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MATHEMATICS EXTENSION 2

2005

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

General Instructions

- Reading time – 5 minutes
- Writing time – 3 hours
- All eight questions should be attempted
- Total marks available - 120
- All questions are worth 15 marks
- An approved calculator may be used
- A table of standard integrals appears on page 13 of this trial exam
- All relevant working should be shown for each question

The release date for this exam is Monday 15 August 2005. Teachers are asked not to release this trial exam to students until this date except under exam conditions where the trial exams are collected by teachers at the end of the exam.

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Question 1 (15 marks) **Marks**

(a) Find $\int \frac{dx}{2x\sqrt{\ln x}}$. **2**

(b) Use integration by parts to find **2**

$$\int 3x^2 \cos^{-1} x \, dx.$$

(c) By using integration by parts twice, evaluate **4**

$$\int_0^{\frac{\pi}{4}} x^2 \cos x \, dx.$$

(d) (i) Find the real numbers A and B such that **2**

$$\frac{3x^2 - x + 12}{(x-2)(x^2+2x+3)} \equiv \frac{A}{x-2} + \frac{Bx-3}{x^2+2x+3}.$$

(ii) Hence find $\int \frac{3x^2 - x + 12}{(x-2)(x^2+2x+3)} dx$. **2**

(e) Show $\int_0^a f(x)dx = \int_0^a f(a-x)dx$, hence find $\int_0^{\frac{\pi}{6}} \frac{\sin^5 3x}{\cos^5 3x + \sin^5 3x} dx$. **3**

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Question 2 (15 marks) **Marks**

(a) (i) Find $1-\sqrt{3}i$ in modulus-argument form. **1**

(ii) Find $(1-\sqrt{3}i)^8$ in the form of $a+ib$. **2**

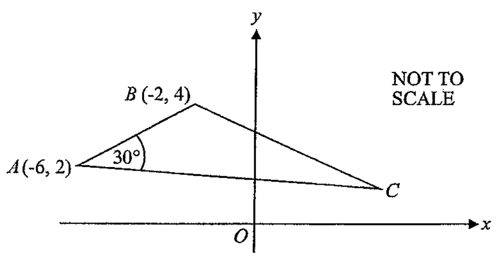
(iii) Find all the values of integers n so that $(1-\sqrt{3}i)^n$ is real. **2**

(b) The complex number $z = x+iy$, where x and y are real, exists such that $|z-2| = \mathrm{Re}(z)$.

(i) Show that the locus of the point P representing z has Cartesian equation $y^2 = 4(x-1)$. Sketch this locus. **2**

(ii) Find the gradients of the tangents to the curve which pass through the origin. Hence find the set of possible values of the principal argument of z. **3**

(c)



ΔABC is drawn in the Argand diagram above where $\angle BAC = 30^\circ$, A and B are the points $(-6,2)$ and $(-2,4)$ respectively. The length of side AC is twice the length of side AB.

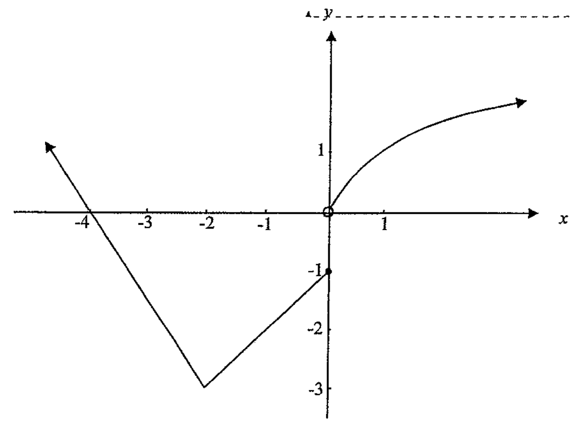
(i) Show that the complex number that represents the vector $\vec{AB}$ is $4+2i$. **1**

(ii) Find the complex number that the point C represents. **2**

(iii) Find the complex numbers that the point D represents such that ΔABD is an equilateral triangle. **2**

Question 3 (15 marks) **Marks**

(a) The diagram below shows the discontinuous function $y = f(x)$.



Draw large (half page), separate sketches of each of the following:

(i) $y = |f(x-1)|$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = \sqrt{-f(x)}$ **2**

(iv) $y = \ln(f(x))$ **2**

(b) An ellipse has parametric equations $x = \sqrt{2}\cos\theta$ and $y = 3\sin\theta$. Find the Cartesian equation, the eccentricity and the foci of the ellipse. **3**

(c) The area between the curve $y = \ln(x+1)$ and the x-axis, between $x = 0$ and $x = 1$, is rotated about the y-axis. Find the volume of the solid of revolution formed using the method of cylindrical shells. **4**

Question 4 (15 marks) **Marks**

(a) (i) Given that $I_n = \int \frac{dx}{\left(1-x^2\right)^{n+\frac{1}{2}}}$, show $I_n = \int \sec^{2n} u\,du$. **3**

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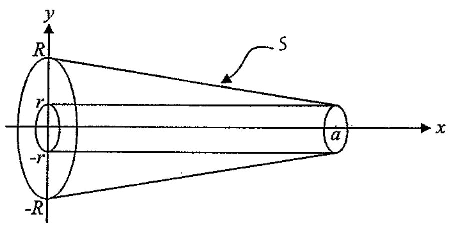
(ii) Hence or otherwise, find $\int \frac{dx}{\left(1-x^2\right)^{\frac{7}{2}}}$. **2**

(b)



A truncated right circular cone with a base radius R has a cylinder of radius r drilled through its centre.

The resulting solid is S.
The x-axis runs through the centre of the cylinder.

(i) Show that the area of a typical cross section of S is given by **2**

$$\pi\left\{\left(\frac{r-R}{a}x+R\right)^2-r^2\right\}.$$

(ii) Find the volume of S. **2**

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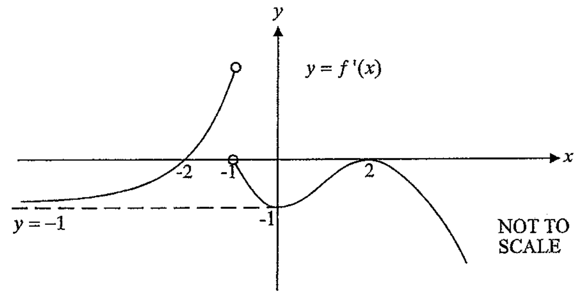
Question 4 continues on the next page.

Question 4 (cont'd) **Marks**

(c)



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The discontinuous function $y = f'(x)$ is shown on the graph above. It is the derivative function of the continuous function $y = f(x)$ and it has a horizontal asymptote $y = -1$.

(i) Find the turning points of the curve $y = f(x)$ and state what type of turning points they are. **2**

(ii) Given that $f(-1) = -1$ and $f(-2) = -2$, sketch the graph of $y = f(x)$. Label clearly on your graph any important features. **4**

Question 5 (15 marks) **Marks**

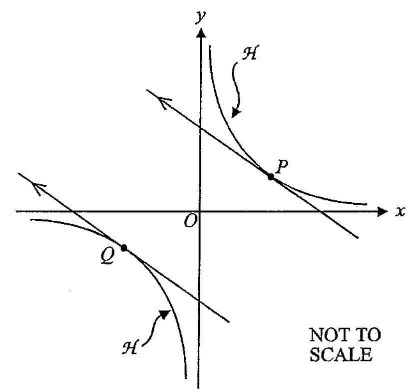
(a) Factorise the polynomial $P(z)=z^4-2z^2+12z-8$ fully over C given that $P(1+\sqrt{3}i)=0$. **3**

(b) The roots of $x^3+px^2+qx+r=0$ form an arithmetic progression. Prove that $2p^3+27r-9pq=0$. **3**

Question 5 continues on the next page.

Question 5 (cont'd) **Marks**

(c)



The points $P\left(cp,\frac{c}{p}\right)$ and $Q\left(cq,\frac{c}{q}\right), c>0$, lie on different branches of the hyperbola H with equation $xy=c^2$. The tangent to H at P and the tangent to H at Q have the same gradient.

(i) Show that the equation of the tangent at P is **2**

$$y=\frac{2c}{p}-\frac{x}{p^2}$$

(ii) Show that $p=-q$ **1**

(iii) Show that PQ passes through the origin. **1**

(iv) Show that the perpendicular distance from P to the tangent through Q is given by **3**

$$\frac{4cp}{\sqrt{(p^4+1)}}$$

(v) Find the coordinates of the points P and Q when the perpendicular distance from P to the tangent through Q is a maximum. **2**

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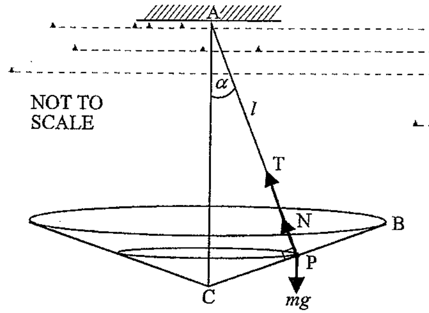
Question 6 (15 marks) — Marks

(a)



A particle P, of mass m kg, is attached to a string of length l m from a point A which is vertically above the vertex of a cone at point C. The particle is in contact with the surface of the cone as the particle rotates as a conical pendulum with angular velocity ω radians per second. The sides of the cone, indicated by BC, make an angle of 90° with the string.
The string makes an angle of α with the vertical.
The forces acting on P are T, the tension force in the string, N, the normal force of the cone on P and mg which is the gravitational force.

