}{BQ}$$
$$PB=\frac{h}{\tan 45°}\qquad BQ=\frac{h}{\tan 30°}$$
$$PB=\frac{h}{1}\qquad BQ=\frac{h}{\frac{1}{\sqrt{3}}}$$
$$\therefore PB=h\qquad BQ=\sqrt{3}h$$

Now,

$$PB^2+BQ^2=40^2$$
$$h^2+3h^2=1600$$
$$4h^2=1600$$
$$h^2=400$$
$$h=\sqrt{400}$$
$$h=20$$

Therefore, the exact height of the tower is 20 metres.

2 marks for complete correct solution

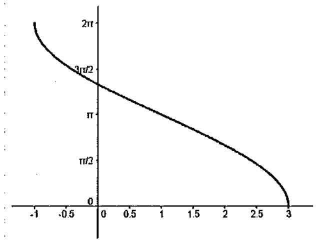
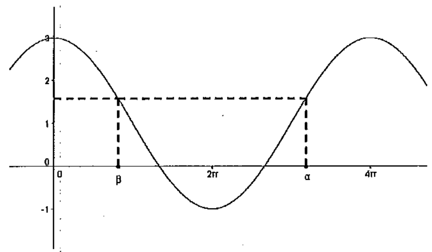
1 mark for partial correct solution.

Year 12 Trial HSC 2012 – Extension 1 Mathematics

Question No: 12 **Solutions and Marking Guidelines**

Outcomes Addressed in this Question:

HE4 Uses the relationship between functions, inverse functions and their derivatives.

Outcome	Sample Solution	Marking Guidelines
HE4	a) Inverse: $x=\frac{1}{4-y}$ $4-y=\frac{1}{x}$ $y=4-\frac{1}{x}$	**1 mark** – correct answer
HE4	b) $\frac{d}{dx}\left(x\cos^{-1}x-\sqrt{1-x^2}\right)=\cos^{-1}x.1+x.\frac{1}{\sqrt{1-x^2}}-\frac{1}{2}(1-x^2)^{-\frac{1}{2}}.2x$ $=\cos^{-1}x+\frac{x}{\sqrt{1-x^2}}-\frac{x}{\sqrt{1-x^2}}$ $=\cos^{-1}x$	**2 marks** – correct solution clearly showing all steps **1 mark** – substantial progress towards correct solution
HE4	c) Shaded area =Area of rectangle - area between curve and y axis bounded by y=0 and y=$\frac{\pi}{2}$ Shaded area $=2\times\frac{\pi}{2}-\int_0^{\frac{\pi}{2}}x\,dy$ Given $y=\sin^{-1}\frac{x}{2}$, $x=2\sin y$ Shaded area $=\pi-\int_0^{\frac{\pi}{2}}2\sin y\,dy$ Shaded area $=\pi-2\int_0^{\frac{\pi}{2}}\sin y\,dy$	**2 marks** – correct explanation, clearly showing all steps **1 mark** – substantial progress towards correct explanation
HE4	d) i) period $=\frac{2\pi}{1/2}=4\pi$ amplitude $=2$	**2 marks** – two correct answers **1 mark** – one correct answer
HE4	d) ii) $P(2\pi,-1)$	**1 mark** – correct answer
HE4	d) iii) $0\le x\le 2\pi$	**1 mark** – correct answer
HE4	d) iv) $y=1+2\cos\frac{x}{2}$ Inverse: $x=1+2\cos\frac{y}{2}$ $\frac{x-1}{2}=\cos\frac{y}{2}$ $y=2\cos^{-1}\left(\frac{x-1}{2}\right)$ $f^{-1}(x)=2\cos^{-1}\left(\frac{x-1}{2}\right)$	**2 marks** – correct solution **1 mark** – substantial progress towards correct solution
HE4	d) v) 	**2 marks** – correct graph, clearly showing all key features **1 mark** – substantial progress towards correct graph
HE4	d) vi)  $\beta=2\pi-(\alpha-2\pi)$ $=4\pi-\alpha$ $\therefore f^{-1}[f(\alpha)]=f^{-1}[f(\beta)]$ since $f(\alpha)=f(\beta)$ $=\beta$ $=4\pi-\alpha$	**2 marks** – correct solution **1 mark** – substantial progress towards correct solution

Year 12	Extension 1 Mathematics	Trial HSC 2012
Question 13	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

PE3	solves problems involving circle geometry
HE2	uses inductive reasoning in the construction of proofs
HE3	uses a variety of strategies to investigate mathematical models of situations involving exponential growth and decay
HE6	determines integrals by reduction to a standard form through a given substitution

Outcome | Solutions | Marking Guidelines

PE3 — (a)(i)(α)

$\angle APB = 90^0$ (angle at the circumference in a semi-circle equals 90^0)

$\therefore \angle APM + 90^0 = 180^0$ (angle sum of straight angle MPB equals 180^0)

$\therefore \angle APM = 90^0$

$\angle AOM = 90^0$ (given)

$\therefore \angle APM = \angle AOM = 90^0$

$\therefore AOPM$ is cyclic (AM subtends equal angles on the same side at O and P)

$\therefore A, O, P$ and M are concyclic.

