Student Name: ____________________

Teacher: ____________________

2012
TRIAL HSC
EXAMINATION

Hurlstone Agricultural High School

Mathematics Extension 2

Examiners

Mr J. Dillon and Mr S. Gee

General Instructions

- Reading time - 5 minutes.
- Working time - 3 hours.
- Write using black or blue pen. Diagrams may be drawn in pencil.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- Show all necessary working in Questions 11-16.
- Start each question in a separate answer booklet.
- Put your student number on each booklet.

Total marks - 100

Section I

10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II

90 marks

- Attempt Questions 11-16. Each of these six questions are worth 15 marks
- Allow about 2 hour 45 minutes for this section

STANDARD INTEGRALS

$\int x^n\,dx$	$=\frac{1}{n+1}x^{n+1},\ n\neq -1;\ x\neq 0,\text{if } n<0$
$\int \frac{1}{x}\,dx$	$=\ln x,\ x>0$
$\int e^{ax}\,dx$	$=\frac{1}{a}e^{ax},\ a\neq 0$
$\int \cos ax\,dx$	$=\frac{1}{a}\sin ax,\ a\neq 0$
$\int \sin ax\,dx$	$=-\frac{1}{a}\cos ax,\ a\neq 0$
$\int \sec^2 ax\,dx$	$=\frac{1}{a}\tan ax,\ a\neq 0$
$\int \sec ax\tan ax\,dx$	$=\frac{1}{a}\sec ax,\ a\neq 0$
$\int \frac{1}{a^2+x^2}\,dx$	$=\frac{1}{a}\tan^{-1}\frac{x}{a},\ a\neq 0$
$\int \frac{1}{\sqrt{a^2-x^2}}\,dx$	$=\sin^{-1}\frac{x}{a},\ a>0,\ -a<x<a$
$\int \frac{1}{\sqrt{x^2-a^2}}\,dx$	$=\ln\left(x+\sqrt{x^2-a^2}\right)\ x>a>0$
$\int \frac{1}{\sqrt{x^2+a^2}}\,dx$	$=\ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE: $\ln x=\log_e x,\ x>0$

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

---

1 Consider the hyperbola with the equation $\frac{x^2}{144} - \frac{y^2}{25} = 1$.

What are the equations of the directrices?

(A) $y = \pm\frac{25}{13}$ (B) $y = \pm\frac{144}{13}$

(C) $x = \pm\frac{25}{13}$ (D) $x = \pm\frac{144}{13}$

2 The points $P(a\cos\theta, b\sin\theta)$ and $Q(a\cos\phi, b\sin\phi)$ lie on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the chord PQ subtends a right angle at $(0,0)$. Which of the following is the correct expression?

(A) $\tan\theta\tan\phi = -\frac{b^2}{a^2}$ (B) $\tan\theta\tan\phi = -\frac{a^2}{b^2}$

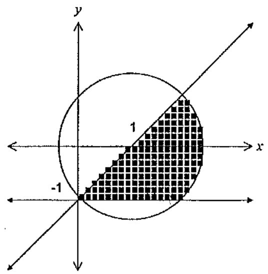
(C) $\tan\theta\tan\phi = \frac{b^2}{a^2}$ (D) $\tan\theta\tan\phi = \frac{a^2}{b^2}$

3 What is $-\sqrt{3} + i$ expressed in modulus-argument form?

(A) $\sqrt{2}(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$ (B) $2(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6})$

(C) $\sqrt{2}(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})$ (D) $2(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})$

4 Consider the Argand diagram below.



Which inequality could define the shaded area?

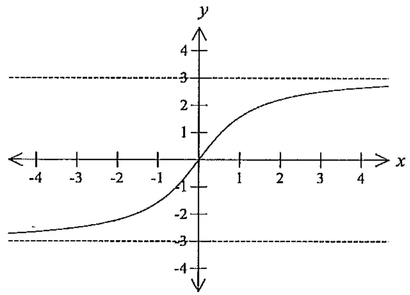
(A) $|z-1| \le \sqrt{2}$ and $0 \le \arg(z-i) \le \frac{\pi}{4}$

(B) $|z-1| \le \sqrt{2}$ and $0 \le \arg(z+i) \le \frac{\pi}{4}$

(C) $|z-1| \le 1$ and $0 \le \arg(z-i) \le \frac{\pi}{4}$

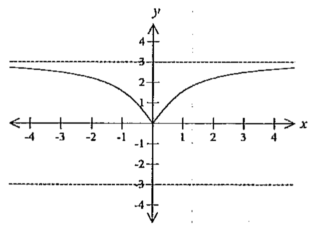
(D) $|z-1| \le 1$ and $0 \le \arg(z+i) \le \frac{\pi}{4}$

5 The diagram shows the graph of the function $y = f(x)$.

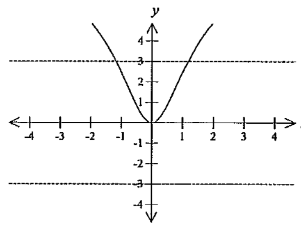


Which of the following is the graph of $y = \sqrt{f(x)}$?

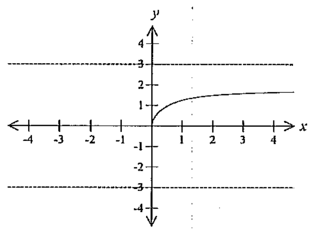
(A)



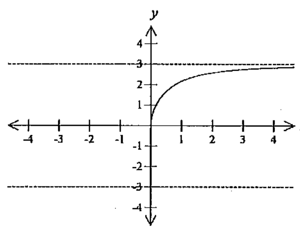
(B)



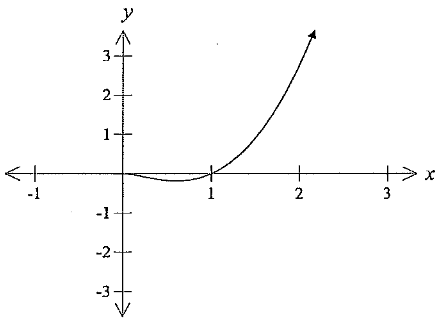
(C)



(D)

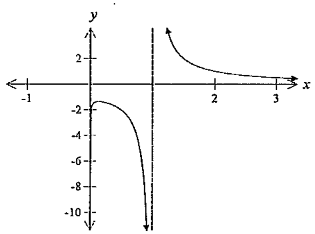


6 The diagram shows the graph of the function $y = f(x)$.

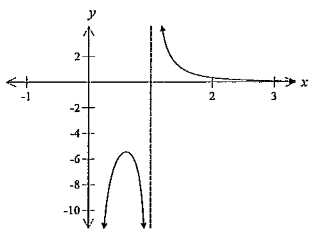


Which of the following is the graph of $y = \dfrac{1}{f(x)}$?