(i) Show that $N+T=\dfrac{mg}{\cos\alpha}$. **1**

(ii) Find an expression for ω in terms of g, l and α. **2**

When the angular velocity of the particle is sufficiently large, the particle is at the point of losing contact with the cone.
At this point $N=0$.

(iii) If the particle is rotating at $\sqrt{\dfrac{g}{2\pi^2 l\cos\alpha}}$ revolutions per second, explain whether or not the particle is still in contact with the surface of the cone. **3**

(iv) Explain why the **angles** at which the sides of the cone are pitched do not prevent the particle from lifting off its surface. **1**

Question 6 (cont'd) — Marks

(b) Use the following identity to answer the following questions.

$$\tan 4\theta=\frac{4\tan\theta-4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta}$$

(i) Solve $x^4+4\sqrt{3}x^3-6x^2-4\sqrt{3}x+1=0$. **3**

(ii) Hence show that

(1) $\tan\dfrac{\pi}{24}+\tan\dfrac{7\pi}{24}+4\sqrt{3}=\tan\dfrac{5\pi}{24}+\tan\dfrac{11\pi}{24}$ **1**

(2) $\tan\dfrac{\pi}{24}\tan\dfrac{5\pi}{24}=\cot\dfrac{7\pi}{24}\cot\dfrac{11\pi}{24}$ **1**

(iii) Find the polynomial of least degree that has zeroes **3**

$$\left(\cot\frac{\pi}{24}\right)^2,\left(\cot\frac{7\pi}{24}\right)^2,\left(\cot\frac{13\pi}{24}\right)^2,\left(\cot\frac{19\pi}{24}\right)^2$$

Question 7 (15 marks) **Marks**

(a) There are sixteen runners to be divided into four teams of four, to run a relay race.

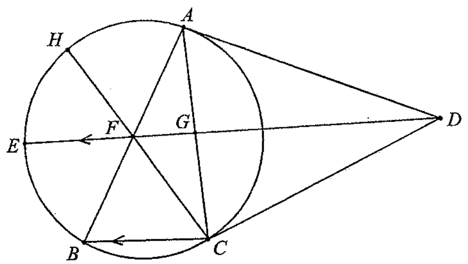
(i) In how many different ways can they be divided into teams if: **2**

(1) there are no restrictions;

(2) no two of four particular runners may be in the same team?

(ii) If the runners are randomly allocated to teams, what is the probability that three particular runners are on the same team? **2**

(b)



The triangle ABC is inscribed in a circle. From an external point D, tangents are drawn to the circle, touching it at A and C. The chord ED is drawn parallel to BC, meeting AB at F and AC at G. The line CF is produced to meet the circle at H.

(i) Prove that $AFCD$ is a cyclic quadrilateral. **3**

(ii) Prove that $HF = AF$. **3**

(c) Prove that for any positive integer m, $\frac{2m+1}{2m+4} \le \sqrt{\frac{3m-2}{3m+1}}$. **5**

Hence deduce that $\left(\frac{1}{2}\right)\left(\frac{5}{8}\right)\left(\frac{7}{10}\right)\ldots\left(\frac{2n+1}{2n+4}\right) \le \frac{1}{\sqrt{3n+1}}$ for any positive integer n.

Question 8 (15 marks) **Marks**

(a) A particle of mass 1 kg moves in a straight line with velocity v against a resistance force equal to $v\sqrt{1-v^2}$. The displacement, x, of the particle from a fixed origin O is initially zero and its velocity at that time is R.

(i) Show that **4**

$$x = \sin^{-1}\left(R\sqrt{1-v^2} - v\sqrt{1-R^2}\right)$$

(ii) The time that has passed since the particle began its movement is given by t. **4**
Given that

$$\frac{d(\operatorname{cosec}\theta - \cot\theta)}{d\theta} = \operatorname{cosec}\theta(\operatorname{cosec}\theta - \cot\theta)$$

and using the substitution $v = \sin\theta$, show that

$$t = \ln\left|\frac{v\left(1-\sqrt{1-R^2}\right)}{R\left(1-\sqrt{1-v^2}\right)}\right|$$

(b) (i) Show that **2**

$$\frac{r+1}{r-1} = 1 + \frac{2}{r} + \frac{2}{r^2} + \frac{2}{r^3} + \ldots + \frac{2}{r^n} + \frac{2}{r^n(r-1)}$$

(ii) Show that **2**

$$\sum_{r=2}^{n}\left(\ln(r+1) - \ln(r-1)\right) = \ln\frac{n(n+1)}{2}$$

(iii) Hence prove by induction that **3**

$$\sum_{r=2}^{n}\ln\left(1 + \frac{2}{r} + \frac{2}{r^2} + \ldots + \frac{2}{r^n} + \frac{2}{r^n(r-1)}\right) = \ln\frac{n(n+1)}{2}$$

for $n = 2,3,4,\ldots$

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STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

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MATHEMATICS EXTENSION 2 HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION SOLUTIONS 2005

Question 1 (15 marks)

(a) $\int \frac{dx}{2x\sqrt{\ln x}}$

Let $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$= \int \frac{u^{-\frac{1}{2}}}{2} du$

$= u^{\frac{1}{2}} + c$

$= \sqrt{\ln x} + c$

2 marks	Correct answer
1 mark	Using the substitution $u = \ln x$

(b) $\int 3x^2 \cos^{-1} x\, dx$

Let $f(x) = 3x^2$ so $F(x) = x^3$

$g(x) = \cos^{-1} x$ so $g'(x) = \frac{-1}{\sqrt{1-x^2}}$

$= x^3 \cos^{-1} x + \int \frac{x^3}{\sqrt{1-x^2}} dx$

Let $u = 1 - x^2$ so $\frac{du}{dx} = -2x$

$dx = \frac{du}{-2x}$

$= x^3 \cos^{-1} x + \int \frac{x^3}{u^{\frac{1}{2}}} \left(\frac{du}{-2x} \right)$

$= x^3 \cos^{-1} x - \frac{1}{2} \int u^{-\frac{1}{2}} (1-u)\, du$

$= x^3 \cos^{-1} x - \frac{1}{2} \int \left(u^{-\frac{1}{2}} - u^{\frac{1}{2}} \right) du$

$= x^3 \cos^{-1} x - u^{\frac{1}{2}} + \frac{1}{3} u^{\frac{3}{2}} + c$

$= x^3 \cos^{-1} x - \sqrt{1-x^2} + \frac{1}{3}(1-x^2)^{\frac{3}{2}} + c$

3 marks	Correct answer
2 marks	Obtaining $x^3 \cos^{-1} x - \frac{1}{2} \int u^{-\frac{1}{2}} (1-u)\, du$
1 mark	Obtaining $x^3 \cos^{-1} x + \int \frac{x^3}{\sqrt{1-x^2}} dx$

(c) $\int_0^{\frac{\pi}{4}} x^2 \cos x\, dx$

Let $u = x^2$ so $\frac{du}{dx} = 2x$

Let $\frac{dv}{dx} = \cos x$ so $v = \sin x$

$= \left[x^2 \sin x \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} 2x \sin x\, dx$

Let $u = 2x$ so $\frac{du}{dx} = 2$

Let $\frac{dv}{dx} = \sin x$ so $v = -\cos x$

$= \frac{\pi^2}{16} \sin \frac{\pi}{4} - \left\{ \left[-2x \cos x \right]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} -2\cos x\, dx \right\}$

$= \frac{\pi^2}{16\sqrt{2}} + \frac{2\pi}{4} \cos \frac{\pi}{4} - 2\left[\sin x \right]_0^{\frac{\pi}{4}}$

$= \frac{\pi^2}{16\sqrt{2}} + \frac{\pi}{2\sqrt{2}} - \frac{2}{\sqrt{2}}$

4 marks	Correct answer
3 marks	Correct integration by parts twice with one mistake
2 marks	Correct integration by parts once including substitution
1 mark	Correct integration by parts once with one mistake

(d) (i) $\frac{3x^2 - x + 12}{(x-2)(x^2+2x+3)} \equiv \frac{A}{x-2} + \frac{Bx-3}{x^2+2x+3}$

$3x^2 - x + 12 \equiv A(x^2+2x+3) + (Bx-3)(x-2)$

Sub $x = 2$ so, $22 = 11A$, $A = 2$

$x = 1$ so, $14 = 12 - (B-3)$, $B = 1$

2 marks	Finding correct values of A and B
1 mark	Obtaining $3x^2 - x + 12 \equiv A(x^2+2x+3) + (Bx-3)(x-2)$

(ii) $\int \frac{3x^2-x+12}{(x-2)(x^2+2x+3)}dx$

$= \int\left(\frac{2}{x-2}+\frac{x-3}{x^2+2x+3}\right)dx$

$= \int\left(\frac{2}{x-2}+\frac{x+1}{x^2+2x+3}-\frac{4}{x^2+2x+3}\right)dx$

$= 2\ln|x-2|+\frac{1}{2}\ln|x^2+2x+3|-\int\frac{4}{(x+1)^2+2}dx$

$= 2\ln|x-2|+\frac{1}{2}\ln|x^2+2x+3|-2\sqrt{2}\tan^{-1}\left(\frac{x+1}{\sqrt{2}}\right)+c$

