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MATHEMATICS EXTENSION 1

2005

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

General Instructions

- Reading time – 5 minutes
- Writing time – 2 hours
- All seven questions should be attempted
- Total marks available - 84
- All questions are worth 12 marks each
- An approved calculator may be used
- A table of standard integrals appears on page 13 of this trial exam
- All relevant working should be shown for each question

The release date for this exam is Monday 15 August 2005. Teachers are asked not to release this trial exam to students until this date except under exam conditions where the trial exams are collected by teachers at the end of the exam.

Question 1 (12 marks) **Marks**

(a) Let A be the point $(-3,8)$ and let B be the point $(5,-6)$. Find the coordinates of the point P that divides the interval AB internally in the ratio $1:3$. **2**

(b) What is the remainder when the polynomial $P(x)=x^3+3x^2-1$ is divided by $x-2$? **2**

(c) Use the table of standard integrals to find the exact value of **2**

$$\int_0^1 \frac{1}{\sqrt{x^2+9}}\,dx.$$

(d) Solve $\dfrac{2}{x+5}\le 1$. **3**

(e) Use the substitution $u=x-1$ to evaluate $\displaystyle\int_2^4 \frac{x}{(x-1)^2}\,dx$. **3**

Question 2 (12 marks) **Marks**

(a) Sketch the graph of $y = 2\sin^{-1} 3x$ showing clearly the domain and range of the function as well as any intercepts. **2**

(b) Let $f(x) = 4x^2 - 1$. **2**

Use the definition $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ to find the derivative of $f(x)$ at $x = a$.

(c) Find $\frac{d}{dx}\left(3x^2 \cos^{-1} x\right)$ **2**

(d) Find $\int 4\cos^2 3x\,dx$. **2**

(e) Solve the equation $\sin 2\theta = \sqrt{2}\cos\theta$ for $0 \le \theta \le 2\pi$. **4**

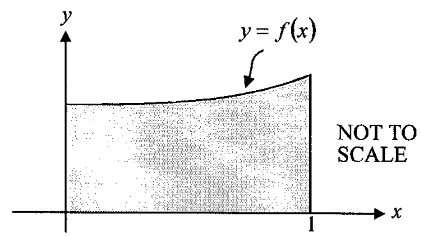
Question 3 (12 marks) **Marks**

(a) Three men and three women are to be seated at a round table. How many seating arrangements are possible if each of the men is to have a woman seated either side of them? **2**

(b) The function $f(x) = \log_e x + 5x$ has a zero near $x = 0\cdot 2$. Using $x = 0\cdot 2$ as a first approximation, use one application of Newton's method to find a second approximation to the zero. Write your answer correct to 3 decimal places. **3**

(c) (i) Find the natural domain of the function $f(x) = \frac{1}{\sqrt{4 - x^2}}$. **1**

(ii) The sketch below shows part of the graph of $y = f(x)$. The area under the curve for $0 \le x \le 1$ is shaded. Find the area of the shaded region. **2**



(d) A particle moves in simple harmonic motion about a fixed point O. The amplitude of the motion is 2 m and the period is $\frac{2\pi}{3}$ seconds. Initially the particle moves from O with a positive velocity.

(i) Explain why the displacement x, in metres, of the particle at time t seconds, is given by **1**

$$x = 2\sin 3t$$

(ii) Find the speed of the particle when it is $\sqrt{3}$ m from O. **2**

(iii) What is the maximum speed reached by the particle? **1**

Question 4 (12 marks) **Marks**

(a) Use mathematical induction to prove that **3**

$$1+6+15+\ldots+n(2n-1)=\frac{1}{6}n(4n-1)(n+1)$$

for all positive integers n.

(b) The population N, of Keystown first reached 25 000 on 1 January 2000.
The population of Keystown is set to increase according to the equation

$$\frac{dN}{dt}=k(N-8000)$$

where t represents time in years after the population first reached 25 000.
On 1 January 2005, the population of Keystown was 29 250.

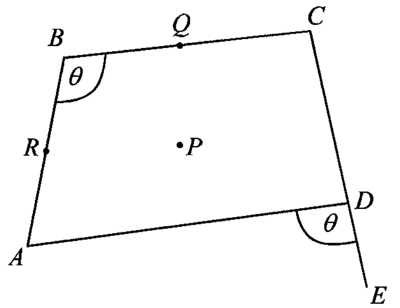
(i) Verify that $N=8000+Ae^{kt}$ is a solution to the above equation where A is a constant. **1**

(ii) Find the values of A and k. **2**

Question 4 continues on the next page.

Question 4 (continued) **Marks**

(c)



NOT TO SCALE

In the diagram, R is the midpoint of AB, Q is the midpoint of BC and $AB = BC$. Point D lies on CE. Let $\angle ADE = \angle ABC = \theta$.