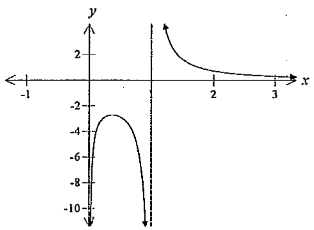
(A)



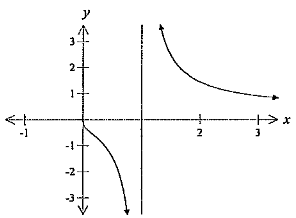
(B)



(C)



(D)



7 Which of the following is an expression for $\int \frac{1}{\sqrt{7-6x-x^2}}dx$?

(A) $\sin^{-1}\left(\frac{x-3}{2}\right)+c$

(B) $\sin^{-1}\left(\frac{x+3}{2}\right)+c$

(C) $\sin^{-1}\left(\frac{x-3}{4}\right)+c$

(D) $\sin^{-1}\left(\frac{x+3}{4}\right)+c$

8 Which of the following is an expression for $\int \frac{1}{\sqrt{x^2-6x+10}}dx$?

(A) $\ln\left(x-3-\sqrt{x^2-6x+10}\right)+c$

(B) $\ln\left(x+3-\sqrt{x^2-6x+10}\right)+c$

(C) $\ln\left(x-3+\sqrt{x^2-6x+10}\right)+c$

(D) $\ln\left(x+3+\sqrt{x^2-6x+10}\right)+c$

9 The equation $4x^3 - 27x + k = 0$ has a double root.
What are the possible values of k?

(A) ± 4

(B) ± 9

(C) ± 27

(D) $\pm\frac{81}{2}$

10 Given that $(x-1)p(x) = 16x^5 - 20x^3 + 5x - 1$, then if $p(x) = (4x^2 + ax - 1)^2$, the value of a is:

(A) 1

(B) 2

(C) $\frac{1}{2}$

(D) 0

Blank page

Section II

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

Answer each question in a new answer booklet.

All necessary working should be shown in every question.

Question 11 (15 marks) **Start a new answer booklet** **Marks**

(a) Using the substitution $u = e^x + 1$ or otherwise, evaluate

$$\int_0^1 \frac{e^x}{(1+e^x)^2}\,dx. \qquad 3$$

(b) Find $\int \frac{1}{x \ln x}\,dx$. 1

(c) (i) Find a, b, and c, such that

$$\frac{16}{(x^2+4)(2-x)} = \frac{ax+b}{x^2+4} + \frac{c}{2-x}. \qquad 2$$

(ii) Find $\int \frac{16}{(x^2+4)(2-x)}\,dx$. 2

(d) Using integration by parts ONLY, evaluate

$$\int_0^1 \sin^{-1} x\,dx. \qquad 3$$

(e) Use the substitution $t = \tan\frac{\theta}{2}$ to show that:

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{4\sin\theta - 2\cos\theta + 6} = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right). \qquad 4$$

Question 12 (15 marks) **Start a new answer booklet** **Marks**

(a) Given $z = \dfrac{\sqrt{3}+i}{1+i}$,

(i) Find the argument and modulus of z. **2**

(ii) Find the smallest positive integer n such that z^n is real. **1**

(b) The complex number z moves such that $\text{Im}\left[\dfrac{1}{\bar{z}-i}\right] = 2$.

Show that the locus of z is a circle. **2**

(c) Sketch the region in the complex plane where the inequalities

$|z+1-i| < 2$ and $0 < \arg(z+1-i) < \dfrac{3\pi}{4}$ hold simultaneously. **3**

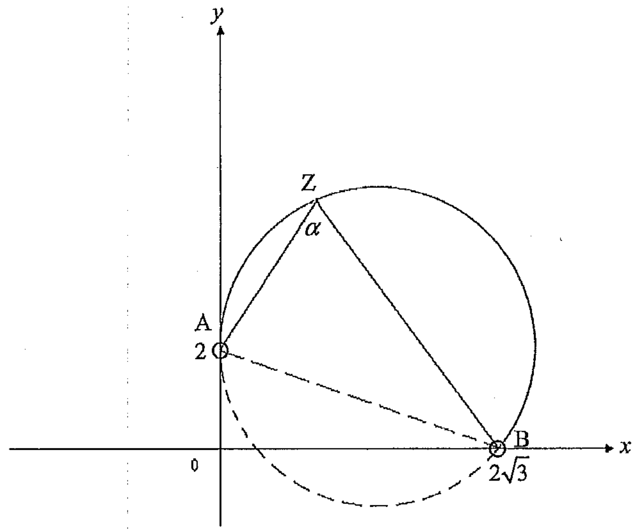
(d) Find the three different values of z for which

$z^3 = \dfrac{1+i}{\sqrt{2}}$. **3**

Question 12 continues on the next page ....

(e) The locus of the complex number Z, moving in the complex plane such that $arg(Z-2\sqrt{3}) - arg(Z-2i) = \dfrac{\pi}{3}$, is a part of a circle.

The angle between the lines from $2i$ to Z and then from $2\sqrt{3}$ to Z is α, as shown in the diagram below.



(i) Show that $\alpha = \dfrac{\pi}{3}$. **2**

(ii) Find the centre and the radius of the circle. **2**

Question 13 (15 marks) **Start a new answer booklet** **Marks**

(a) Consider the polynomial equation

$x^4 + ax^3 + bx^2 + cx + d = 0$

where a, b, c, and d are all integers. Suppose the equation has a root of the form $x = ki$, where k is real, and $k \neq 0$.

(i) State why the conjugate $x = -ki$ is also a root. **1**

(ii) Show that $c = k^2 a$. **2**

(iii) Show that $c^2 + a^2 d = abc$. **2**

(iv) If $x = 2$ is also a root of the equation, and $b = 0$, show that d and c are both even. **2**

(b) (i) Solve $z^5 + 1 = 0$ by De Moivre's Theorem, leaving your solutions in modulus-argument form. **2**

(ii) Prove that the solutions of $z^4 - z^3 + z^2 - z + 1 = 0$ are the non-real solutions of $z^5 + 1 = 0$. **1**

(iii) Show that if $z^4 - z^3 + z^2 - z + 1 = 0$ where $z = cis\theta$ then $4\cos^2\theta - 2\cos\theta - 1 = 0$. **3**

Hint: $z^4 - z^3 + z^2 - z + 1 = 0 \Rightarrow z^2 - z + 1 - \frac{1}{z} + \frac{1}{z^2} = 0$

(iv) Hence, find the exact value of $\sec\frac{3\pi}{5}$. **2**

Question 14 (15 marks) **Start a new answer booklet** **Marks**

(a) (i) Determine the real values of λ for which the equation

$\frac{x^2}{4-\lambda} + \frac{y^2}{2-\lambda} = 1$ defines

(α) an ellipse **1**

(β) a hyperbola **1**

(ii) Sketch the curve corresponding to the value $\lambda = 1$, indicating the positions of the foci and directrices and stating their coordinates and equations respectively. Also mark any axes intercepts on your diagram. **3**