2 marks	Correct answer
1 mark	Obtaining $2\ln\|x-2\|+\frac{1}{2}\ln\|x^2+2x+3\|-\int\frac{4}{(x+1)^2+2}dx$

(e) $\int_0^a f(x)dx$

$= \int_a^0 f(a-u)(-du)$

$= -\int_a^0 f(a-u)du$

$= \int_0^a f(a-x)dx$

Let $u=a-x$ so $\frac{du}{dx}=-1$ and $du=-dx$

Also $x=a$ so $u=0$

and $x=0$ so $u=a$

Now let $u=x$ so $du=dx$

Let $I=\int_0^{\frac{\pi}{6}}\frac{\sin^5 3x}{\cos^5 3x+\sin^5 3x}dx$

$= \int_0^{\frac{\pi}{6}}\frac{\sin^5 3\left(\frac{\pi}{6}-x\right)}{\cos^5 3\left(\frac{\pi}{6}-x\right)+\sin^5 3\left(\frac{\pi}{6}-x\right)}dx$

$= \int_0^{\frac{\pi}{6}}\frac{\sin^5\left(\frac{\pi}{2}-3x\right)}{\cos^5\left(\frac{\pi}{2}-3x\right)+\sin^5\left(\frac{\pi}{2}-3x\right)}dx$

$= \int_0^{\frac{\pi}{6}}\frac{\cos^5 3x}{\sin^5 3x+\cos^5 3x}dx$

$= I$

$2I = \int_0^{\frac{\pi}{6}}\frac{\sin^5 3x+\cos^5 3x}{\sin^5 3x+\cos^5 3x}dx$

$= [x]_0^{\frac{\pi}{6}}$

$= \frac{\pi}{6}$

$I = \frac{\pi}{12}$

3 marks	Correct answer
2 marks	Obtaining $\int_0^{\frac{\pi}{6}}\frac{\cos^5 3x}{\sin^5 3x+\cos^5 3x}dx$
1 mark	Showing that $\int_0^a f(x)dx=\int_0^a f(a-x)dx$

Question 2 (15 marks)

(a) (i) $1-\sqrt{3}i = 2\text{cis}\left(-\frac{\pi}{3}\right)$

1 mark	Correct answer

(ii) $\left(1-\sqrt{3}i\right)^8 = \left[2\text{cis}\left(-\frac{\pi}{3}\right)\right]^8$

$= 2^8\text{cis}\left(-\frac{8\pi}{3}\right)$ (De Moivre)

$= 2^8\text{cis}\left(-\frac{2\pi}{3}\right)$

$= 2^8\left(\cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3}\right)$

$= 2^8\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

$= -128 - 128\sqrt{3}i$

2 marks	Correct answer
1 mark	Obtaining $2^8\text{cis}\left(\frac{-2\pi}{3}\right)$

(iii) $\left(1-\sqrt{3}i\right)^n = 2^n\text{cis}\left(-\frac{n\pi}{3}\right)$ (De Moivre)

$= 2^n\left(\cos\frac{n\pi}{3} - i\sin\frac{n\pi}{3}\right)$ is real

So $\sin\frac{n\pi}{3} = 0$ (imaginary part equals zero)

$\frac{n\pi}{3} = k\pi$ where $n, k \in J$

so $n = 3k$

2 marks	Correct answer
1 mark	Obtaining $\sin\frac{n\pi}{3} = 0$

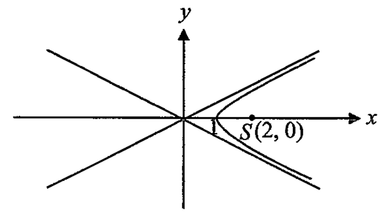
(b) (i) $|z-2| = \text{Re}(z),\ z = x+iy$

$\sqrt{(x-2)^2 + y^2} = x$

$x^2 - 4x + 4 + y^2 = x^2$

$y^2 = 4(x-1)$

Parabola with vertex $(1,0)$ and focus $(2,0)$



2 marks	Obtaining correct Cartesian equation and graph
1 mark	Obtaining correct Cartesian equation

(ii) let $y = mx$ be the equation of tangent

$y^2 = 4(x-1)$ −(1)

$y = mx$ −(2)

Sub (2) into (1)

$m^2x^2 = 4x - 4$

$m^2x^2 - 4x + 4 = 0$

$\Delta = 16 - 16m^2 = 0$ (since it is a tangent)

$m^2 = 1$

so $m = \pm 1$

Since $\tan\theta = m$

$\tan\theta = \pm 1$

so $\theta = \pm\frac{\pi}{4}$

Hence $-\frac{\pi}{4} \le \text{Arg}\, z \le \frac{\pi}{4}$

3 marks	Finding correct values of z
2 marks	Finding $m = \pm 1$ where m is the gradient of the tangent
1 mark	Obtaining $m^2x^2 - 4x + 4 = 0$

(c) (i) $\vec{AB} = \vec{AO} + \vec{OB}$

$\vec{OB} = -2 + 4i$

$\vec{OA} = -6 + 2i$

$\vec{AB} = 6 - 2i - 2 + 4i$

$= 4 + 2i$

1 mark	Correct answer

(ii) $\vec{OC} = \vec{OB} + \vec{BA} + \vec{AC}$

$= \vec{OB} - \vec{AB} + \vec{AC}$

$\vec{AC} = 2\vec{AB}\,cis\left(-\frac{\pi}{6}\right)$

$\vec{OC} = -2 + 4i - (4 + 2i) + 2(4 + 2i)\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$

$= \left(-4 + 4\sqrt{3}\right) + i\left(2\sqrt{3} - 2\right)$

2 marks	Correct answer
1 mark	Finding $\vec{AC}$ correctly

(iii) $\vec{OD} = \vec{OA} + \vec{AD}$

$\vec{AD} = \vec{AB}\,cis\frac{\pi}{3}$

OR $\vec{AD} = \vec{AB}\,cis\left(-\frac{\pi}{3}\right)$

$\vec{OD} = -6 + 2i + (4 + 2i)cis\frac{\pi}{3}$

$= -6 + 2i + (4 + 2i)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right)$

$= -6 + 2i + 2 - \sqrt{3} + i + 2\sqrt{3}i$

$= \left(-4 - \sqrt{3}\right) + i\left(3 + 2\sqrt{3}\right)$

$\vec{OD} = -6 + 2i + (4 + 2i)cis\left(-\frac{\pi}{3}\right)$

$= -6 + 2i + (4 + 2i)\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)$

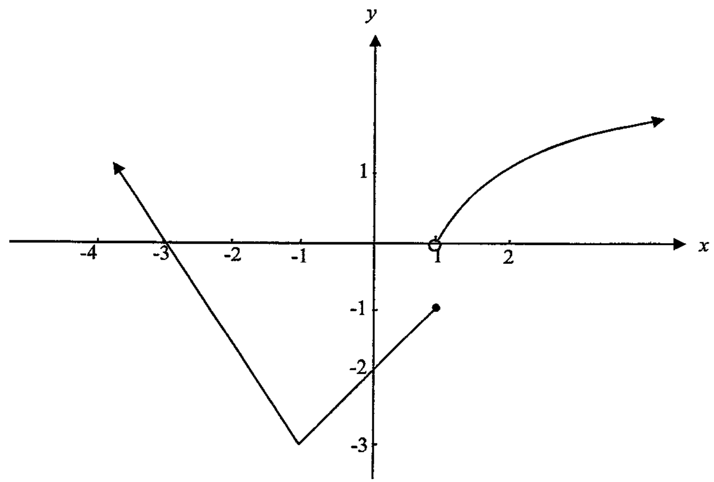
$= -6 + 2i + 2 + \sqrt{3} + i - 2\sqrt{3}i$

$= \left(-4 + \sqrt{3}\right) + i\left(3 - 2\sqrt{3}\right)$

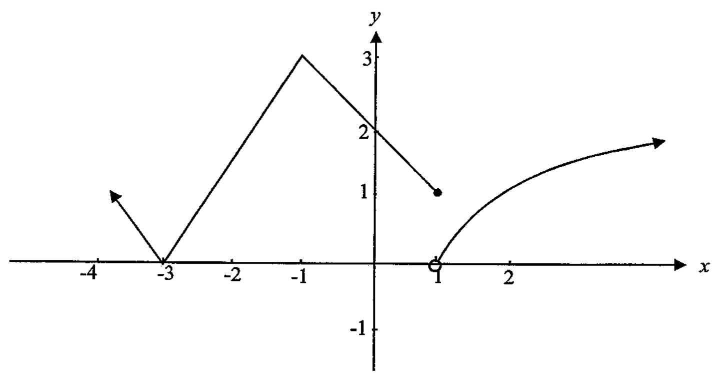
2 marks	Finding both correct answers
1 mark	Finding one correct answer

Question 3 (15 marks)

(a) (i) The graph of $y = f(x-1)$ is shown below.

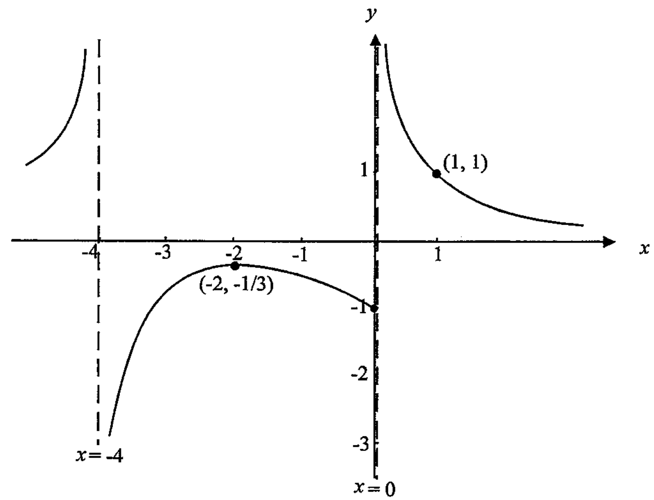


The graph of $y = |f(x-1)|$ is shown below.