(i) Explain why $ABCD$ is a cyclic quadrilateral. **1**

(ii) Given that P is the centre of the circle that passes through points A, B, C and D, show that $BQPR$ is a cyclic quadrilateral. **2**

(iii) Show that $\angle APR = \dfrac{180^\circ - \theta}{2}$. **3**

Question 5 (12 marks) **Marks**

(a) The Wilson's have a morning newspaper delivered to their home seven days a week. They have noted over time, that on one day out of every twenty, a paper is not delivered. **3**

The probability that over n consecutive mornings the paper is not delivered on just one or two of those mornings, is given by r where $0 \le r \le 1$ and $n = 1,2,3,\ldots$.

Show that

$$r = \frac{n}{20}\left(\frac{n+37}{38}\right)\left(\frac{19}{20}\right)^{n-1}.$$

(b) Consider the function $f(x) = x(x-2)^2$, $x \le a$ where a is a constant.

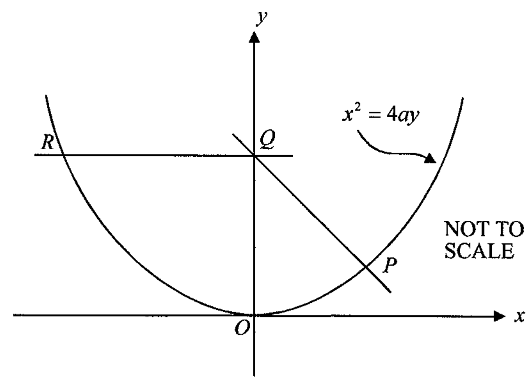
(i) Find the values of a given that the inverse function $f^{-1}(x)$ of $f(x)$ exists. **2**

(ii) State the domain of $f^{-1}(x)$. **1**

Question 5 continues on the next page.

Question 5 (continued) **Marks**

(c)



The diagram above shows the graph of the parabola $x^2 = 4ay$. The normal to the parabola at the variable point $P(2at, at^2)$, $t > 0$, cuts the y-axis at Q. Point R lies on the parabola.

(i) Show that the equation of the normal to the parabola at P is $x + ty = at^3 + 2at$. **2**

(ii) Find the coordinates of R given that QR is parallel to the x-axis and $\angle PQR > 90°$. **2**

(iii) Let M be the midpoint of RQ. Find the Cartesian equation of the locus of M. **2**

Question 6 (12 marks) **Marks**

(a) A particle moves in a straight line with an acceleration given by

$$\frac{d^2x}{dt^2} = 9(x-2)$$

where x is the displacement in metres from an origin O after t seconds. Initially, the particle is 4 metres to the right of O so that $x = 4$ and has velocity $v = -6$.

(i) Show that $v^2 = 9(x-2)^2$. **2**

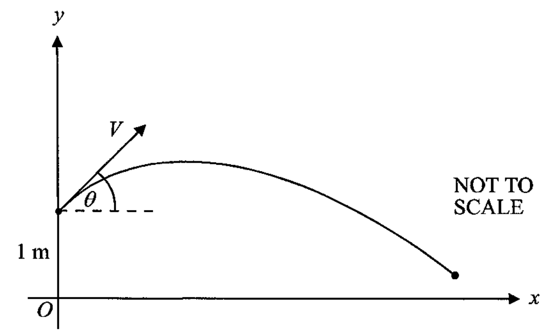
(ii) Find an expression for v and hence find x as a function of t. **2**

(iii) Explain whether the velocity of the particle is ever zero. **2**

Question 6 continues on the next page.

Question 6 (continued) **Marks**

(b) A boy throws a ball and projects it with a speed of V m s^{-1} from a point 1 metre above the ground. The ball lands on top of a flowerpot in a neighbour's yard.



The angle of projection is θ as indicated in the diagram.
The equations of motion of the ball are

$$\ddot{x} = 0 \text{ and } \ddot{y} = -10$$

where x and y are shown on the axes on the diagram. The position of the ball t seconds after it is thrown by the boy is described by the coordinates (x, y).

It has been found that $y = Vt\sin\theta - 5t^2 + 1$.

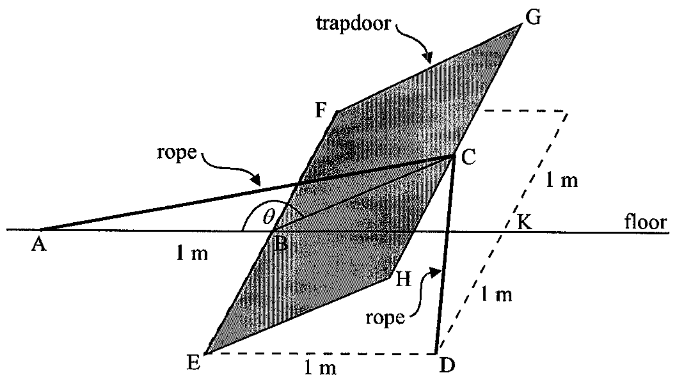
(i) Show that $x = Vt\cos\theta$. **2**

(ii) When the ball is at its maximum height above the ground, it passes directly above a 1.5 metre high fence and clears the fence by 0.5 metres.
Find an expression for V in terms of θ. **2**

(iii) Find the value of V given that $\theta = \tan^{-1}\frac{9}{40}$. **2**
Give your answer in m s^{-1}, correct to 2 decimal places.