(iii) Describe how the shape of this curve changes as λ increases from 1 towards 2. What is the limiting position of the curve as 2 is approached? **3**

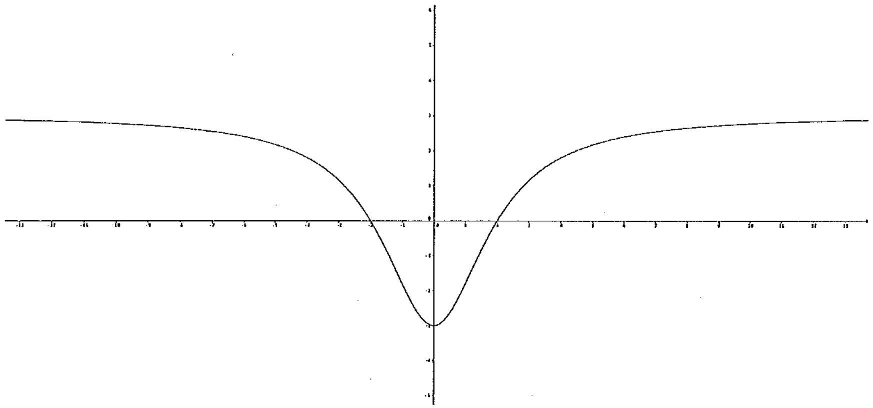
(b) (i) Show that the equation of the normal to the hyperbola $xy = c^2$ at $P(cp, \frac{c}{p})$ is $p^3x - py = c(p^4 - 1)$. **2**

(ii) The normal at $P(cp, \frac{c}{p})$ meets the hyperbola $xy = c^2$ again at $Q(cq, \frac{c}{q})$. Prove that $p^3 q = -1$. **2**

(iii) Hence, show that the locus of the midpoint R of PQ is given by $c^2(x^2 - y^2)^2 + 4x^3y^3 = 0$. **3**

Question 15 (15 marks) **Start a new answer booklet** **Marks**

(a) Given below is the graph of $f(x) = 3 - \dfrac{24}{x^2+4}$.



Use the graph of $y = f(x)$ to sketch, on separate axes, the graphs of

(i) $y = \left[f(x)\right]^2$ **2**

(ii) $y = \sqrt{f(x)}$ **2**

(iii) $y = f'(x)$ **2**

Each graph should be at least one – third of a page in size.

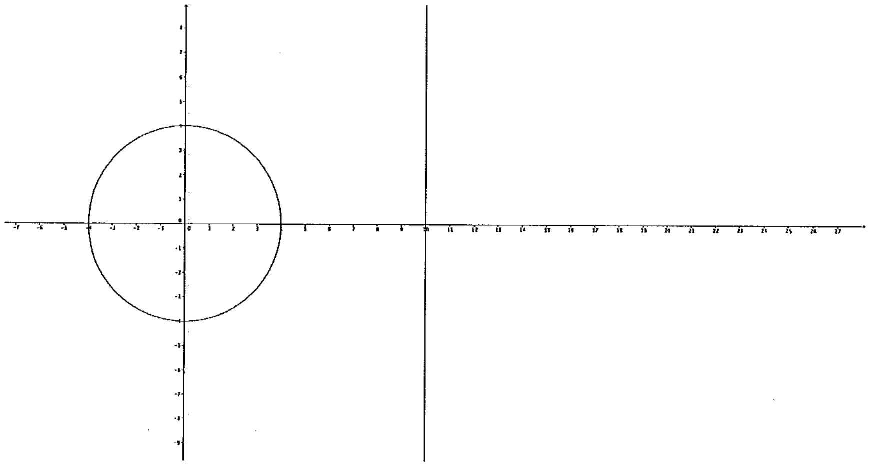
(b) Consider the curve that is defined by $4x^2 - 2xy + y^2 - 6x = 0$

(i) Show that $\dfrac{dy}{dx} = \dfrac{3-4x+y}{y-x}$ **2**

(ii) Find the coordinates of all points where the tangent is vertical. **2**

Question 15 continues on the next page …….

(c) A solid is formed by rotating the area enclosed by the curve $x^2 + y^2 = 16$ through one complete revolution about the line $x = 10$.



(i) Use the method of slicing to show that the volume of this solid is

$$V = 40\pi \int_{-4}^{4} \sqrt{16 - y^2}\, dy$$ **3**

(ii) Find the exact volume of the solid. **2**

Question 16 (15 marks) **Start a new answer booklet** **Marks**

(a) Let $f(x) = (1 - \frac{x^2}{2}) - \cos x$

(i) Show that $f(x)$ is an even function. **1**

(ii) Find expressions for $f'(x)$ and $f''(x)$. **2**

(iii) Deduce that $f'(x) \le 0$ for $x \ge 0$. **2**

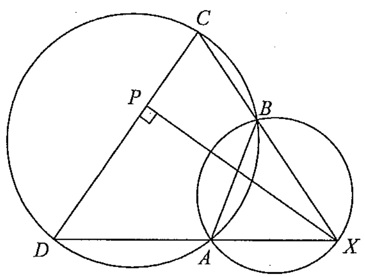
(iv) Hence, show that $\cos x \ge 1 - \frac{x^2}{2}$. **2**

(b) (i) Use the principle of mathematical induction to prove that

$$(1+x)^n > 1 + nx \text{ for } n > 1 \text{ and } x > -1$$ **3**

(ii) Hence, deduce that $\left(1 - \frac{1}{2n}\right)^n > \frac{1}{2}$ for $n > 1$. **1**

(c)



In the diagram above, $AB = AD = AX$ and $XP \perp DC$.

(i) Prove that $\angle DBX = 90°$ **2**

(ii) Hence, or otherwise, prove that $AB = AP$. **2**

Year 12 Mathematics Extension 2

Section I - Answer Sheet

Student Number ANSWERS

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

- If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

- If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.

correct

A ● B ⊗ C ○ D ○

1.	A ○	B ○	C ○	D ●
2.	A ○	B ●	C ○	D ○
3.	A ○	B ○	C ○	D ●
4.	A ○	B ●	C ○	D ○
5.	A ○	B ○	C ●	D ○
6.	A ○	B ○	C ●	D ○
7.	A ○	B ○	C ○	D ●
8.	A ○	B ○	C ●	D ○
9.	A ○	B ○	C ●	D ○
10.	A ○	B ●	C ○	D ○

2012 X2 Trial HSC ~ Multiple Choice Answers

1. $b^2 = a^2(e^2-1)$ $\quad a^2 = 144$ and $b^2 = 25$.