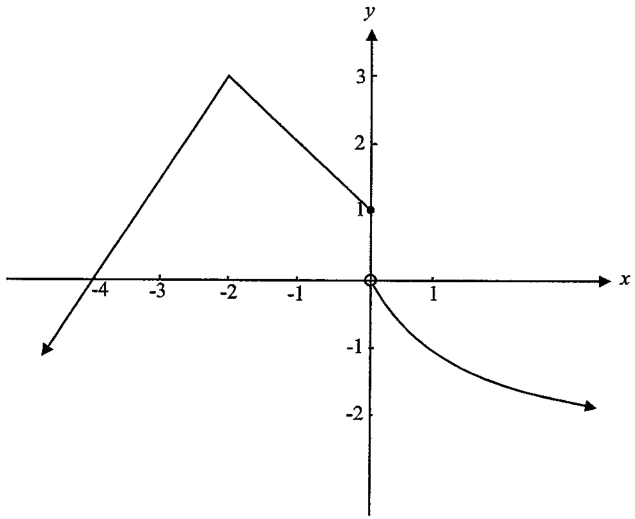
2 marks	Correct graph of $y = \lvert f(x-1)\rvert$
1 mark	Correct graph of $y = f(x-1)$

(ii) The graph of $y = \frac{1}{f(x)}$ is shown below.

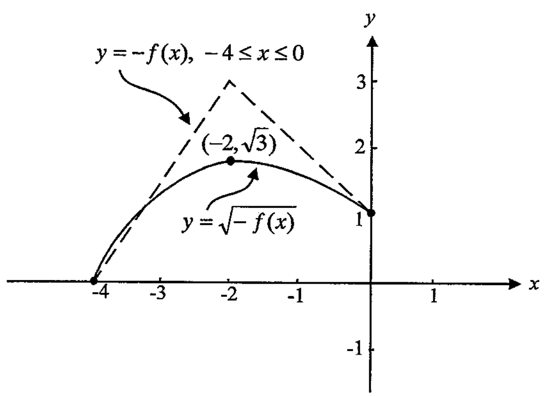


2 marks	Correct graph of $y = \frac{1}{f(x)}$ including clear marking of the two asymptotes at $x = 0$ and $x = -4$ as well as the points $(-2, -\frac{1}{3})$, $(0, -1)$ and $(1, 1)$
1 mark	Correct graph of $y = \frac{1}{f(x)}$ with one or more of the above details missing

(iii) The graph of $y=-f(x)$ is shown below.

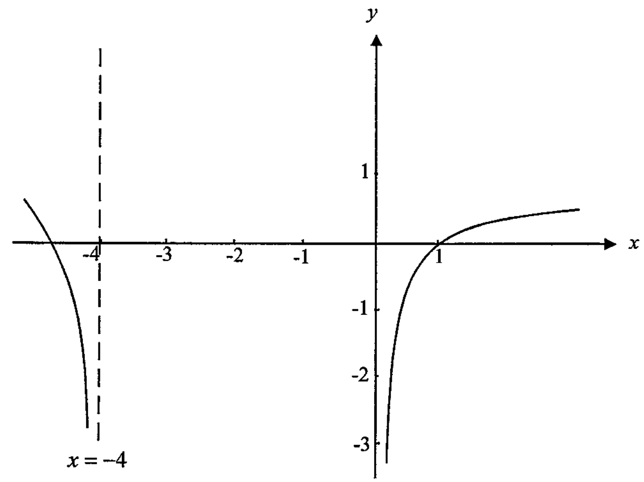


Only those values in the domain of $y=-f(x)$ for which $y \geq 0$ are included in the graph of $y=\sqrt{-f(x)}$. That is, only values of x where $-4 \leq x \leq 0$ are included.
For values of $y=-f(x)$ that are less than 1, $\sqrt{-f(x)} > -f(x)$.
For values of $y=-f(x)$ that are greater than 1, $\sqrt{-f(x)} < -f(x)$.
Note that $\sqrt{-f(0)}=1$ and $\sqrt{-f(-4)}=0$.
The graph of $y=-f(x)$ for $-4 \leq x \leq 0$ together with the graph of $y=\sqrt{-f(x)}$ are shown below.



2 marks	Correct graph including endpoints, and the point $(-2,\sqrt{3})$
1 mark	Correct graph with important points not marked

(iv) The graph of $y=\ln(f(x))$ only exists for $f(x)>0$, that is, for $x<-4$ and $x>0$.
The graph of $y=\ln(f(x))$ is shown below.



2 marks	Correct graph including the asymptote at $x=-4$ and the point $(1,0)$
1 mark	One correct branch of the graph shown OR Correct graph shown with the asymptote and point $(1,0)$ not shown clearly

(b) Now,

$x=\sqrt{2}\cos\theta \qquad y=3\sin\theta$

$x^2=2\cos^2\theta \qquad y^2=9\sin^2\theta$

$\frac{x^2}{2}=\cos^2\theta \qquad \frac{y^2}{9}=\sin^2\theta$

So $\frac{x^2}{2}+\frac{y^2}{9}=1$ is the Cartesian equation of the ellipse. The major axis runs along the y-axis.

$a=3$ and $b=\sqrt{2}$

Now, $b^2=a^2(1-e^2)$

$2=9(1-e^2)$

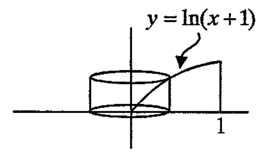
$e^2=\frac{7}{9}$

$e=\frac{\sqrt{7}}{3} \quad (e>0)$

The foci are located at $(0,\pm ae)$ that is, at $(0,\pm\sqrt{7})$.

3 marks	Correctly finding the Cartesian equation, the foci and e.
2 marks	Finding the Cartesian equation and any of the other two above including a correct value of e or correct foci from an incorrect Cartesian equation.
1 mark	Finding the correct Cartesian equation

(c) Consider a small strip running parallel to the y-axis which is to be rotated around the y-axis. Its rotation forms a cylindrical shell of radius x, height $\ln(x+1)$ and thickness δx.



$V=2\pi x\times\ln(x+1)\times\delta x$

$=\lim_{\delta x\to 0}\sum_{x=0}^{1}2\pi x\ln(x+1)\times\delta x$

$=2\pi\int_0^1 x\ln(x+1)dx$

$=2\pi\left\{\left[\frac{x^2}{2}\ln(x+1)\right]_0^1-\int_0^1\frac{x^2}{2}\times\frac{1}{x+1}dx\right\}$

$=2\pi\left\{\frac{1}{2}\ln 2-\frac{1}{2}\int_0^1\frac{x^2}{x+1}dx\right\}$

$=2\pi\left\{\frac{1}{2}\ln 2-\frac{1}{2}\int_0^1\left(x-1+\frac{1}{x+1}\right)dx\right\}$

$=\pi\ln 2-\pi\left[\frac{x^2}{2}-x+\ln(x+1)\right]_0^1$

$=\pi\ln 2-\pi\left\{\frac{1}{2}-1+\ln 2\right\}$

$=\pi\ln 2-\frac{\pi}{2}+\pi-\pi\ln 2$

$=\frac{\pi}{2}$ cubic units

4 marks	Correct answer
3 marks	Correct method with one mistake
2 marks	Correct integrand OR Correct terminals and one mistake in integrand
1 mark	Indicates an understanding of cylindrical shells

Question 4 (15 marks)

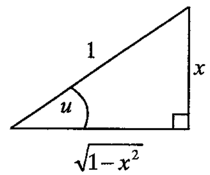
(a) (i) $I_n = \int \frac{dx}{(1-x^2)^{n+\frac{1}{2}}}$

let $x = \sin u$ so $dx = \cos u du$

Also, $\cos u = \sqrt{1-x^2}$

$\cos^2 u = 1 - x^2$

$$(1-x^2)^{n+\frac{1}{2}} = (\cos^2 u)^{n+\frac{1}{2}}$$
$$= (\cos u)^{2n+1}$$
$$I_n = \int \frac{\cos u}{(\cos u)^{2n+1}} du$$
$$= \int \frac{1}{(\cos u)^{2n}} du$$
$$= \int \sec^{2n} u \, du$$



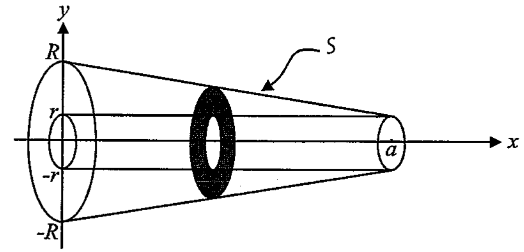
3 marks	Showing $I_n = \int \sec^{2n} u \, du$
2 marks	Showing $(1-x^2)^{n+\frac{1}{2}} = (\cos u)^{2n+1}$
1 mark	Showing $1 - x^2 = \cos^2 u$

(ii) $I_3 = \int \frac{dx}{(1-x^2)^{\frac{7}{2}}}$

$$= \int \sec^6 x dx$$
$$= \int \sec^2 x (1 + \tan^2 x)^2 dx$$
$$= \int \sec^2 x (1 + 2\tan^2 x + \tan^4 x) dx$$
$$= \int (\sec^2 x + 2\tan^2 x \sec^2 x + \tan^4 x \sec^2 x) dx$$
$$= \tan x + \frac{2}{3}\tan^3 x + \frac{1}{5}\tan^5 x + c$$

2 marks	Correct answer
1 mark	Showing $I_3 = \int \sec^2 x (1 + \tan^2 x)^2 dx$

(b) (i)



A typical cross-section is an annulus.
The upper straight edge of S passes through (O, R) and (a, r).
The gradient of the line segment through these two points is

$$\frac{r-R}{a}$$

The equation of the line passing through these two points is

$$y - R = \frac{r-R}{a}(x)$$
$$y = \frac{r-R}{a}x + R$$

The outer radius of the annulus, which is the cross-section, is therefore

$$\frac{r-R}{a}x + R.$$

The inner radius is r.