Question 7 (12 marks) **Marks**

(a)



A rectangular trapdoor is shown in the diagram as $EFGH$ where $EH = 1$ m, $EF = 2$ m and BC divides the trapdoor in half.
A rope is anchored at point A on the floor 1 metre from point B and points A, B and K lie on a straight line.
The rope passes through a small loop on the edge of the trapdoor at point C and is anchored to the floor at point D.
As the trapdoor is being opened or closed, the rope running from A through C to D is kept taut by pulling it tight or letting it out through anchor point A.

Let $\angle ABC = \theta$, $0^\circ \le \theta \le 180^\circ$.

(i) Show that $AC = \sqrt{2 - 2\cos\theta}$. **1**

(ii) Show that $CD = \sqrt{3 + 2\cos\theta}$. **2**

(iii) Let l equal the length of the rope from A through C to D. find the maximum value of l. Justify your answer. **4**

Question 7 (continued) **Marks**

(b) The binomial expansion is given by

$$(x-1)^n = \sum_{k=0}^{n} {}^nC_k\, x^{n-k}(-1)^k, \quad n > k.$$

(i) By integrating the expansion above, show that **2**

$$\frac{(x-1)^{n+1}}{n+1} = \frac{(-1)^{n+1}}{n+1} + \sum_{k=0}^{n} \frac{{}^nC_k\, x^{n-k+1}(-1)^k}{n-k+1}$$

where n and k are both positive integers and $n > k$.

(ii) By differentiating twice, the binomial expansion given above, obtain an expansion for $n(n-1)(x-1)^{n-2}$, and hence show that **3**

$$n(n-1) = 2 \times {}^nC_{n-2}.$$

STANDARD INTEGRALS

$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \; x \neq 0, \text{ if } n < 0$

$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$

$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$

$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$

$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$

$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$

$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$

$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$

$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, a > 0, \; -a < x < a$

$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$

$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$

NOTE : $\ln x = \log_e x, \quad x > 0$

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MATHEMATICS EXTENSION 1 HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION SOLUTIONS 2005

Question 1 (12 marks)

(a) $P = \left(\dfrac{lx_1 + kx_2}{k+l}, \dfrac{ly_1 + ky_2}{k+l}\right)$

where $k = 1,\ l = 3,\ x_1 = -3,\ y_1 = 8,\ x_2 = 5$ and $y_2 = -6$.

So $P = \left(\dfrac{3\times -3 + 1\times 5}{4}, \dfrac{3\times 8 + 1\times -6}{4}\right)$

$= \left(-1,\ 4\dfrac{1}{2}\right)$

2 marks	Correct answer
1 mark	Either the x or the y coordinate of P correct

(b) $P(x) = x^3 + 3x^2 - 1$

Using the Remainder Theorem,

$P(2) = 8 + 12 - 1$

$= 19$

The remainder is 19.

2 marks	Correct answer
1 mark	Correct method with an arithmetic mistake

(c) From the table of standard integrals, we have

$$\int_0^1 \frac{1}{\sqrt{x^2+9}}\,dx = \left[\ln\left(x + \sqrt{x^2+9}\right)\right]_0^1$$

$$= \ln\left(1+\sqrt{10}\right) - \ln(0+3)$$

$$= \ln\frac{1+\sqrt{10}}{3}$$

2 marks	Correct answer
1 mark	Correct first line above

Question 1 (continued)

(d) Method 1 Multiplying both sides by $(x+5)^2$

$\dfrac{2}{x+5} \le 1, \qquad x \ne -5$

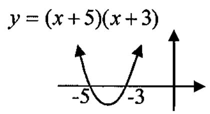
$2(x+5) \le (x+5)^2$

$2x + 10 \le x^2 + 10x + 25$

$0 \le x^2 + 8x + 15$

$0 \le (x+5)(x+3)$

From the graph of $y = (x+5)(x+3)$ we see that $y \ge 0$ for $x \le -5$ or $x \ge 3$ but $x \ne -5$ from above so $x < -5$ or $x \ge 3$.



3 marks	Correct answer
2 marks	Correct method but with one mistake
1 mark	Finds either of the critical points $x = -5$ or $x = -3$ or states that $x \ne -5$

Method 2 Critical points method

$\dfrac{2}{x+5} \le 1, \qquad x \ne -5$

Consider $\dfrac{2}{x+5} = 1$

$2 = x + 5$

$x = -3$

Look at the number line.