Marking Guidelines:

Award 2
Correct solution.

Award 1
Correct solution with insufficient reasoning provided

PE3 — (a)(i) (β)

$AO = OP$ (radii)

$\therefore \angle PAO = \angle OPA$ (equal angles are opposite equal sides in ΔOPA)

But, $\angle OAP = \angle OMP$ (angles at the circumference in the same segment are equal)

$\therefore \angle OPA = \angle OMB \ (= \angle OMP)$

Marking Guidelines:

Award 2
Correct solution.

Award 1
Correct solution with insufficient reasoning provided

HE2 — (a) (ii)

Show true for $n = 0$,

$4 \times 2^0 + 3^{3.0} = 4 + 1 = 5$ which is divisible by 5.

$\therefore$ True for $n = 0$

Show true for $n = 1$,

$4 \times 2^1 + 3^{3.1} = 8 + 27 = 35$ which is divisible by 5.

$\therefore$ True for $n = 1$

Assume true for $n = k$, i.e. $4 \times 2^k + 3^{3k} = 5M$, M is an integer.

Prove true for $n = k+1$, i.e. Show $4 \times 2^{k+1} + 3^{3(k+1)} = 5J$, J is an integer.

$$
\begin{aligned}
4 \times 2^{k+1} + 3^{3(k+1)} &= 4 \times 2 \times 2^k + 3^{3(k+1)} \\
&= 2 \times \left(5M - 3^{3k}\right) + 3^{3k} \times 3^3 \\
&= 2 \times 5M - 2 \times 3^{3k} + 27 \times 3^{3k} \\
&= 2 \times 5M + 25 \times 3^{3k} \\
&= 5\left(2M + 5 \times 3^{3k}\right) \\
&= 5J, \ J \text{ is an integer}
\end{aligned}
$$

$\therefore$ True for $n = k+1$.

$\therefore$ True by mathematical induction for $n \geq 0$

Marking Guidelines:

Award 3
Correct solution.

Award 2
Attempts to prove true for $n = k+1$, after proving true for $n = 0$ and using assumption.

Award 1
Proves true for $n = 0$.

HE3 — (b)(i)(α)

$M = 200 - 198e^{-kt}$

$$\text{LHS} = \frac{dM}{dt} = -198. - ke^{-kt} = k.198e^{-kt}$$

$$\text{RHS} = k(200 - M) = k\left(200 - \left(200 - 198e^{-kt}\right)\right) = k.198e^{-kt} = \text{LHS}$$

$\therefore M = 200 - 198e^{-kt}$ is a solution to $\frac{dM}{dt} = k(200 - M)$

Marking Guidelines:

Award 1 for correct solution.

HE3 — (b)(i)(β)

$t = 6$, $M = 68$

$68 = 200 - 198e^{-6k}$

$198e^{-6k} = 132$

$e^{-6k} = \frac{132}{198} = \frac{2}{3}$

$-6k = \ln\left(\frac{2}{3}\right)$

$k = \frac{1}{6}\ln\left(\frac{3}{2}\right) \approx 0.06757751802$

$\therefore k = 0.068$ (to 3 decimal places)

Marking Guidelines:

Award 2 for correct solution.

Award 1 for substantial progress towards solution.

HE3 — (b)(i)(γ)

As $t \to \infty$, $M \to 200$

$\therefore$ Limiting mass $= 200\,\text{kg}$

Marking Guidelines:

Award 1 for correct solution.

(b)(ii)(α)

$$\text{LHS} = \frac{u^3}{u+1}$$

$$
\begin{aligned}
\text{RHS} &= u^2 - u + 1 - \frac{1}{u+1} \\
&= \frac{u^2(u+1) - u(u+1) + 1(u+1) - 1}{u+1} \\
&= \frac{u^3 + u^2 - u^2 - u + u + 1 - 1}{u+1} \\
&= \frac{u^3}{u+1} \\
&= \text{LHS}
\end{aligned}
$$

Marking Guidelines:

Award 1 for correct solution.