The area of the annulus is therefore $A = \pi\left(\left(\frac{r-R}{a}x + R\right)^2 - r^2\right)$ as required.

2 marks	Correct derivation
1 mark	Finds correctly the equation of the line segment OR Gives correct formula for area of an annulus without correct expression for the outer radius

(ii) Volume of S

$$= \lim_{\delta x \to 0} \sum_{\delta x = 0}^{a} \pi \left(\left(\frac{r-R}{a} x + R \right)^2 - r^2 \right) \delta x$$

$$= \pi \int_0^a \left(\left(\frac{r-R}{a} x + R \right)^2 - r^2 \right) dx$$

$$= \pi \int_0^a \left(\frac{(r-R)^2 x^2}{a^2} + \frac{2R(r-R)x}{a} + R^2 - r^2 \right) dx$$

$$= \pi \left[\frac{(r-R)^2 x^3}{3a^2} + \frac{2R(r-R)x^2}{2a} + R^2 x - r^2 x \right]_0^a$$

$$= \pi \left(\frac{a^3 (r-R)^2}{3a^2} + \frac{a^2 R(r-R)}{a} + R^2 a - r^2 a \right)$$

$$= \frac{\pi a}{3} \left((r-R)^2 + 3R(r-R) + 3R^2 - 3r^2 \right)$$

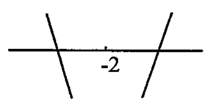
$$= \frac{\pi a}{3} \left(r^2 - 2rR + R^2 + 3rR - 3R^2 + 3R^2 - 3r^2 \right)$$

$$= \frac{\pi a}{3} \left(R^2 + rR - 2r^2 \right) \text{ cubic units}$$

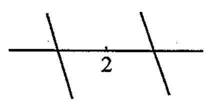
2 marks	Correct answer
1 mark	Correct integrand and terminals

(c) (i) Now, $f'(-2) = 0$ and $f'(2) = 0$.

At $x = -2$, we have a minimum turning point.

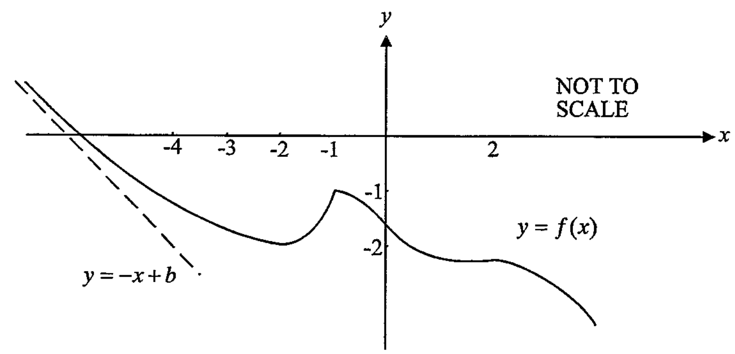


At $x = 2$, we have a stationary point of inflection.



2 marks	Correct answer
1 mark	Identifying but not classifying both stationary points OR identifying and classifying just one stationary point

(ii)



The oblique asymptote $y = -x + b$ should be indicated.
There should be a minimum turning point at $(-2, -2)$.
There should be a cusp at $x = -1$.
There should be a point of inflection at $x = 0$.
There should be a stationary point of inflection at $x = 2$.

4 marks	Correct graph showing all points mentioned
3 marks	A graph that leaves out one of the above points mentioned
2 marks	A graph that leaves out two of the above points mentioned
1 mark	A graph that leaves out three of the above points mentioned

Question 5 (15 marks)

(a) $P(z)=z^4-2z^2+12z-8$

Let $\alpha=1+\sqrt{3}i$ so $\overline{\alpha}=1-\sqrt{3}i$ is also a root since all coefficients of $P(z)$ are real

Since $z-1-\sqrt{3}i$ and $z-1+\sqrt{3}i$ are factors of $P(z)$

then z^2-2z+4 is also a factor of $P(z)$.

$$\begin{aligned}P(z)&=z^4-2z^2+12z-8\\&=(z^2-2z+4)(z^2+2z-2)\quad\text{(by inspection)}\\&=(z-1-\sqrt{3}i)(z-1+\sqrt{3}i)((z+1)^2-3)\\&=(z-1-\sqrt{3}i)(z-1+\sqrt{3}i)(z+1-\sqrt{3})(z+1+\sqrt{3})\text{ over }C\end{aligned}$$

3 marks	Correct answer
2 marks	Finding 2 correct quadratic factors
1 mark	Finding z^2-2z+4 as a factor

(b) $x^3+px^2+qx+r=0$

Let α,β and γ be the roots.

Now $\beta-\alpha=\gamma-\beta$ since we have an arithmetic progression.

So $\alpha+\gamma=2\beta$ $-(1)$

Also $\alpha+\beta+\gamma=-p$

from (1) we have $3\beta=-p$

$\beta=-\frac{p}{3}$ $-(2)$

Also $\alpha\beta+\alpha\gamma+\beta\gamma=q$

$\beta(\alpha+\gamma)+\alpha\gamma=q$ $-(3)$

Also $\alpha\beta\gamma=-r$

$\alpha\gamma=-\frac{r}{\beta}$ $-(4)$

Sub (1) and (4) into (3)

$2\beta^2=q+\frac{r}{\beta}$

from (2), $\frac{2p^2}{9}=q-\frac{3r}{p}$

$2p^3=9pq-27r$

So, $2p^3+27r-9pq=0$

3 marks	Correctly reasoned proof
2 marks	Showing $\alpha\gamma=-\frac{r}{\beta}$
1 mark	Showing $\beta=\frac{-p}{3}$

(c) (i) At P, $x=cp$ and $y=\frac{c}{p}$

$\frac{dx}{dp}=c$ $\quad\frac{dy}{dp}=\frac{-c}{p^2}$

So $\frac{dy}{dx}=\frac{dy}{dp}\cdot\frac{dp}{dx}$

$=\frac{-1}{p^2}$

Equation of tangent at P is

$y-\frac{c}{p}=\frac{-1}{p^2}(x-cp)$

$y=\frac{-x}{p^2}+\frac{c}{p}+\frac{c}{p}$

$y=\frac{2c}{p}-\frac{x}{p^2}$ as required

2 marks	Correct derivation
1 mark	Correct derivation of gradient

(ii) The gradient of the tangent at P is $-\frac{1}{p^2}$ from part (i).

So, the gradient of the tangent at Q is $-\frac{1}{q^2}$.

Since the gradients are equal, we have $-\frac{1}{p^2}=-\frac{1}{q^2}$.

$q^2=p^2$

$(q-p)(q+p)=0$

$q=p$ or $q=-p$

Since P and Q are distinct points, $q\neq p$ so $p=-q$ as required.

1 mark	Correct derivation including reason for dismissing $p=q$ as an answer

(iii) Method 1

gradient of $PO = \dfrac{\frac{c}{p} - 0}{cp - 0}$

gradient of $QO = \dfrac{\frac{c}{q} - 0}{cq - 0}$

Since $p = -q$ from part (ii),

gradient of $QO = \dfrac{-\frac{c}{p}}{-cp}$

$= \dfrac{\frac{c}{p}}{cp}$

= gradient of PO

Since PO and QO have the same gradient and have a common point, PQ is a straight line.

Method 2

gradient of PQ $= \dfrac{\frac{c}{q} - \frac{c}{p}}{cq - cp}$

since $p = -q$, we have $= \dfrac{\frac{c}{-p} - \frac{c}{p}}{-cp - cp}$

$= -\dfrac{2c}{p} \times \dfrac{-1}{2cp}$

$= \dfrac{1}{p^2}$

Equation of PQ $\quad y - \dfrac{c}{p} = \dfrac{1}{p^2}(x - cp)$

Let $x = 0$ and $y = 0$

$LS = 0 - \dfrac{c}{p}$ $\qquad RS = \dfrac{-cp}{p^2}$

$= -\dfrac{c}{p}$ $\qquad = \dfrac{-c}{p}$

$= LS$

So $(0,0)$ lies on PQ and hence PQ passes through the origin.

1 mark	Correct working for either method

(iv) The gradient of the tangent at P is $-\dfrac{1}{p^2}$ from part (i).