$25 = 144(e^2-1)$ $\quad a = 12 \quad b = 5$

$(e^2-1) = \dfrac{25}{144}$ or $e^2 = \dfrac{169}{144}$ or $e = \dfrac{13}{12}$

Equation of the directrices are $x = \pm\dfrac{a}{e} = \pm\dfrac{144}{13}$. **(D)**

2. POQ is a right-angled triangle. Therefore $OP^2 + OQ^2 = PQ^2$.

$a^2\cos^2\theta + b^2\sin^2\theta + a^2\cos^2\phi + b^2\sin^2\phi = a^2(\cos\theta-\cos\phi)^2 + b^2(\sin\theta-\sin\phi)^2$

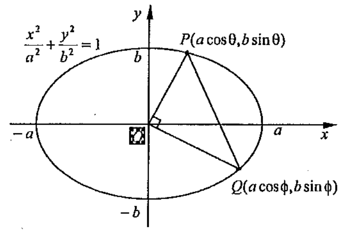
$a^2(\cos^2\theta+\cos^2\phi) + b^2(\sin^2\theta+\sin^2\phi) = a^2(\cos\theta-\cos\phi)^2 + b^2(\sin\theta-\sin\phi)^2$

Hence

$0 = -2a^2\cos\theta\cos\phi - 2b^2\sin\theta\sin\phi$

$2b^2\sin\theta\sin\phi = -2a^2\cos\theta\cos\phi$

$\dfrac{\sin\theta\sin\phi}{\cos\theta\cos\phi} = \dfrac{-2a^2}{2b^2}$ or $\tan\theta\tan\phi = -\dfrac{a^2}{b^2}$



(B)

3. $\tan\theta = \dfrac{1}{-\sqrt{3}}$

$\theta = \dfrac{5\pi}{6}$

$r^2 = x^2 + y^2$

$= (\sqrt{3})^2 + 1^2$

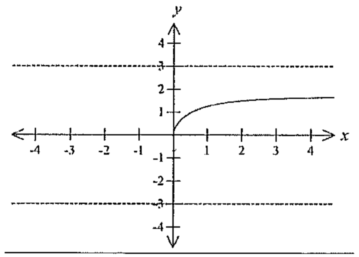
$r = 2$

$-\sqrt{3} + i = 2(\cos\dfrac{5\pi}{6} + i\sin\dfrac{5\pi}{6})$ **(D)**

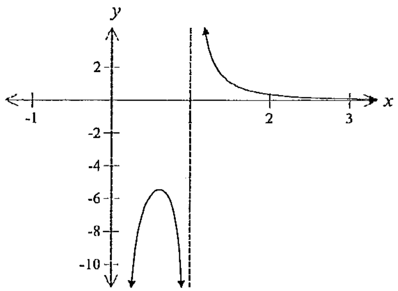
4. $|z-1|\le\sqrt{2}$ represents a region with a centre is (1, 0) and radius is less than or equal to $\sqrt{2}$.

$0\le\arg(z+i)\le\frac{\pi}{4}$ represents a region between angle 0 and $\frac{\pi}{4}$ whose vertex is $(-1,0)$ not including the vertex $\therefore |z-1|\le\sqrt{2}$ and $0\le\arg(z+i)\le\frac{\pi}{4}$ **(B)**

5. **(C)**



6. **(C)**



7. **(D)**

$$\int\frac{1}{\sqrt{7-6x-x^2}}dx=\int\frac{1}{\sqrt{16-9-6x-x^2}}dx$$
$$=\int\frac{1}{\sqrt{16-(x+3)^2}}dx$$
$$=\sin^{-1}\left(\frac{x+3}{4}\right)+c$$

8. **(C)**

$$\int\frac{dx}{\sqrt{x^2-6x+10}}=\int\frac{dx}{\sqrt{x^2-6x+9+1}}=\frac{dx}{\sqrt{(x-3)^2+1}}$$
$$=\ln\left(x-3+\sqrt{(x-3)^2+1}\right)+c$$
$$=\ln\left(x-3+\sqrt{x^2-6x+10}\right)+c$$

9. **(C)**

Let $P(x)=4x^3-27x+k$

$P'(x)=12x^2-27$

Let α be the double root.

Hence $P(\alpha)=0$ and $P'(\alpha)=0$

When $P'(\alpha)=0$ then $12\alpha^2-27=0$

$$\alpha^2=\frac{9}{4}$$
$$\alpha=\pm\frac{3}{2}$$

When $P(\alpha)=0$ then $4\alpha^3-27\alpha+k=0$

$$k=27\alpha-4\alpha^3$$
$$=\alpha(27-4\alpha^2)$$
$$=\pm\frac{3}{2}(27-4\times\frac{9}{4})$$
$$=\pm27$$

10. **(B)**

$(x-1)(4x^2+ax-1)^2=16x^5-20x^3+5x-1$

Let $x=2$, $1.(15+2a)^2=16.2^5-20.2^3+5.2-1=361$

$\therefore 15+2a=\pm19$

$\therefore 2a=-15\pm19=4$ or -34

$\therefore a=2$ or -17.

Year 12	Mathematics Extension 2	Trial HSC Examination 2012
Question 11	Solutions and Marking Guidelines	

Outcome Addressed in this Question

E8 applies further techniques of integration, including partial fractions, integration by parts and recurrence formulae, to problems