For $x = -6,\ \dfrac{2}{-6+5} \le 1$

For $x = -4,\ \dfrac{2}{-4+5} \ge 1$

For $x = -1,\ \dfrac{2}{-1+5} \le 1$

So $\dfrac{2}{x+5} \le 1$ for $x \le -5$ or $x \ge 3$, but $x \ne -5$

So $x < -5$ or $x \ge 3$

3 marks	Correct answer
2 marks	Correct method but with one mistake
1 mark	Finds either of the critical points $x = -5$ or $x = -3$ or states that $x \ne -5$

Question 1 (continued)

Method 3 Consider the cases where $x+5>0$ and $x+5<0$

$$\frac{2}{x+5}\le 1, \qquad x\ne -5$$

Now, $2\le x+5$ if $x+5>0$

$x\ge -3$ if $x>-5$

So $x\ge -3$

Also, $2\ge x+5$ if $x+5<0$

$x\le -3$ if $x<-5$

So $x<-5$

So $x<-5$ or $x\ge -3$

3 marks	Correct answer
2 marks	Correct method but with one mistake
1 mark	Finds either of the critical points $x=-5$ or $x=-3$ or states that $x\ne -5$

(e)

$$\int_2^4 \frac{x}{(x-1)^2} = \int_1^3 (u+1)u^{-2}\frac{du}{dx}dx$$

$$= \int_1^3 \left(u^{-1}+u^{-2}\right)du$$

$$= \left[\log_e u - u^{-1}\right]_1^3$$

$$= \left(\log_e 3-\frac{1}{3}\right)-\left(\log_e 1-1\right)$$

$$= \log_e 3+\frac{2}{3} \qquad (\text{since } \log_e 1 = 0)$$

where $u=x-1$,

$\frac{du}{dx}=1$,

and $x=u+1$

Also, $x=4$ so $u=3$

and $x=2$ so $u=1$

3 marks	Correct answer
2 marks	Finds correct integrand and terminals OR Follows correct method but forgets to change terminals
1 mark	Finds correct terminals OR finds correct integrand

Question 2 (12 marks)

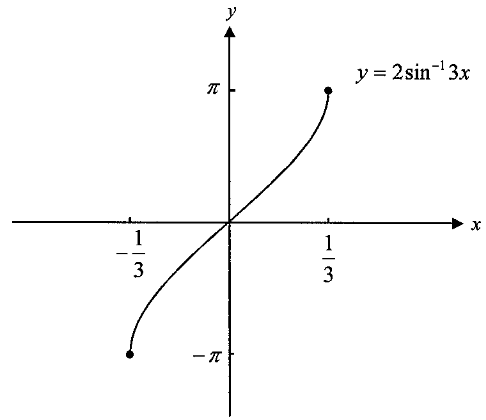
(a) The function $y=2\sin^{-1}3x$ is defined for

$$-1\le 3x\le 1$$

$$-\frac{1}{3}\le x\le\frac{1}{3}$$

The coefficient 2 has the effect of stretching vertically the basic graph of $y=\sin^{-1}3x$ by a factor of 2.

Since the graph of $y=\sin^{-1}3x$ has a range of $-\frac{\pi}{2}\le y\le\frac{\pi}{2}$, the graph of $y=2\sin^{-1}3x$ has a range of $-\pi\le y\le\pi$.



2 marks	Showing correct graph with endpoints clearly marked
1 mark	Correct shape of graph without clearly marked endpoints

Question 2 (continued)

(b)

$$f(x) = 4x^2 - 1$$
$$f(a) = 4a^2 - 1$$
$$f(a+h) = 4(a+h)^2 - 1$$
$$= 4a^2 + 8ah + 4h^2 - 1$$
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$
$$= \lim_{h \to 0} \frac{4a^2 + 8ah + 4h^2 - 1 - 4a^2 + 1}{h}$$
$$= \lim_{h \to 0} \frac{8ah + 4h^2}{h}$$
$$= \lim_{h \to 0} \frac{h(8a + 4h)}{h}$$
$$= \lim_{h \to 0} (8a + 4h)$$
$$= 8a$$

2 marks	Correct derivation
1 mark	Correct substitution into definition

(c) $\frac{d}{dx}\left(3x^2 \cos^{-1} x\right) = 6x \cos^{-1} x + 3x^2 \times \frac{-1}{\sqrt{1-x^2}}$ (Product rule)

$$= 6x \cos^{-1} x - \frac{3x^2}{\sqrt{1-x^2}}$$

2 marks	Correct answer
1 mark	Reasonable attempt to use product rule OR Showing $\frac{d}{dx}\left(\cos^{-1} x\right) = \frac{-1}{\sqrt{1-x^2}}$

(d) $\int 4\cos^2 3x \, dx = 4\int\left(\frac{1}{2} + \frac{1}{2}\cos 6x\right)dx$ since $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$

$$= 4\left(\frac{x}{2} + \frac{1}{12}\sin 6x\right) + c$$
$$= 2x + \frac{1}{3}\sin 6x + c$$

2 marks	Correct answer
1 mark	Correct substitution of identity for $\cos^2 3x$

Question 2 (continued)

(e)

$$\sin 2\theta = \sqrt{2}\cos\theta$$
$$2\sin\theta\cos\theta = \sqrt{2}\cos\theta$$
$$\sqrt{2}\sin\theta\cos\theta - \cos\theta = 0$$
$$\cos\theta\left(\sqrt{2}\sin\theta - 1\right) = 0$$

S	A
T	C

$\cos\theta = 0$ or $\sin\theta = \frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

4 marks	Four correct answers
3 marks	Correct factorisation and finding $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$ OR Correct factorisation and finding $\theta = \frac{\pi}{4}, \frac{3\pi}{4}$
2 marks	Correct factorisation
1 mark	Correct replacement of $\sin 2\theta$

Question 3 (12 marks)

(a) The six people are to be seated at a round table and effectively the men and women sit alternately.