The gradient of the normal at P is therefore p^2.
The equation of the normal at P is

$$y - \frac{c}{p} = p^2(x - cp)$$

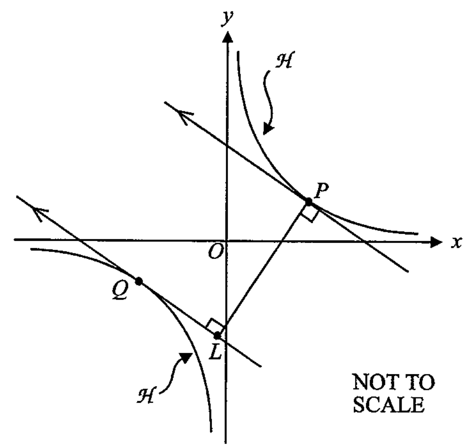
$$y = p^2x - cp^3 + \frac{c}{p} \qquad -(A)$$

The equation of the tangent at Q is $y = \dfrac{2c}{q} - \dfrac{x}{q^2}$ (using part (i)).

Since $p = -q$,

$$y = -\frac{2c}{p} - \frac{x}{p^2} \qquad -(B)$$

Let the normal at P intersect the tangent at Q at point L.



Now, equating (A) and (B) gives

$$p^2x - cp^3 + \frac{c}{p} = -\frac{2c}{p} - \frac{x}{p^2}$$

$$p^2x + \frac{x}{p^2} = -\frac{2c}{p} - \frac{c}{p} + cp^3$$

$$x\left(p^2 + \frac{1}{p^2}\right) = -\frac{3c}{p} + cp^3$$

$$x\frac{(p^4+1)}{p^2} = \frac{-3c + cp^4}{p}$$

$$x = \frac{p^2}{p^4+1} \times \frac{c(p^4-3)}{p}$$

$$= \frac{pc(p^4-3)}{p^4+1}$$

In (B) gives $y = -\frac{2c}{p} - \frac{pc(p^4-3)}{p^2(p^4+1)}$

$$= \frac{-2c(p^4+1) - c(p^4-3)}{p(p^4+1)}$$

$$= \frac{-2cp^4 - 2c - cp^4 + 3c}{p(p^4+1)}$$

$$= \frac{-3cp^4 + c}{p(p^4+1)}$$

$$= \frac{c(1-3p^4)}{p(p^4+1)}$$

So L is the point $\left(\frac{pc(p^4-3)}{p^4+1}, \frac{c(1-3p^4)}{p(p^4+1)}\right)$

The perpendicular distance from P to L is given by

$$\sqrt{\left(cp - \frac{cp(p^4-3)}{p^4+1}\right)^2 + \left(\frac{c}{p} - \frac{c(1-3p^4)}{p(p^4+1)}\right)^2}$$

$$= \sqrt{\left(\frac{cp(p^4+1) - cp(p^4-3)}{p^4+1}\right)^2 + \left(\frac{c(p^4+1) - c(1-3p^4)}{p(p^4+1)}\right)^2}$$

$$= \sqrt{\left(\frac{cp^5 + cp - cp^5 + 3cp}{p^4+1}\right)^2 + \left(\frac{cp^4 + c - c + 3cp^4}{p(p^4+1)}\right)^2}$$

$$= \sqrt{\left(\frac{4cp}{p^4+1}\right)^2 + \left(\frac{4cp^4}{p(p^4+1)}\right)^2}$$

$$= \sqrt{\frac{16c^2p^2}{(p^4+1)^2} + \frac{16c^2p^8}{p^2(p^4+1)^2}}$$

$$= \sqrt{\frac{16c^2p^4 + 16c^2p^8}{p^2(p^4+1)^2}}$$

$$= \sqrt{\frac{16c^2p^4(1+p^4)}{p^2(p^4+1)^2}}$$

$$= \sqrt{\frac{16c^2p^2}{(p^4+1)}}$$

$$= \frac{4cp}{\sqrt{p^4+1}}$$ as required

3 marks	Correct derivation
2 marks	Correctly finding point L
1 mark	Correctly finding equation of normal at P and tangent at Q

(v) Let D equal the perpendicular distance from P to the tangent through Q.

$$D = \frac{4cp}{\sqrt{p^4+1}} \qquad \text{from part (iv)}$$

$$\frac{dD}{dp} = 4c\left(p^4+1\right)^{-\frac{1}{2}} + 4cp \times -\frac{1}{2}\left(p^4+1\right)^{-\frac{3}{2}} \times 4p^3$$

$$= \frac{4c}{\left(p^4+1\right)^{\frac{1}{2}}} - \frac{8cp^4}{\left(p^4+1\right)^{\frac{3}{2}}}$$

$$= \frac{4c\left(p^4+1\right) - 8cp^4}{\left(p^4+1\right)^{\frac{3}{2}}}$$

$$= \frac{4c - 4cp^4}{\left(p^4+1\right)^{\frac{3}{2}}}$$

$$\frac{dD}{dp} = 0$$

$$4c - 4cp^4 = 0$$

$$p^4 = 1$$

$$p = \pm 1 \qquad p \in R$$

Test for max/min.

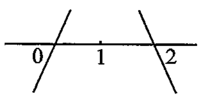
When $p = 0$, $\dfrac{dD}{dp} = 4c$

> 0 since $c > 0$

$p = 2$ $\dfrac{dD}{dp} = \dfrac{4c - 64c}{17^{\frac{3}{2}}}$

$= \dfrac{-60c}{17^{\frac{3}{2}}}$

< 0 since $c > 0$



We have a maximum when $p = 1$.

When $p = 0$ $\dfrac{dD}{dp} = 4c$

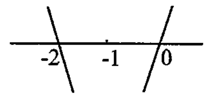
> 0

When $p = -2$ $\dfrac{dD}{dp} = \dfrac{4c - 64c}{17^{\frac{3}{2}}}$

$= \dfrac{-60c}{17^{\frac{3}{2}}}$

< 0 since $c > 0$

We have a minimum when $p = -1$



So the perpendicular distance from P to the tangent through Q is a maximum when $p = 1$.

This occurs when P is (c, c) and Q is $(-c, -c)$.

2 marks	Correctly finding P as well as P and Q including testing for a maximum
1 mark	Correctly finding P and Q but not testing for a maximum OR Finding $p = \pm 1$

Question 6 (15 marks)

(a) (i) Resolving vertically

$T\cos\alpha + N\cos\alpha = mg$

$$N + T = \frac{mg}{\cos\alpha}$$

1 mark	Correct derivation

(ii) Resolving radially (which allows us to introduce ω)

$N\cos(90-\alpha)+T\cos(90-\alpha) = mr\omega^2$

$$N\sin\alpha + T\sin\alpha = ml\sin\alpha\,\omega^2$$

since $r = l\sin\alpha$

$$N + T = ml\omega^2$$

From (i), $N + T = \dfrac{mg}{\cos\alpha}$

So, $l\omega^2 = \dfrac{g}{\cos\alpha}$

$$\omega = \sqrt{\frac{g}{l\cos\alpha}}$$

2 marks	Correct answer
1 mark	Finding $N + T = ml\omega^2$

(iii) Now, $n = \sqrt{\dfrac{g}{2\pi^2 l\cos\alpha}}$ and $\omega = 2\pi n$

$$\omega = 2\pi\sqrt{\frac{g}{2\pi^2 l\cos\alpha}}$$

$$= \sqrt{\frac{2g}{l\cos\alpha}}$$

When the particle is at the point of losing contact, $N = 0$.

From (ii) $N + T = ml\omega^2$

When $N = 0$, $\omega = \sqrt{\dfrac{T}{ml}}$

$$= \sqrt{\frac{mg}{ml\cos\alpha}}$$

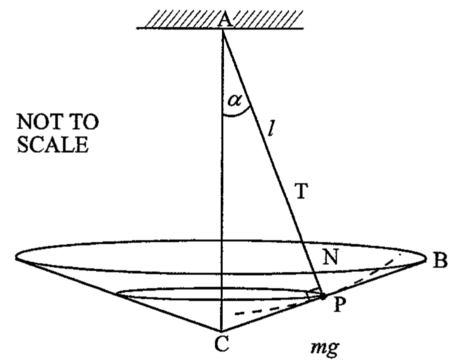
Since from (i), $N + T = \dfrac{mg}{\cos\alpha}$ and $N = 0$

So, $\omega = \sqrt{\dfrac{g}{l\cos\alpha}}$

Since when $n = \sqrt{\dfrac{g}{2\pi^2 l\cos\alpha}}$, $\omega = \sqrt{\dfrac{2g}{l\cos\alpha}}$, the angular velocity is greater than when the particle is at the point of losing contact with the surface. Therefore, the particle is no longer in contact with the surface.

3 marks	Correct explanation
2 marks	Finding ω when $N = 0$ and when $n = \sqrt{\dfrac{g}{2\pi^2 l\cos\alpha}}$ but not giving a correct conclusion
1 mark	Finding $\omega = \sqrt{\dfrac{2g}{l\cos\alpha}}$ OR Finding $\omega = \sqrt{\dfrac{g}{l\cos\alpha}}$

(iv) At any point of contact that the particle has with the cone, the side of the cone BC acts as a tangent to a circle with length l. As ω increases and the particle lifts off the surface, the vertical height of P below A decreases. Since BC is a tangent it does not again come in contact with P which is moving in a circle with centre A and radius l.