HE6 — (b)(ii)(β)

$\int_0^4 \frac{x}{1+\sqrt{x}}\,dx \quad u = \sqrt{x} \Rightarrow dx = 2\sqrt{x}\,du = 2u\,du$

$$
\begin{aligned}
&= \int_0^2 \frac{u^2}{1+u} . 2u\,du \\
&= 2\int_0^2 \frac{u^3}{u+1}\,du \\
&= 2\int_0^2 \left(u^2 - u + 1 - \frac{1}{u+1}\right) du \\
&= 2\left[\frac{u^3}{3} - \frac{u^2}{2} + u - \ln(u+1)\right]_0^2 \\
&= 2\left[\frac{8}{3} - \frac{4}{2} + 2 - \ln(3) - \left(0 - 0 + 0 - \ln(1)\right)\right] \\
&= 2\left[\frac{8}{3} - \ln(3)\right] \\
&= \frac{16}{3} - 2\ln(3) \\
&= \frac{16}{3} - \ln(9)
\end{aligned}
$$

Marking Guidelines:

Award 3
Correct solution.

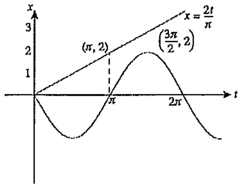
Award 2
Substantial progress towards solution

Award 1
Limited progress towards solution

Year 12	Mathematics Extension 1	TRIAL EXAM 2012
Question No. 14	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

PE3 - solves problems involving permutations and combinations, inequalities, polynomials, circle geometry and parametric representations
H4 - expresses practical problems in mathematical terms based on simple given models
H5 - applies appropriate techniques from the study of calculus, geometry, probability, trigonometry and series to solve problems
H9 - communicates using mathematical language, notation, diagrams and graphs
HE7 - evaluates mathematical solutions to problems and communicates them in an appropriate form

Outcome	Solutions	Marking Guidelines
PE3	(a) Arrangements $=\frac{8!}{3!2!}=\frac{40320}{12}=3360$ ways	**2 marks** : correct answer **1 mark** : partially correct solution (1 for 8! or 1 for division)
PE3	(b) (i) $40!\ (=8{\cdot}159\times10^{47})$	**1 mark** : correct solution
	(ii) $3!\times37!\ (=8{\cdot}258\times10^{43})$	**1 mark** : correct solution
H4, H5	(c) (i) $v_1=\frac{2}{\pi}$ $x_1=\frac{2t}{\pi}+C_1$ when $t=0, x=0$ $\Rightarrow C_1=0$ $\therefore\ x_1=\frac{2t}{\pi}$ $v_2=-2\cos t$ $x_2=-2\sin t+C_2$ when $t=0, x=0$ $\Rightarrow C_2=0$ $\therefore\ x_2=-2\sin t$	**2 marks** : correct solution **1 mark** : substantially correct solution
H4, H5, H9	(ii)  The graphs don't interesect again. $x=\frac{2t}{\pi}$ has a value greater than 2 for $x>\pi$, and the maximum value of $x=-2\sin t$ is 2.	**2 marks** : correct solution **1 mark** : substantially correct solution
PE3	(d) (i) To find the gradient of the tangent $y=\frac{1}{4}x^2$ $\frac{dy}{dx}=\frac{1}{2}x$ At $P(2p,p^2)$ $\frac{dy}{dx}=\frac{1}{2}\times2p=p$ Equation of the tangent at $P(2p,p^2)$ $y-y_1=m(x-x_1)$ $y-p^2=p(x-2p)$ $px-y-p^2=0$ (1)	**2 marks** : correct solution **1 mark** : substantially correct solution
PE3	(ii) Similarly at $Q(2q,q^2)$ the equation of the tangent is $qx-y-q^2=0$ (2) Eqn (1) − (2) $px-qx-p^2+q^2=0$ $(p-q)x=p^2-q^2$ $=(p-q)(p+q)$ $x=(p+q)$ Substitute $(p+q)$ for x into Eqn(1) $p(p+q)-y-p^2=0$ $y=p^2+pq-p^2$ $=pq$ The coordinates of T is $(p+q, pq)$	**1 mark** : correct solution
PE3	(iii) To find the equation of the locus eliminate p and q. Now $x=p+q$ and $y=pq$ (Coordinates of T) Given $\frac{1}{p}+\frac{1}{q}=2$ or $\frac{p+q}{pq}=2$ Therefore $\frac{x}{y}=2$ $y=\frac{x}{2}$	**1 mark** : correct solution
HE7	(e) $\frac{dV}{dt}=\frac{1}{2}\sin\left(\frac{2\pi}{5}t\right)$ $V=\int_0^{\frac{5}{2}}\frac{1}{2}\sin\left(\frac{2\pi}{5}t\right)dt$ $=-\frac{1}{2}.\frac{5}{2\pi}\left[\cos\left(\frac{2\pi}{5}t\right)\right]_0^{\frac{5}{2}}$ $=-\frac{5}{4\pi}[\cos\pi-\cos0]$ $=-\frac{5}{4\pi}.[-2]$ $=\frac{5}{2\pi}$ litres.	**3 marks** : correct solution **2 marks** : substantially correct solution **1 mark** : partially correct solution