1^{st} woman	1^{st} man	2^{nd} woman	2^{nd} man	3^{rd} woman	3^{rd} man
1	3	2	2	1	1

Hence the number of arrangements possible is $1 \times 3 \times 2 \times 2 \times 1 \times 1 = 12$.

2 marks	Correct answer
1 mark	Taking into account that the arrangement is in a circle and not a straight line

Question 3 (continued)

(b) Now, $f(x) = \log_e x + 5x$

$$f'(x) = \frac{1}{x} + 5$$

and $x_0 = 0 \cdot 2$

Using Newton's method

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$= 0 \cdot 2 - \frac{\log_e 0 \cdot 2 + 5 \times 0 \cdot 2}{\frac{1}{0 \cdot 2} + 5}$$

$= 0 \cdot 261$ (correct to 3 decimal places)

3 marks	Correct answer
2 marks	Finds $f(0 \cdot 2)$ and $f'(0 \cdot 2)$ correctly and substitutes into formula
1 mark	Correctly evaluates $f'(0 \cdot 2)$

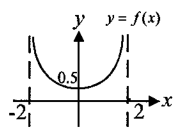
(c) (i) Method 1

Because of the square root sign, $4 - x^2 \geq 0$ so $-2 \leq x \leq 2$.
Also however, because $\sqrt{4 - x^2} \neq 0$, $x \neq -2, 2$.
So the natural domain of f is $-2 < x < 2$.

1 mark	Correctly reasoned answer

Method 2

Sketch the graph.



The natural domain of f is $-2 < x < 2$.

1 mark	Correct answer and correct graph

Question 3 (continued)

(ii) Area $= \int_0^1 \frac{1}{\sqrt{4 - x^2}} dx$

$$= \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^1$$

$$= \sin^{-1}\frac{1}{2} - \sin^{-1} 0$$

$$= \frac{\pi}{6}$$

Area required is $\frac{\pi}{6}$ square units.

2 marks	Correct answer
1 mark	Correct first line above

(d) (i) The particle starts from the centre of motion and moves with positive velocity so the general form of the displacement time function is $x = a \sin nt$.

Now $a = 2$ and period $= \frac{2\pi}{n}$

so $n = 3$

So the required equation is $x = 2 \sin 3t$ as required.

1 mark	Correctly derived equation

Question 3 (continued)

(ii) $x = 2\sin 3t$

When $x = \sqrt{3}$,

$$\sin 3t = \frac{\sqrt{3}}{2}$$

$$3t = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}, \ldots$$

$$t = \frac{\pi}{9}, \frac{2\pi}{9}, \frac{7\pi}{9}, \ldots$$

S	A
T	C

Now $x = 2\sin 3t$

$$\frac{dx}{dt} = 6\cos 3t$$

When $t = \frac{\pi}{9}$,

$$\frac{dx}{dt} = 6\cos\frac{\pi}{3}$$

$$= 3$$

S	A
T	C

(Check when $t = \frac{2\pi}{9}$,

$$\frac{dx}{dt} = 6\cos\frac{2\pi}{3}$$

$$= 6 \times -\frac{1}{2}$$

$$= -3$$

speed $= |-3|$

$= 3$)

So speed is $3\,\text{m}\,\text{s}^{-1}$.

2 marks	Correct answer
1 mark	Finding correct values of t when particle is $\sqrt{3}$ m from O OR Substituting incorrect value of t into correct expression for $\frac{dx}{dt}$ OR Giving an answer of $-3\,\text{m}\,\text{s}^{-1}$

Question 3 (continued)

(iii) Method 1

For *SHM* the maximum speed occurs at the centre of motion i.e. at O.

$$x = 2\sin 3t$$

$$0 = 2\sin 3t$$

$$3t = 0, \pi, 2\pi \ldots$$

$$t = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \ldots$$

S	A
T	C

$$\frac{dx}{dt} = 6\cos 3t$$

When $t = 0$,

$$\frac{dx}{dt} = 6\cos 0$$

$$= 6$$

(Check when $t = \frac{\pi}{3}$,

$$\frac{dx}{dt} = 6\cos\pi$$

$$= -6$$

Speed $= |-6|$

$= 6$)

Maximum speed reached by particle is $6\,\text{m}\,\text{s}^{-1}$.

1 mark	Correct answer

Method 2

Maximum speed occurs when $\frac{d^2x}{dt^2} = 0$

$$\frac{d^2x}{dt^2} = -18\sin 3t$$

$$0 = -18\sin 3t$$

$$3t = 0, \pi, 2\pi, \ldots$$

$$t = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \ldots$$

$$\frac{dx}{dt} = 6\cos 3t$$

When $t = 0$,

$$\frac{dx}{dt} = 6\cos 0$$

$$= 6$$

(Check when $t = \frac{\pi}{3}$,

$$\frac{dx}{dt} = 6\cos\pi$$

$$= -6$$

Speed $= |-6|$

$= 6$)

Maximum speed reached by particle is $6\,\text{m}\,\text{s}^{-1}$.