1 mark	Correct explanation

(b) (i) Let $x = \tan\theta$ and

$$\frac{4\tan\theta - 4\tan^3\theta}{1-6\tan^2\theta+\tan^4\theta} = \frac{1}{\sqrt{3}}$$

$$\text{so } \frac{4x-4x^3}{1-6x^2+x^4} = \frac{1}{\sqrt{3}}$$

$$x^4 + 4\sqrt{3}x^3 - 6x^2 - 4\sqrt{3}x + 1 = 0$$

$$\tan 4\theta = \frac{1}{\sqrt{3}}$$

$$4\theta = n\pi + \frac{\pi}{6}$$

$$\theta = \frac{n\pi}{4} + \frac{\pi}{24}, n = J$$

$n=0$, $\theta = \frac{\pi}{24}$ so $x = \tan\frac{\pi}{24}$

$n=1$, $\theta = \frac{\pi}{4} + \frac{\pi}{24}$ so $x = \tan\frac{7\pi}{24}$

$n=-1$, $\theta = -\frac{\pi}{4} + \frac{\pi}{24}$ so $x = -\tan\frac{5\pi}{24}$ or $\tan\frac{19\pi}{24}$

$n=2$, $\theta = \frac{\pi}{2} + \frac{\pi}{24}$ so $x = \tan\frac{13\pi}{24}$ or $-\tan\frac{11\pi}{24}$

3 marks	Finding all correct solutions
2 marks	Finding two correct solutions
1 mark	Finding $\theta = \frac{n\pi}{4} + \frac{\pi}{24}, n \in J$

(ii) (1) From part (i) $\tan\frac{\pi}{24} + \tan\frac{7\pi}{24} - \tan\frac{5\pi}{24} - \tan\frac{11\pi}{24} = -4\sqrt{3}$ (sum of roots)

$$\tan\frac{\pi}{24} + \tan\frac{7\pi}{24} + 4\sqrt{3} = \tan\frac{5\pi}{24} + \tan\frac{11\pi}{24}$$

1 mark	Correct answer

(2) Also, $\left(\tan\frac{\pi}{24}\right)\left(\tan\frac{7\pi}{24}\right)\left(-\tan\frac{5\pi}{24}\right)\left(-\tan\frac{11\pi}{24}\right) = 1$ (product of roots)

$$\tan\frac{\pi}{24}\tan\frac{5\pi}{24} = \frac{1}{\tan\frac{7\pi}{24}\tan\frac{11\pi}{24}}$$

$$\text{So, } \tan\frac{\pi}{24}\tan\frac{5\pi}{24} = \cot\frac{7\pi}{24}\cot\frac{11\pi}{24}$$

1 mark	Correct answer

(c) (iii) let $\alpha = \tan\frac{\pi}{24}$

$$\frac{1}{\alpha^2} = \left(\cot\frac{\pi}{24}\right)^2$$

Let $\frac{1}{\alpha^2} = x$ so, $\alpha = \frac{1}{\sqrt{x}}$

$$\left(\frac{1}{\sqrt{x}}\right)^4 + 4\sqrt{3}\left(\frac{1}{\sqrt{x}}\right)^3 - 6\left(\frac{1}{\sqrt{x}}\right)^2 - 4\sqrt{3}\left(\frac{1}{\sqrt{x}}\right) + 1 = 0$$

$$\frac{1}{x^2} + \frac{4\sqrt{3}}{x\sqrt{x}} - \frac{6}{x} - \frac{4\sqrt{3}}{\sqrt{x}} + 1 = 0$$

$$x^2 - 4\sqrt{3}x\sqrt{x} - 6x + 4\sqrt{3}\sqrt{x} + 1 = 0$$

$$4\sqrt{3}x\sqrt{x} - 4\sqrt{3}\sqrt{x} = x^2 - 6x + 1$$

$$4\sqrt{3}\sqrt{x}(x-1) = x^2 - 6x + 1$$

$$48x(x-1)^2 = (x^2-6x+1)^2$$

$$48x(x^2-2x+1) = x^4 - 2x^2(6x-1) + (6x-1)^2$$

$$48x^3 - 96x^2 + 48x = x^4 - 12x^3 + 2x^2 + 36x^2 - 12x + 1$$

$$x^4 - 60x^3 + 134x^2 - 60x + 1 = 0$$

3 marks	Finding correct polynomial
2 marks	Obtaining $4\sqrt{3}\sqrt{x}(x-1) = x^2 - 6x + 1$
1 mark	Obtaining $\left(\frac{1}{\sqrt{x}}\right)^4 + 4\sqrt{3}\left(\frac{1}{\sqrt{x}}\right)^3 - 6\left(\frac{1}{\sqrt{x}}\right)^2 - 4\sqrt{3}\left(\frac{1}{\sqrt{x}}\right) + 1 = 0$

Question 7 (15 marks)

(a) (i) (1) $\dfrac{\binom{16}{4}\binom{12}{4}\binom{8}{4}\binom{4}{4}}{4!} = 2627625$

1 mark	Correct answer

(2) $\dfrac{\binom{4}{1}\binom{12}{3}\times\binom{3}{1}\binom{9}{3}\times\binom{2}{1}\binom{6}{3}\times\binom{1}{1}\binom{3}{3}}{4!} = 369600$

1 mark	Correct answer

(ii) Pr(3 particular runners on the same team)

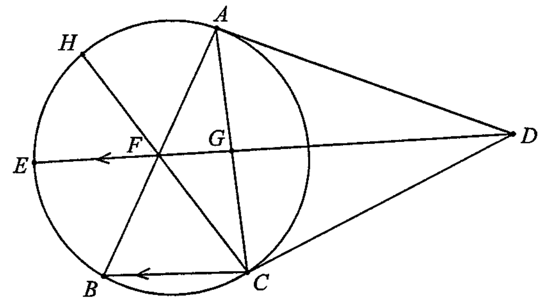
$$= \frac{\dfrac{\binom{3}{3}\binom{13}{1}\times\binom{12}{4}\times\binom{8}{4}\times\binom{4}{4}}{3!}}{\dfrac{\binom{16}{4}\binom{12}{4}\binom{8}{4}\binom{4}{4}}{4!}}$$

$$= \frac{75075}{2627625}$$

$$= \frac{1}{35}$$

2 marks	Correct answer
1 mark	Obtaining $\binom{3}{3}\binom{13}{1}\times\binom{12}{4}\times\binom{8}{4}\times\binom{4}{4}$ without 3! in the denominator

(b)



(i) $\angle CAD = \angle ABC$ (alternate segment theorem)
$\angle ABC = \angle AFD$ (corresponding angles are equal since ED is parallel to BC)
$AD = CD$ (tangents from an external point are equal in length)
So, $\angle CAD = \angle ACD$ (since ΔACD is isosceles)

Hence $\angle AFD = \angle ACD$. Therefore $AFCD$ is a cyclic quadrilateral since angles standing on the same arc AD are equal.

3 marks	Correct proof
2 marks	Showing $\angle CAD = \angle ACD$ with correct reasons
1 mark	Showing $\angle CAD = \angle ABC$ with correct reason

(ii) Since $AFCD$ is a cyclic quad,
$\angle DAC = \angle DFC$ (angles subtended at circumference by arc CD are equal)
$\angle FCB = \angle DFC$ (alternate angles are equal since ED is parallel to BC)
So ΔBCF is isosceles ($\angle CAD = \angle ABC$ from part (i)).
Now $\angle AFH = \angle BFC$ (vertically opposite)
and $\angle AHF = \angle ABC$ (angles subtended at circumference by arc AC are equal)
so ΔHFA is similar to ΔBFC (equiangular)
therefore ΔHFA is isosceles.
So $HF = AF$ (Sides of isosceles triangles opposite equal corresponding sides.)

3 marks	Correct proof
2 marks	Showing $\Delta HFA \equiv \Delta BFC$
1 mark	$\angle FCB = \angle DFC$

(c) Consider $\left(\frac{2m+1}{2m+4}\right)^2 - \frac{3m-2}{3m+1}$

$$= \frac{(2m+1)^2(3m+1)-(3m-2)(2m+4)^2}{(2m+4)^2(3m+1)}$$

$$= \frac{(4m^2+4m+1)(3m+1)-(3m-2)(4m^2+16m+16)}{(2m+4)^2(3m+1)}$$

$$= \frac{12m^3+16m^2+7m+1-(12m^3+40m^2+16m-32)}{(2m+4)^2(3m+1)}$$

$$= \frac{-24m^2-9m+33}{(2m+4)^2(3m+1)}$$

$$= -\frac{3(8m+11)(m-1)}{(2m+4)^2(3m+1)}$$

≤ 0 since $8m+11>0$ and $m-1\geq 0$

We then have $\left(\frac{2m+1}{2m+4}\right)^2 \leq \frac{3m-2}{3m+1}$

So $\frac{2m+1}{2m+4} \leq \sqrt{\frac{3m-2}{3m+1}}$

Substituting $m = 1,2...,n$ successively into the above inequality, we obtain

$$\frac{3}{6}=\frac{1}{2}\leq\sqrt{\frac{1}{4}},\quad \frac{5}{8}\leq\sqrt{\frac{4}{7}},\quad \frac{7}{10}\leq\sqrt{\frac{7}{10}}$$

$$\frac{2n+1}{2n+4}\leq\sqrt{\frac{3n-2}{3n+1}}$$

Multiplying together

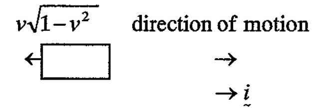
$$\left(\frac{1}{2}\right)\left(\frac{5}{8}\right)\left(\frac{7}{10}\right)...\left(\frac{2n+1}{2n+4}\right)\leq\sqrt{\frac{1}{4}}\times\sqrt{\frac{4}{7}}\times\sqrt{\frac{7}{10}}\times...\times\sqrt{\frac{3n-2}{3n+1}}$$