Part	Solutions	Marking Guidelines
(a)	$\int_0^1 \frac{e^x}{(1+e^x)^2}\,dx \qquad u=e^x+1 \Rightarrow du=e^x dx$ $=\int_2^{e+1}\frac{du}{u^2}$ $=\left[-\frac{1}{u}\right]_2^{e+1}$ $=-\frac{1}{e+1}-\left(-\frac{1}{2}\right)$ $=\frac{1}{2}-\frac{1}{e+1}$	**Award 3** Correct solution. **Award 2** Substantial progress towards solution. **Award 1** Attempts to manipulate integrand and find primitive.
(b)	$\int \frac{1}{x\ln x}\,dx \qquad u=\ln x \Rightarrow du=\frac{dx}{x}$ $=\int\frac{du}{u}$ $=\ln u+c$ $=\ln(\ln x)+c$	**Award 1** Correct solution.
(c) (i)	$16=(ax+b)(2-x)+c(x^2+4)$ $x=2 \Rightarrow 16=8c \therefore c=2$ ...... (1) $x=1 \Rightarrow 16=a+b+10 \therefore a+b=6$ ...... (2) $x=0 \Rightarrow 16=2b+8 \therefore b=4$ ...... (3) $(3)\to(2) \Rightarrow a=2$	**Award 2** Correct answers for a, b and c. **Award 1** Correct answers for two of a, b or c.
(ii)	$\int\frac{16}{(x^2+4)(2-x)}\,dx$ $=\int\left(\frac{2x+4}{x^2+4}+\frac{2}{2-x}\right)dx$ $=\int\left(\frac{2x}{x^2+4}+\frac{4}{x^2+4}-\frac{2}{x-2}\right)dx$ $=\ln(x^2+4)+2\tan^{-1}\left(\frac{x}{2}\right)-2\ln(x-2)+c$	**Award 2** Correct solution. **Award 1** Substantial progress towards solution.
(d)	$\int_0^1 \sin^{-1}x\,dx=\int_0^1 1.\sin^{-1}x\,dx$ $=\left[x\sin^{-1}x\right]_0^1-\int_0^1 x.\frac{1}{\sqrt{1-x^2}}\,dx$ $=\frac{\pi}{2}-0+\frac{1}{2}\int_0^1 -2x.(1-x^2)^{-\frac{1}{2}}\,dx$ $=\frac{\pi}{2}+\frac{1}{2}\left[\frac{(1-x^2)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^1$ $=\frac{\pi}{2}+(0-1)$ $=\frac{\pi}{2}-1$	**Award 3** Correct solution **Award 2** Substantial progress towards solution **Award 1** Attempts to use integration by parts
(e)	$\int_0^{\frac{\pi}{2}}\frac{d\theta}{4\sin\theta-2\cos\theta+6}$ $=\int_0^1\frac{\frac{2dt}{1+t^2}}{4\times\frac{2t}{1+t^2}-2\times\frac{1-t^2}{1+t^2}+6}$ $=\int_0^1\frac{2dt}{8t^2+8t+4}$ $=\frac{1}{4}\int_0^1\frac{dt}{t^2+t+\frac{1}{2}}$ $=\frac{1}{4}\int_0^1\frac{dt}{\left(t+\frac{1}{2}\right)^2+\frac{1}{4}}$ $=\frac{1}{4}\left[\frac{1}{\frac{1}{2}}\tan^{-1}\left(\frac{t+\frac{1}{2}}{\frac{1}{2}}\right)\right]_0^1$ $=\frac{1}{2}\left[\tan^{-1}\left(\frac{1+\frac{1}{2}}{\frac{1}{2}}\right)-\tan^{-1}\left(\frac{0+\frac{1}{2}}{\frac{1}{2}}\right)\right]$ $=\frac{1}{2}\left[\tan^{-1}(3)-\tan^{-1}(1)\right]$ $=\frac{1}{2}\left[\tan^{-1}\left(\frac{3-1}{1+3\times1}\right)\right]$ $=\frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)$	**Award 4** Correct solution **Award 3** Substantial progress towards solution **Award 2** Limited progress towards solution **Award 1** Attempts to use manipulate integrand and determine primitive

Year 12	Mathematics Extension 2	Trial HSC Examination 2012
Question 12	Solutions and Marking Guidelines	

Outcome Addressed in this Question

E3 uses the relationship between algebraic and geometric representations of complex numbers

Part	Solutions	Marking Guidelines

(a) (i)

$$z=\frac{\sqrt{3}+i}{1+i}$$

$$\arg z=\arg\left(\sqrt{3}+i\right)-\arg\left(1+i\right)$$

$$=\frac{\pi}{6}-\frac{\pi}{4}$$

$$=-\frac{\pi}{12}$$

$$|z|=\frac{2}{\sqrt{2}}=\sqrt{2}$$

Award 2
Correct answers.

Award 1
Substantial progress towards solution **or** only one correct answer

(ii)

$$z=\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{12}\right)$$

$$z^{12}=\left(\sqrt{2}\right)^{12}\operatorname{cis}\left(-\pi\right)=-2^6=-64$$

$$\therefore n=12.$$

Award 1
Correct solution.

(b)

$$\operatorname{Im}\left[\frac{1}{\bar{z}-i}\right]=2$$

Let $z=x+iy$

$$\frac{1}{\bar{z}-i}=\frac{1}{x-iy-i}$$

$$=\frac{1}{x-i\left(y+1\right)}\times\frac{x+i\left(y+1\right)}{x+i\left(y+1\right)}$$

$$=\frac{x+i\left(y+1\right)}{x^2+\left(y+1\right)^2}$$

$$\operatorname{Im}\left[\frac{1}{\bar{z}-i}\right]=2\Rightarrow\frac{\left(y+1\right)}{x^2+\left(y+1\right)^2}=2$$

$$\therefore y+1=2x^2+2\left(y+1\right)^2=2x^2+2y^2+4y+2$$

$$2x^2+2y^2+4y+2-y-1=0$$

$$2x^2+2\left(y^2+\frac{3y}{2}+\frac{9}{16}\right)-\frac{1}{8}=0$$

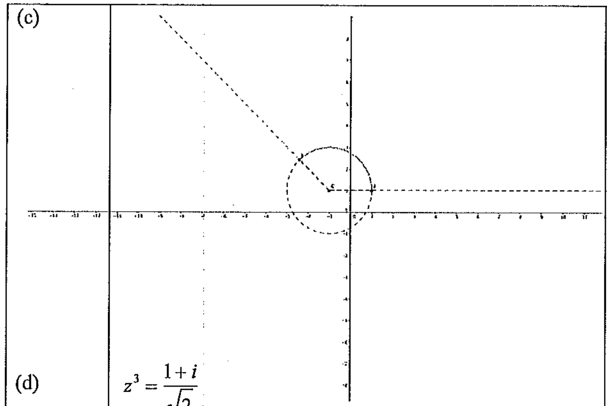
$$\therefore x^2+\left(y+\frac{3}{4}\right)^2=\frac{1}{16}$$

$\therefore$ The locus is a circle

Award 2
Correct solution.

Award 1
Substantial progress towards solution.

(c)



Award 3
Correct region shaded, with centre of circle and angular region clearly indicated. Exclusions must be shown.

Award 2
Substantial progress towards solution.

Award 1
Limited progress towards solution.

(d)

$$z^3=\frac{1+i}{\sqrt{2}}$$

$$=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$$

$$=1.\operatorname{cis}\left(\frac{\pi}{4}+2k\pi\right)$$

$$z=1.\operatorname{cis}\left(\frac{\pi}{12}+\frac{2k\pi}{3}\right)$$

$$k=0\Rightarrow z=\operatorname{cis}\frac{\pi}{12}$$

$$k=1\Rightarrow z=\operatorname{cis}\frac{9\pi}{12}=\operatorname{cis}\frac{3\pi}{4}$$

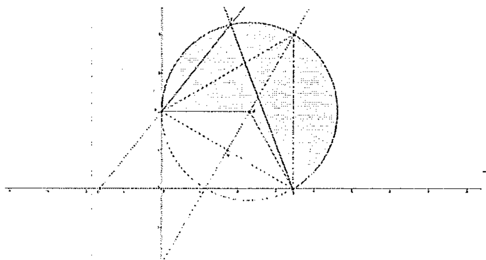
$$k=2\Rightarrow z=\operatorname{cis}\frac{17\pi}{12}=\operatorname{cis}\left(-\frac{7\pi}{12}\right)$$

Award 3
Correct solution.

Award 2
Substantial progress towards solution.

Award 1
Limited progress towards solution.