1 mark	Correct answer.

Question 4 (12 marks)

(a) Prove that $1+6+15+...+n(2n-1)=\frac{1}{6}n(4n-1)(n+1)$ using mathematical induction.

When $n=1$, $RHS=\frac{1}{6}\times 3\times 2$

$=1$

$=LHS$

So the statement is true for $n=1$.

Suppose that it is true for $n=k$.

That is, suppose that $1+6+15+...+k(2k-1)=\frac{1}{6}k(4k-1)(k+1)$ - (A)

Then, when $n=k+1$, we have

$$1+6+15+...+(k+1)\{2(k+1)-1\}=\frac{1}{6}(k+1)\{4(k+1)-1\}\{(k+1)+1\}$$
$$=\frac{1}{6}(k+1)(4k+3)(k+2)$$

$$\begin{aligned}LHS &= 1+6+15+...+(k+1)\{2(k+1)-1\}\\ &= 1+6+15+...+k(2k-1)+(k+1)\{2(k+1)-1\}\\ &= \frac{1}{6}k(4k-1)(k+1)+(k+1)(2k+1) \quad \text{from } (A)\\ &= (k+1)\left\{\frac{1}{6}k(4k-1)+(2k+1)\right\}\\ &= (k+1)\left\{\frac{1}{6}k(4k-1)+\frac{1}{6}(12k+6)\right\}\\ &= \frac{1}{6}(k+1)\{4k^2-k+12k+6\}\\ &= \frac{1}{6}(k+1)(4k+3)(k+2)\\ &= RHS\end{aligned}$$

It is true for $n=1$ and when it is true for $n=k$ it has been proven that it is true for $n=k+1$. Hence by mathematical induction it must be true for all positive integers n.

3 marks	Shows correct proof
2 marks	Shows proof for $n=1$ AND shows correct assumption statement for $n=k$ and substitutes $n=k+1$ correctly into the expression
1 mark	One of either of the points mentioned immediately above

Question 4 (continued)

(b) (i) Consider $\frac{dN}{dt}=k(N-8000)$

$$\begin{aligned}LS &= \frac{dN}{dt}\\ &= \frac{d}{dt}(8000+Ae^{kt})\\ &= Ake^{kt}\\ RS &= k(N-8000)\\ &= k(8000+Ae^{kt}-8000)\\ &= Ake^{kt}\\ &= LS\end{aligned}$$

Have verified.

1 mark	Shows that $N=8000+Ae^{kt}$ satisfies the differential equation OR Finds the general solution to the differential equation

(ii) When $t=0$, $N=25000$

So, $25000=8000+Ae^0$

$A=17000$

When $t=5$, $N=29250$

So, $29250=8000+17000e^{5k}$

$21250=17000e^{5k}$

$1\cdot 25=e^{5k}$

$5k=\log_e 1\cdot 25$

$k=0\cdot 2\log_e 1\cdot 25$

So $A=17000$ and $k=0\cdot 2\log_e 1\cdot 25$

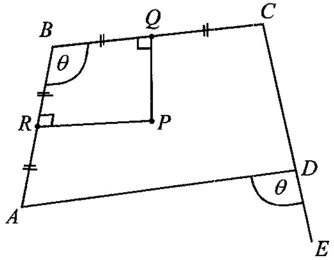
2 marks	Correct answers for A and k
1 mark	Correct answer for A OR Correct answer for k

Question 4 (continued)

(c) (i) $\angle ADE = \angle ABC = \theta$ and hence $ABCD$ is a cyclic quadrilateral because the exterior angle at a vertex of a cyclic quadrilateral equals the interior opposite angle.

1 mark	Correct answer

(ii)



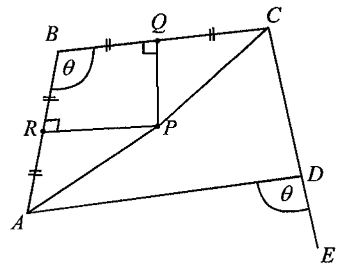
NOT TO SCALE

Since P is the centre of the circle that passes through the points A, B, C and D, then QP is the perpendicular bisector of BC and PR is the perpendicular bisector of AB.
(The line from the centre of a circle to the midpoint of a chord is perpendicular to the chord.)
So $\angle BQP = \angle BRP = 90^\circ$
So $BQPR$ is a cyclic quadrilateral. (The opposite sides of a cyclic quadrilateral are supplementary).

2 marks	Correct reasoning
1 mark	Stating with reasons that QP and PR are the perpendicular bisectors of BC and AB respectively.

Question 4 (continued)

(iii)



NOT TO SCALE

Now, $\angle QPR = 180^\circ - \theta$ (Opposite sides of a cyclic quadrilateral are supplementary.

and $\angle APC = 2\theta$ (A, B and C lie on a circle and the angle at the centre of a circle is twice the angle at the circumference subtended by the same arc.