So $\left(\frac{1}{2}\right)\left(\frac{5}{8}\right)\left(\frac{7}{10}\right)...\left(\frac{2n+1}{2n+4}\right)\leq\frac{1}{\sqrt{3n+1}}$

5 marks	Correct proof
4 marks	Showing $\frac{2n+1}{2n+4}\leq\sqrt{\frac{3n-2}{3n+1}}$
3 marks	Showing $\frac{2m+1}{2m+4}\leq\sqrt{\frac{3m-2}{3m+1}}$
2 marks	Showing $\left(\frac{2m+1}{2m+4}\right)^2-\frac{3m-2}{3m+1}\leq 0$
1 mark	Showing $\left(\frac{2m+1}{2m+4}\right)^2-\frac{3m-2}{3m+1}=\frac{12m^3+16m^2+7m+1-(12m^3+40m^2+16m-32)}{(2m+4)^2(3m+1)}$

Question 8 (15 marks)

(a) (i) $\underset{\sim}{R} = m\underset{\sim}{a}$



$-v\sqrt{1-v^2}\,\underset{\sim}{i} = a\,\underset{\sim}{i}$

So $a = -v\sqrt{1-v^2}$

$\ddot{x} = -v\sqrt{1-v^2}$

$v\frac{dv}{dx} = -v\sqrt{1-v^2}$

$\frac{dv}{dx} = -\sqrt{1-v^2}$

$\frac{dx}{dv} = \frac{-1}{\sqrt{1-v^2}}$

$x = -\sin^{-1}v + c$

When $x=0,\ v=R$ so

$0 = -\sin^{-1}R + c$

$x = -\sin^{-1}v + \sin^{-1}R$

Let $\alpha = \sin^{-1}v$ and $\beta = \sin^{-1}R$

So, $v = \sin\alpha$ $\quad R = \sin\beta$

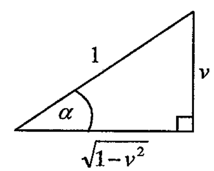
So $x = \beta - \alpha$

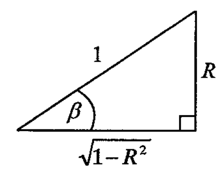
$\sin x = \sin(\beta-\alpha)$

$= \sin\beta\cos\alpha - \cos\beta\sin\alpha$

Since

$\sin\alpha = v$ and $\sin\beta = R$





$\cos\alpha = \sqrt{1-v^2}$ $\quad \cos\beta = \sqrt{1-R^2}$

So, $\sin x = R\sqrt{1-v^2} - v\sqrt{1-R^2}$

$x = \sin^{-1}\left(R\sqrt{1-v^2} - v\sqrt{1-R^2}\right)$

as required.

4 marks	Correct explanation
3 marks	Establishing that $\sin x = \sin\beta\cos\alpha - \cos\beta\sin\alpha$ or equivalent
2 marks	Finding $x = -\sin^{-1}v + \sin^{-1}R$
1 mark	Finding $\ddot{x} = -v\sqrt{1-v^2}$

(ii) From (i), $\ddot{x} = -v\sqrt{1-v^2}$

$$\frac{dv}{dt} = -v\sqrt{1-v^2}$$

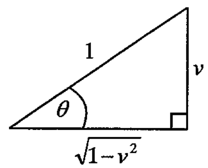
$$\frac{dt}{dv} = \frac{-1}{v\sqrt{1-v^2}}$$

$$t = -1\int \frac{1}{v\sqrt{1-v^2}}dv$$

Let $v = \sin\theta$

$$\frac{dv}{d\theta} = \cos\theta$$

and $\cos\theta = \sqrt{1-v^2}$



So, $$t = -\int \frac{1}{\sin\theta\cos\theta}\cdot\frac{dv}{d\theta}\cdot d\theta \quad \text{-(*)}$$

$$= -\int \frac{1}{\sin\theta\cos\theta}\cdot\cos\theta\, d\theta$$

$$= -\int \text{cosec}\theta\, d\theta$$

$$= -\int \frac{\text{cosec}\theta(\text{cosec}\theta - \cot\theta)}{\text{cosec}\theta - \cot\theta}d\theta$$

Now since

$$\frac{d(\text{cosec}\theta - \cot\theta)}{d\theta} = \text{cosec}\theta(\text{cosec}\theta - \cot\theta)$$

$$t = -\ln|\text{cosec}\theta - \cot\theta| + c$$

Now from (*)

$\text{cosec}\theta = \frac{1}{v}$ and $\cot\theta = \frac{\sqrt{1-v^2}}{v}$

So,

$$t = -\ln\left|\frac{1}{v} - \frac{\sqrt{1-v^2}}{v}\right| + c$$

$$= -\ln\left|\frac{1-\sqrt{1-v^2}}{v}\right| + c$$

$$t = \ln\left|\frac{v}{1-\sqrt{1-v^2}}\right| + c$$

When $t = 0$, $v = R$ so

$$c = -\ln\left|\frac{R}{1-\sqrt{1-R^2}}\right|$$

$$t = \ln\left|\frac{v}{1-\sqrt{1-v^2}}\right| - \ln\left|\frac{R}{1-\sqrt{1-R^2}}\right|$$

$$t = \ln\left|\frac{v(1-\sqrt{1-R^2})}{R(1-\sqrt{1-v^2})}\right|$$

as required.

4 marks	Correctly derived answer
3 marks	Obtaining $t = -\ln\left\|\frac{1}{v} - \frac{\sqrt{1-v^2}}{v}\right\| + c$
2 marks	Obtaining $t = -\ln\|\text{cosec}\theta - \cot\theta\| + c$
1 mark	Obtaining $t = -\int \text{cosec}\theta\, d\theta$

(b) (i) Using long division

$$1 + \frac{2}{r} + \frac{2}{r^2} + \ldots + \frac{2}{r^n}$$

$r-1\overline{)r+1}$

$r-1$

2

$2 - \frac{2}{r}$

$\frac{2}{r}$

$\frac{2}{r} - \frac{2}{r^2}$

$\frac{2}{r^2}\ldots$

.

.

.

$\frac{2}{r^{n-1}}$

$\frac{2}{r^{n-1}} - \frac{2}{r^n}$

$\frac{2}{r^n}$

So $\frac{r+1}{r-1}=1+\frac{2}{r}+\frac{2}{r^2}+...+\frac{2}{r^n}+\frac{2}{r^n}\div(r-1)$

$=1+\frac{2}{r}+\frac{2}{r^2}+...+\frac{2}{r^n}+\frac{2}{r^n(r-1)}$

2 marks	Correct derivation including last term
1 mark	Attempt at long division

(ii) To show: $\sum_{r=2}^{n}(\ln(r+1)-\ln(r-1))=\ln\frac{n(n+1)}{2}$

$LS=\sum_{r=2}^{n}(\ln(r+1)-\ln(r-1))$

$=\ln 3-\ln 1+\ln 4-\ln 2+\ln 5-\ln 3+\ln 6-\ln 4+...$

$+\ln(n-1)-\ln(n-3)+\ln(n)-\ln(n-2)+\ln(n+1)-\ln(n-1)$

$=-\ln 1-\ln 2+\ln(n)+\ln(n+1)$

$=\ln\frac{n(n+1)}{2}$

$=RS$

Have shown.

(iii) Now $\sum_{r=2}^{n}\ln(1+\frac{2}{r}+\frac{2}{r^2}+...+\frac{2}{r^n}+\frac{2}{r^n(r-1)}=\sum_{r=2}^{n}\ln\frac{r+1}{r-1}$ from part (i)

So we wish to prove the statement

$\sum_{r=2}^{n}\ln\frac{r+1}{r-1}=\ln\frac{n(n+1)}{2}$ by induction for $n=2,3,4...$

When $n=2$,

$LS=\sum_{r=2}^{n}\ln\frac{r+1}{r-1}$

$=\ln 3$

$RS=\ln 3$

$=RS$

So the statement is true for $n=2$.

Let us assume that the statement is true for $n=k$.

That is, let us assume that

$\sum_{r=2}^{k}\ln\frac{r+1}{r-1}=\ln\frac{k(k+1)}{2}$ ...(*)

Now, when $n=k+1$, we have

$\sum_{r=2}^{k+1}\ln\frac{r+1}{r-1}=\ln\frac{(k+1)(k+2)}{2}$

$\sum_{r=2}^{k}\ln\frac{r+1}{r-1}+\ln\frac{k+2}{k}=\ln\frac{(k+1)(k+2)}{2}$

$\ln\frac{k(k+1)}{2}+\ln\frac{k+2}{k}=\ln\frac{(k+1)(k+2)}{2}$ from ...(*)

$\ln\frac{(k+1)(k+2)}{2}=\ln\frac{(k+1)(k+2)}{2}$

If the statement is true for $n=k$ then it is true for $n=k+1$. Since the statement is true when $n=2$, then by mathematical induction it is also true for $n=3$ and so on.

3marks	Correctly reasoned proof
2 marks	Showing the statement is true for $n=2$ and making the assumption given by ...(*)
1 mark	Showing the statement is true for $n=2$