(e) (i)



Let $\arg\left(Z-2\sqrt{3}\right)=\beta$

Let $\arg\left(Z-2i\right)=\gamma$

(produce ray to x axis to create triangle arguments mearured to positive x axis)

$$\therefore\arg\left(Z-2\sqrt{3}\right)-\arg\left(Z-2i\right)=\beta-\gamma$$

$$=\alpha\ \text{(Exterior Angle of triangle)}$$

$$=\frac{\pi}{3}$$

Award 2
Correct solution.

Award 1
Substantial progress towards solution.

(ii)	Let H be the centre of the circle. $BA^2+2^2=\left(2\sqrt{3}\right)^2$ $\therefore BA=4$ $\angle AHB=\frac{2\pi}{3}$ (Angle at centre is twice angle at the circumference) $\angle AHM=\frac{\pi}{3}$ $\frac{2}{AH}=\cos\frac{\pi}{6}$ (MH is the perpendicular bisector of BA) $\therefore AH(radius)=\frac{4}{\sqrt{3}}$ $\angle DBA=\frac{\pi}{6}=\angle HAM$ $\therefore HA \parallel DB$ Centre $=\left(\frac{4}{\sqrt{3}},2\right)$ (x coordinate length is the radius as the radius is perpendicular to the tangent at (0,2) and the centre height is 2 the value at the y axis)	**Award 2** Correct solution. **Award 1** Substantial progress towards solution.

Outcome Addressed in this Question

E4 uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving polynomials.

Part	Solutions	Marking Guidelines
(a) (i)	The coefficients are real. By the conjugate root theorem, $x=-ki$ is also a root.	**Award 1** Correct explanation.
(ii)	$(ki)^4+a(ki)^3+b(ki)^2+c(ki)+d=0$ $k^4-ak^3i-bk^2+cki+d=0$ $\left(k^4-bk^2+d\right)-i\left(ak^3-ck\right)=0$ Equating imaginary parts, $ak^3-ck=0$ $k\left(ak^2-c\right)=0$ $\therefore k=0$ (which is not a solution) or $\therefore c=ak^2$	**Award 2** Correct solution. **Award 1** Substantial progress towards solution.
(iii)	Equating real parts, $k^4-bk^2+d=0$ $\left(\frac{c}{a}\right)^2-b\left(\frac{c}{a}\right)+d=0$ $\frac{c^2}{a^2}-\frac{bc}{a}+d=0$ $\frac{c^2-abc+da^2}{a^2}=0$ $\therefore c^2-abc+da^2=0$ $\therefore c^2+a^2d=abc$	**Award 2** Correct solution. **Award 1** Substantial progress towards solution.
(iv)	Substitute $x=2$, $16+8a+2c+d=0$ i.e. $d=-16-8a-2c=2\left(-8-4a-2\right)$ $\therefore d$ is even Substitute $b=0$ into $c^2+a^2d=abc$ $c^2+a^2d=0$ Since d is even, c^2 is even. Since c^2 is even, then c is even.	**Award 2** Correct solution. **Award 1** Substantial progress towards solution.

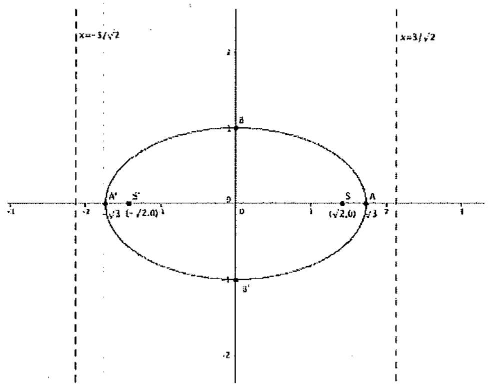
(b) (i)	$z^5=-1$ $\therefore z^5=\text{cis}(\pi+2k\pi)$ $\therefore z=\text{cis}\left(\frac{\pi+2k\pi}{5}\right)$ i.e. $z=\text{cis}\left(\frac{\pi+2.0.\pi}{5}\right),\text{cis}\left(\frac{\pi+2.1.\pi}{5}\right),\text{cis}\left(\frac{\pi+2.2.\pi}{5}\right),\text{cis}\left(\frac{\pi+2.3.\pi}{5}\right),\text{cis}\left(\frac{\pi+2.4.\pi}{5}\right)$ $=\text{cis}\left(\frac{\pi}{5}\right),\text{cis}\left(\frac{3\pi}{5}\right),\text{cis}\left(\frac{5\pi}{5}\right),\text{cis}\left(\frac{7\pi}{5}\right),\text{cis}\left(\frac{9\pi}{5}\right)$ $=\text{cis}\left(\frac{\pi}{5}\right),\text{cis}\left(\frac{3\pi}{5}\right),\text{cis}(\pi),\text{cis}\left(-\frac{3\pi}{5}\right),\text{cis}\left(-\frac{\pi}{5}\right)$ $=\text{cis}\left(\frac{\pi}{5}\right),\text{cis}\left(\frac{3\pi}{5}\right),-1,\text{cis}\left(-\frac{3\pi}{5}\right),\text{cis}\left(-\frac{\pi}{5}\right)$	**Award 2** Correct solution. **Award 1** Substantial progress towards solution.
(ii)	$z^5+1=(z+1)(z^4-z^3+z^2-z+1)$ The only real root is $z=-1$ From (i), the other roots must be the solutions to $z^4-z^3+z^2-z+1=0$	**Award 1** Correct explanation.
(iii)	$z^4-z^3+z^2-z+1=0$ becomes $z^2-z+1-\frac{1}{z}+\frac{1}{z^2}=0$ (dividing through by z^2) $\therefore z^2+z^{-2}-(z+z^{-1})+1=0$ $\therefore 2\cos 2\theta-2\cos\theta+1=0$ Using the result $z^n+z^{-n}=2\cos n\theta$ (where $z=\cos\theta+i\sin\theta$) i.e. $2(2\cos^2\theta-1)-2\cos\theta+1=0$ $\therefore\ 4\cos^2\theta-2\cos\theta-1=0$	**Award 3** Correct solution. **Award 2** Substantial progress towards solution. **Award 1** Limited progress towards solution.
(iv)	$z=\text{cis}\frac{3\pi}{5}$ is a solution of $z^4-z^3+z^2-z+1=0$ $\therefore\theta=\frac{3\pi}{5}$ is a solution of $4\cos^2\theta-2\cos\theta-1=0$ $\cos\theta=\frac{-(-2)\pm\sqrt{(-2)^2-4.4.-1}}{2.4}$ $=\frac{2\pm\sqrt{20}}{8}$ $=\frac{1\pm\sqrt{5}}{4}$ But $\cos\frac{3\pi}{5}<0\Rightarrow\sec\frac{3\pi}{5}<0$ $\therefore\sec\frac{3\pi}{5}=\frac{4}{1-\sqrt{5}}=-(1+\sqrt{5})$	**Award 2** Correct solution. **Award 1** Substantial progress towards solution.