Also, $PQ = PR$ (Equal chords are equidistant from the centre of the circle.)

$CQ = AR$ (Since $AB = BC$ and R and Q are respective midpoints.)

$AP = CP$ (Radii of a circle are equal.)

So $\Delta APR \equiv \Delta CPQ$ (Three pairs of sides are equal in length.)

So $\angle CPQ = \angle APR$

Now, $\angle QPR + \angle CPQ + \angle CPA + \angle APR = 360^\circ$

So, $180^\circ - \theta + \angle APR + 2\theta + \angle APR = 360^\circ$

$$\angle APR = \frac{180^\circ - \theta}{2}$$

as required.

3marks	Correct reasoning including showing $\Delta APR \equiv \Delta CPQ$
2 marks	Showing congruence correctly but incorrectly reasoning the rest of the case
1 mark	Showing correct reasoning without showing $\Delta APR \equiv \Delta CPQ$

Question 5 (12 marks)

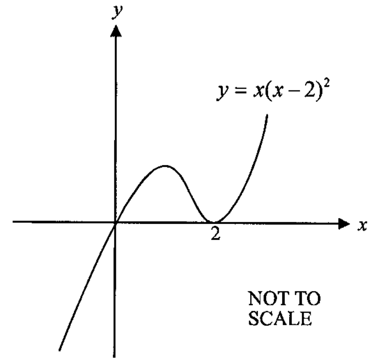
(a)
$$r={}^nC_1\left(\frac{1}{20}\right)^1\left(\frac{19}{20}\right)^{n-1}+{}^nC_2\left(\frac{1}{20}\right)^2\left(\frac{19}{20}\right)^{n-2}$$
$$=\frac{n}{20}\left(\frac{19}{20}\right)^{n-1}+\frac{n(n-1)}{2\times 1}\left(\frac{1}{20}\right)^2\left(\frac{19}{20}\right)^{n-2}$$
$$=\frac{n}{20}\left(\frac{19}{20}\right)^{n-1}\left\{1+\frac{(n-1)}{40}\left(\frac{19}{20}\right)^{-1}\right\}$$
$$=\frac{n}{20}\left(\frac{19}{20}\right)^{n-1}\left\{1+\frac{n-1}{40}\times\frac{20}{19}\right\}$$
$$=\frac{n}{20}\left(\frac{19}{20}\right)^{n-1}\left\{1+\frac{n-1}{38}\right\}$$
$$=\frac{n}{20}\left(\frac{n+37}{38}\right)\left(\frac{19}{20}\right)^{n-1}$$

as required.

3 marks	Correct derivation of answer
2 marks	Gives first line in above solution but "fudges" simplification
1 mark	Gives ${}^nC_1\left(\frac{1}{20}\right)^1\left(\frac{19}{20}\right)^{n-1}$ OR Gives ${}^nC_2\left(\frac{1}{20}\right)^2\left(\frac{19}{20}\right)^{n-2}$

Question 5 (continued)

(b) (i) The inverse function $f^{-1}(x)$ exists if the graph of $y=f(x)$ is $1:1$.
From the graph we see that $f(x)$ is not a 1:1 function.
That is, a horizontal line can be drawn that will cut the graph at more than one point.



One of the turning points is $(2,0)$.
Find the other turning point.

$$y=x(x-2)^2$$
$$=x^3-4x^2+4x$$
$$\frac{dy}{dx}=3x^2-8x+4$$
$$=(3x-2)(x-2)$$

So, $(3x-2)(x-2)=0$

$$x=\frac{2}{3}\text{ or }x=2$$

So $f^{-1}(x)$ exists if $a\le\frac{2}{3}$.

2 marks	Correct answer
1 mark	Giving answer $a\le\frac{2}{3}$ OR Giving an incorrect inequality as an answer after an arithmetic mistake in the differentiation or solution to the quadratic

(ii) The domain of $f(x)$ is $x\le\frac{2}{3}$.

The range of $f(x)$ is $y\le\frac{32}{27}$.

(Since $f\left(\frac{2}{3}\right)=\frac{2}{3}\left(\frac{2}{3}-2\right)^2$
$=\frac{32}{27}$)

So the domain of $f^{-1}(x)$ is $x\le\frac{32}{27}$.

1 mark	Correct answer

Question 5 (continued)

(c) (i) Method 1 – Using the Cartesian equation

$x^2 = 4ay$

$2x = 4a\dfrac{dy}{dx}$ (implicit differentiation)

$\dfrac{dy}{dx} = \dfrac{x}{2a}$

At $P(2at, at^2)$,

$\dfrac{dy}{dx} = \dfrac{2at}{2a}$

$= t$

So the gradient of the normal at P is $-\dfrac{1}{t}$.