Year 12	Mathematics Extension 2	Trial HSC Examination 2012
Question 14	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

E3 uses the relationship between algebraic and geometric representations of conic sections

E4 uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections

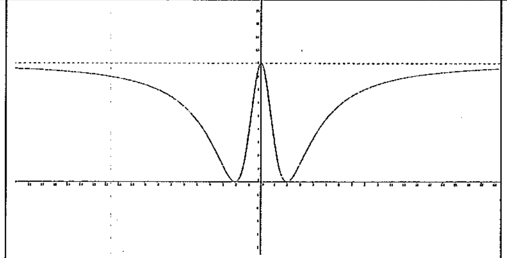
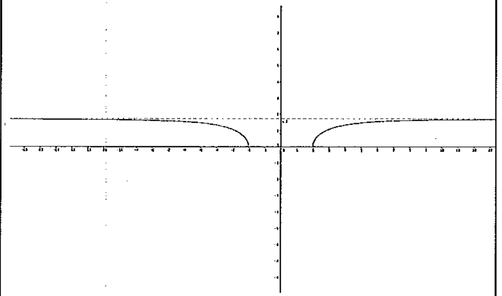
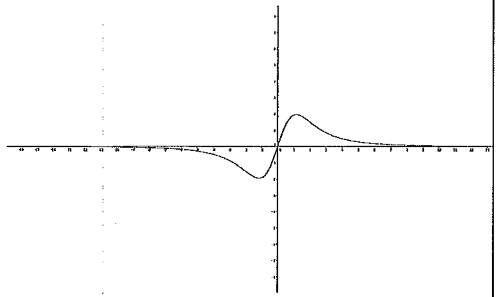
Part	Solutions	Marking Guidelines
(a) (i)(α)	$4-\lambda>0$ and $2-\lambda>0$ $\therefore\lambda<4$ and $\lambda<2$ Hence, $\lambda<2$.	**Award 1** Correct answer.
(i)(β)	$4-\lambda>0$ and $2-\lambda<0$ or $4-\lambda<0$ and $2-\lambda>0$ Hence, $2<\lambda<4$. (Not possible to have $\lambda<2$ and $\lambda>4$)	**Award 1** Correct answer.
(ii)	$\lambda=1: \frac{x^2}{3}+\frac{y^2}{1}=1$ $e^2=1-\frac{b^2}{a^2}=1-\frac{1}{3}=\frac{2}{3}\Rightarrow e=\sqrt{\frac{2}{3}}$ Foci $=(\pm ae,0)=(\pm\sqrt{2},0)$ Directrices: $x=\pm\frac{a}{e}=\pm\frac{3}{\sqrt{2}}$ 	**Award 3** Correct graph, with foci, directrices with intercepts with axes clearly indicated. **Award 2** Correct graph, with any two of foci, directrices and intercepts with axes indicated. **Award 1** Correct graph, with only one of foci, directrices and intercepts with axes indicated.
(iii)	As λ increases from 1 to 2, $4-\lambda$ decreases from 3 to 2 while $2-\lambda$ decreases from 1 to 0. The curve remains an ellipse with the semi – major axis reducing from $\sqrt{3}$ to $\sqrt{2}$ and the semi – minor axis from 1 to 0. As 2 is approached, $b\to 0$, the ellipse becomes a line segment joining $(-\sqrt{2},0)$ to $(\sqrt{2},0)$	**Award 3** Correct solution with all reasoning provided. **Award 2** Solution with substantial reasoning provided. **Award 1** Solution with limited reasoning provided

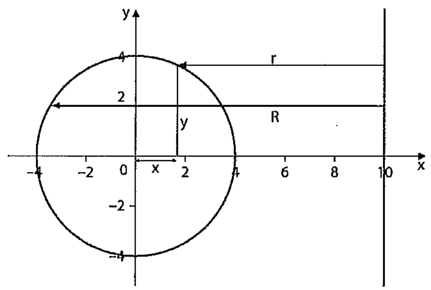
(b) (i)	$xy = c^2$ $y = \frac{c^2}{x}$ $\frac{dy}{dx} = -\frac{c^2}{x^2}$ At $P\left(cp, \frac{c}{p}\right)$, $\frac{dy}{dx} = -\frac{c^2}{(cp)^2} = -\frac{1}{p^2}$ $\therefore m_{\text{tangent}} = -\frac{1}{p^2}$ $\therefore m_{\text{normal}} = p^2$ Equation of normal is $y - \frac{c}{p} = p^2(x - cp)$ $py - c = p^3(x - cp)$ $\therefore p^3x - py = cp^4 - c = c(p^4 - 1)$	**Award 2** Correct solution. **Award 1** Substantial progress towards solution.
(ii)	$m_{PQ} = \frac{\frac{c}{p} - \frac{c}{q}}{cp - cq} = \frac{\frac{cq - cp}{pq}}{cp - cq} = -\frac{1}{pq}$ Hence, $-\frac{1}{pq} = p^2$ $\therefore p^3q = -1$	**Award 2** Correct solution. **Award 1** Substantial progress towards solution.
(iii)	$R = \left(\frac{cp + cq}{2}, \frac{\frac{c}{p} + \frac{c}{q}}{2}\right) = \left(\frac{c}{2}(p+q), \frac{c}{2}\left(\frac{p+q}{pq}\right)\right)$ Let $x = \frac{c}{2}(p+q)$ and $y = \frac{c}{2}\left(\frac{p+q}{pq}\right)$ $\therefore \frac{x}{y} = \frac{\frac{c}{2}(p+q)}{\frac{c}{2}\left(\frac{p+q}{pq}\right)} = pq$ From (ii) $pq = -\frac{1}{p^2}$ $\therefore \frac{x}{y} = -\frac{1}{p^2}$ Using the equation of the normal, $p^2x - y = \frac{c}{p}(p^4 - 1)$	**Award 3** Correct solution. **Award 2** Substantial progress towards solution **Award 1** Limited progress towards solution
(iv)	$-\left(\frac{y}{x}\right)x - y = \frac{c}{p}\left(\left(-\frac{y}{x}\right)^2 - 1\right)$ $-2y = \frac{c}{p}\left(\frac{y^2 - x^2}{x^2}\right)$ Square both sides, $4y^2 = c^2 \times \left(-\frac{x}{y}\right)\left(\frac{x^2 - y^2}{x^2}\right)^2$ $4y^3x^3 = -c^2(x^2 - y^2)^2$ $\therefore 4y^3x^3 + c^2(x^2 - y^2)^2 = 0$	

Outcomes Addressed in this Question

E6 combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions

E7 uses the techniques of slicing to determine volumes

Part	Solutions	Marking Guidelines
(a) (i)		**Award 2** Correct graph. **Award 1** Substantially correct graph.
(ii)		**Award 2** Correct graph. **Award 1** Substantially correct graph.
(iii)		**Award 2** Correct graph. **Award 1** Substantially correct graph
(b) (i)	$4x^2 - 2xy + y^2 - 6x = 0$ Implicit differentiation yields $8x - 2x\frac{dy}{dx} - y.2 + 2y\frac{dy}{dx} - 6 = 0$ $4x - x\frac{dy}{dx} - y + y\frac{dy}{dx} - 3 = 0$ $(y - x)\frac{dy}{dx} = 3 - 4x + y$ $\frac{dy}{dx} = \frac{3 - 4x + y}{y - x}$	**Award 2** Correct solution **Award 1** Substantial progress towards solution

(ii)	If the tangent is vertical, $\frac{dy}{dx}$ is undefined. $\therefore y-x=0 \Rightarrow x=y$ Substitute into the equation of the curve $4x^2-2x.x+x^2-6x=0$ $3x^2-6x=0$ $3x\left(x-2\right)=0$ $\therefore x=0$ or $x=2$ $\therefore$ Points where tangents are vertical are $(0,0)$ and $(2,2)$	**Award 2** Correct solution. **Award 1** Substantial progress towards solution.
(c) (i)	 Slices are taken perpendicular to the axis of rotation ($x=10$). The base is an annulus. $A=\pi(R^2-r^2)$ $=\pi(R+r)(R-r)$ Now $r=10-x$ and $R=10+x$ $=10-\sqrt{16-y^2}$ and $=10+\sqrt{16-y^2}$ Area of the annulus is $A=\pi(R+r)(R-r)$ $=\pi(10+\sqrt{16-y^2}+10-\sqrt{16-y^2})(10+\sqrt{16-y^2}-10+\sqrt{16-y^2})$ $=\pi(20)(2\sqrt{16-y^2})$ $=40\pi\sqrt{16-y^2}$ $\Delta V=40\pi\sqrt{16-y^2}.\Delta y$ $V=\lim_{\Delta y\to 0}\sum_{y=-4}^{4}40\pi\sqrt{16-y^2}\Delta y$ $=\int_{-4}^{4}40\pi\sqrt{16-y^2}\,dy$ $=40\pi\int_{-4}^{4}\sqrt{16-y^2}\,dy$	**Award 3** Correct solution. **Award 2** Correctly calculates the area of the annulus and attempts to determine the volume. **Award 1** Attempts to calculate the area of the annulus.

(ii)	$\int_{-4}^{4}\sqrt{16-y^2}dy$ is the area of a semicircle with a radius of 4. $\int_{-4}^{4}\sqrt{16-y^2}\,dy=\frac{1}{2}\times\pi\times 4^2$ $=8\pi$ $V=40\pi\int_{-4}^{4}\sqrt{16-y^2}\,dy$ $=40\pi\times 8\pi$ $=320\pi^2$ unit^3	**Award 2** Correct answer. **Award 1** Using area of semi circle or appropriate integration.

Year 12	Mathematics Extension 2	Trial HSC Examination 2012
Question 16	Solutions and Marking Guidelines	

Outcomes Addressed in this Question

E2 chooses appropriate strategies to construct arguments and proofs in both concrete and abstract settings

E9 communicates abstract ideas and relationships using appropriate notation and logical argument

Part	Solutions	Marking Guidelines
(a) (i)	$f(x)=\left(1-\frac{(x)^2}{2}\right)-\cos(x)$ $f(-x)=\left(1-\frac{(-x)^2}{2}\right)-\cos(-x)$ $f(x)=\left(1-\frac{(x)^2}{2}\right)-\cos(x)$ as $y=\cos x$ is an even function $\therefore f(-x)=f(x)$ $\therefore f(x)$ is an even function.	**Award 1** Correct solution.
(ii)	$f'(x)=\left(-\frac{2x}{2}\right)-(-\sin x)=\sin x-x$ $f''(x)=\cos x-1$	**Award 2** Correct expressions for $f'(x)$ and $f''(x)$ **Award 1** Only one of $f'(x)$ or $f''(x)$ correct
(iii)	$f''(x)\le 0$ for $x\ge 0$ because $-1\le\cos x\le 1$, hence, $\cos x-1\le 0$ This means that $f'(x)$ is an decreasing function for $x\ge 0$. $f'(0)=0$ $\therefore f'(x)\le f'(0)$ i.e. $f'(x)\le 0$	**Award 2** Correct solution. **Award 1** Substantial progress towards solution
(iv)	Since $f'(x)\le 0$ then $f(x)\le f(0)$ for $x\ge 0$. $f(0)=0$ $\therefore f(x)\le 0$ for $x\ge 0$ But $f(x)$ is an even function $\therefore f(x)\le 0$ for $x\le 0$ $\therefore f(x)\le 0$ for all x $\therefore \cos x-\left(1-\frac{x^2}{2}\right)\le 0$ $\therefore \cos x\le 1-\frac{x^2}{2}$	**Award 2** Correct solution. **Award 1** Substantial progress towards solution

(b) (i)	Test the result for $n=2$ $(1+x)^2>1+2x$ $1+2x+x^2>1+2x$ Since $x^2>0$ the result is true for $n=2$ Assume the result is true for $n=k$ $(1+x)^k>1+kx$ To prove the result is true for $n=k+1$ i.e we want to establish that $(1+x)^{k+1}>1+(k+1)x$ $\text{LHS}=(1+x)^{k+1}$ $=(1+x)(1+x)^k$ $>(1+x)(1+kx)$ Assumption for $n=k$ $>1+kx+x+kx^2$ $x>-1$ hence $(1+x)>0$ $>1+kx+x$ $kx^2>0$ $>1+(k+1)x$ $=\text{RHS}$ Therefore the result holds true for $n=k+1$ Hence the result is true for $n\ge 2$ by mathematical induction.	**Award 3** Correct solution. **Award 2** Attempts to prove the result true for $n=k+1$ **Award 1** Establishes the result for $n=2$
(ii)	From part (i) with $x=-\frac{1}{2n}$ ($n>1$ it satisfies $x>-1$) $(1-\frac{1}{2n})^n>1+n\times-\frac{1}{2n}$ $>\frac{1}{2}$ for $n>1$	**Award 1** Correct solution.
(c) (i)	The circle through D, B and X has centre A, since $AD=AB=AX$. Hence, DAX is a diameter. Thus, $\angle DBX=90°$ (angle at circumference in semi-circle equals 90^0).	**Award 2** Correct solution with full reasoning **Award 1** Recognises that D, B and X lie on a circle centred at A.
(ii)	By the converse of the angle in a semicircle, since $\angle DPX$ is a right angle, the circle with diameter DAX also passes through P. Hence $AP=AB$ (radii).	**Award 2** Correct solution with full reasoning **Award 1** Argues that the circle with diameter DAX also passes through P without giving reasons.