Equation of the normal at P:

$y - at^2 = -\dfrac{1}{t}(x - 2at)$

$ty - at^3 = -x + 2at$

$x + ty = at^3 + 2at$ as required

2 marks	Derives correct gradient of normal AND Derives correct equation of normal
1 mark	Derives correct gradient of normal

Method 2 – Using parametric equations

At $P(2at, at^2)$,

$x = 2at$ and $y = at^2$

$\dfrac{dx}{dt} = 2a$ $\quad \dfrac{dy}{dt} = 2at$

So $\dfrac{dy}{dx} = \dfrac{dy}{dt} \cdot \dfrac{dt}{dx}$ (Chain rule)

$= 2at \cdot \dfrac{1}{2a}$

$= t$

So the gradient of the normal at P is $-\dfrac{1}{t}$.

Equation of the normal at P:

$y - at^2 = -\dfrac{1}{t}(x - 2at)$

$ty - at^3 = -x + 2at$

$x + ty = at^3 + 2at$ as required

2 marks	Derives correct gradient of normal AND Derives correct equation of normal
1 mark	Derives correct gradient of normal

Question 5 (continued)

(ii) The normal at P passes through Q which lies on the y-axis.

Now, $x + ty = at^3 + 2at$,

when $x = 0$, $y = at^2 + 2a$

So Q and therefore R have a y-coordinate of $y = at^2 + 2a$.

Since R lies on the parabola, substitute $y = at^2 + 2a$ into $x^2 = 4ay$.

$$x^2 = 4a(at^2 + 2a)$$
$$= 4a^2t^2 + 8a^2$$
$$= 4a^2(t^2 + 2)$$

Since $\angle PQR > 90°$

$$x = -2a\sqrt{t^2 + 2}$$

If $\angle PQR < 90°$, $x = 2a\sqrt{t^2 + 2}$

So R has coordinates $\left(-2a\sqrt{t^2 + 2}, at^2 + 2a\right)$

2 marks	Finding two correct coordinates
1 mark	Finding one correct coordinate

(iii) $M = \left(\dfrac{-2a\sqrt{t^2+2} + 0}{2}, \dfrac{at^2 + 2a + at^2 + 2a}{2}\right)$

$= \left(-a\sqrt{t^2 + 2}, at^2 + 2a\right)$

$x = -a\sqrt{t^2 + 2}$

$y = at^2 + 2a$

So $x = -a\sqrt{\dfrac{y - 2a}{a} + 2}$

$= -a\sqrt{\dfrac{y - 2a + 2a}{a}}$

$x = -a\sqrt{\dfrac{y}{a}}$ is the Cartesian equation of the locus of M.

($x = -\sqrt{ay}$ is also an acceptable equation)

2 marks	Correct equation
1 mark	Correct coordinates of M

Question 6 (12 marks)

(a) (i)

$$\frac{d^2x}{dt^2} = 9(x-2)$$

$$\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 9(x-2)$$

$$\frac{1}{2}v^2 = 9\int(x-2)dx$$

$$= 9\left(\frac{x^2}{2} - 2x\right) + c$$

Initially $x = 4$ and $v = -6$

So, $18 = 9(8-8) + c$

$c = 18$

$$\frac{1}{2}v^2 = 9\left(\frac{x^2}{2} - 2x\right) + 18$$

$$v^2 = 9x^2 - 36x + 36$$

$$= 9(x^2 - 4x + 4)$$

$$= 9(x-2)^2$$

as required

2 marks	Correct integration AND Correct evaluation of c
1 mark	Statement that $\frac{d\left(\frac{1}{2}v^2\right)}{dx} = 9(x-2)$

Question 6 (continued)

(ii) From (i), $v^2 = 9(x-2)^2$

So $v = \pm 3(x-2)$, but

when $t = 0$, $x = 4$ and $v = -6$.

So, $v = -3(x-2)$

So, $\frac{dx}{dt} = -3(x-2)$

$$\frac{dt}{dx} = \frac{-1}{3(x-2)}$$

$$t = -\frac{1}{3}\int\frac{1}{x-2}dx$$

$$= -\frac{1}{3}\log_e(x-2) + c$$

$t = 0$, $x = 4$

$$0 = -\frac{1}{3}\log_e 2 + c$$

$$t = \frac{1}{3}\log_e\frac{2}{x-2}$$

$$3t = \log_e\frac{2}{x-2}$$

$$e^{3t} = \frac{2}{x-2}$$

$$x - 2 = 2e^{-3t}$$

$$x = 2(1 + e^{-3t})$$

2 marks	Correct expression for $x(t)$
1 mark	Correct expression for $v(x)$

(iii) From (ii), $v = -3(x-2)$

If $v = 0$, $0 = -3(x-2)$

So $x = 2$

From (ii) also, $x = 2(1 + e^{-3t})$

If $x = 2$, $2 = 2(1 + e^{-3t})$

$e^{-3t} = 0$ which has no solution.

So v is never zero.

Alternatively, the graph of $x = 2(1 + e^{-3t})$ has an asymptote of $x = 2$ and so $x \neq 2$ so $v \neq 0$.

2 marks	Correct explanation involving $v(x)$ and $x(t)$
1 mark	Correct explanation only as far as involving $v(x)$