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MATHEMATICS

2005

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

General Instructions

- Reading time – 5 minutes
- Writing time – 3 hours
- All questions should be attempted
- Total marks available - 120
- All questions are worth 12 marks
- An approved calculator may be used
- A table of standard integrals can be found on page 14
- All relevant working should be shown for each question

The release date for this exam is Monday 15 August 2005. Teachers are asked not to release this trial exam to students until this date except under exam conditions where the trial exams are collected by teachers at the end of the exam.

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Question 1 (12 marks) **Marks**

(a) Evaluate $\sqrt{\dfrac{e^2 - 1\cdot 2}{2\pi}}$ correct to 4 significant figures. **2**

(b) Differentiate $3x^2(4x-1)$ with respect to x. **2**

(c) Express $0\cdot\dot{1}\dot{8}$ as a fraction in simplest form. **2**

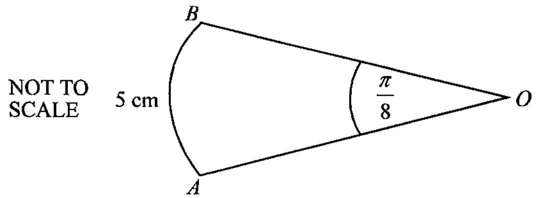
(d) Solve the equation $\dfrac{x-1}{2} - \dfrac{x}{3} = \dfrac{1}{2}$. **2**

(e) Expand and simplify $(2\sqrt{3}-1)^2 - 4(1-\sqrt{3})$. **2**

(f) Solve $|3x-5| \le 4$. **2**

Question 2 (12 marks) **Marks**

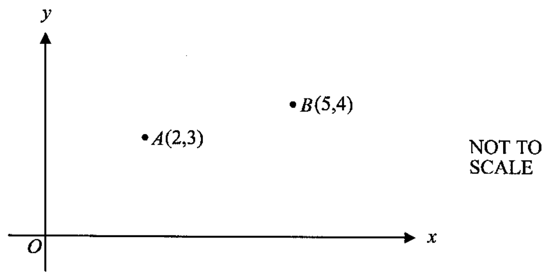
(a)



In the diagram, AB is an arc of a circle with centre O. The length of the arc AB is 5cm and angle AOB is $\frac{\pi}{8}$ radians. Find the length of AO. **2**

(b) Two fair dice are thrown simultaneously. One of the dice is red and the other is black. What is the probability that exactly one 3 is thrown? **2**

(c) The diagram shows the points $A(2,3)$ and $B(5,4)$.



(i) Show that the equation of AB is $x-3y+7=0$. **2**

(ii) Find the coordinates of M, the midpoint of AB. **1**

(iii) Show that the equation of the perpendicular bisector of AB is $3x+y-14=0$. **2**

(iv) The perpendicular bisector of AB cuts the x-axis at C. Find the coordinates of C. **1**

(v) Find the area of triangle BCO. **2**

Question 3 (12 marks) **Marks**

(a) Differentiate

(i) $\left(e^{2x+1}\right)\sin x$ **2**

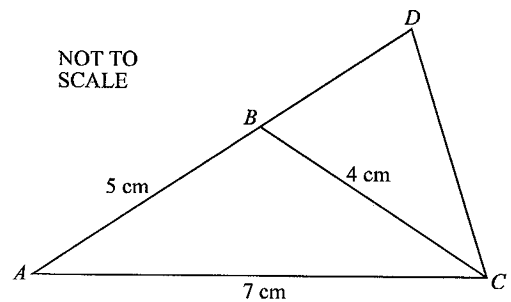
(ii) $\dfrac{\tan 3x}{3x+2}$ **2**

(b) Find

(i) $\int \frac{3}{\sqrt{e^x}}\,dx$ **2**

(ii) $\int \frac{\cos 2x}{\sin 2x}\,dx$ **2**

(c)



In the above diagram, ABC is a triangle in which $AB = 5$ cm, $BC = 4$ cm, $AC = 7$ cm and AB is produced to D such that $AD = AC$.

(i) Find the size of the smallest angle in ΔABC. Express your answer to the nearest degree. **2**

(ii) Hence, find CD. Express your answer to 2 decimal places. **2**

Question 4 (12 marks) **Marks**

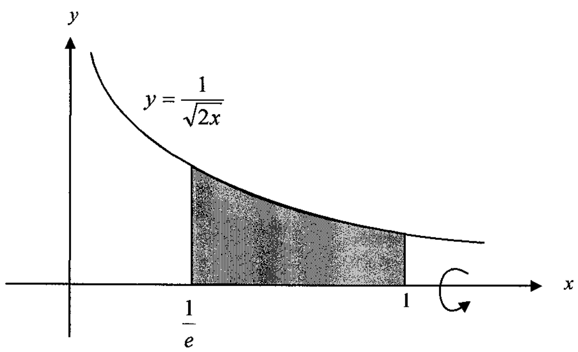
(a) Solve $3\sin x = 1$ for $0 < x \le 2\pi$. **2**
Express your answer in radians, correct to 2 decimal places.

(b) (i) Sketch the curve $y = 2\cos x$ for $0 \le x \le \pi$. **2**

(ii) On the diagram for part (i), shade the region enclosed by the curve $y = 2\cos x$ and the x and y-axis as well as the region enclosed by the curve $y = 2\cos x$, the line $x = \pi$ and the x-axis. **2**

(iii) Find the total area of the shaded regions described in part (ii). **2**

(c)



The shaded region shown in the diagram above is bounded by the curve $y = \frac{1}{\sqrt{2x}}$, the lines $x = \frac{1}{e}$ and $x = 1$ and the x-axis. This shaded region is rotated about the x-axis. **4**

Find the exact volume of the solid of revolution that is formed.

Question 5 (12 marks) **Marks**

(a) Consider the function defined by $f(x) = 2x^3 - 3x^2 - 36x + 26$.

(i) Find the coordinates of the stationary points of the curve $y = f(x)$ and determine their nature. **3**

(ii) Find the coordinates of any point of inflexion. **1**

(iii) Sketch the graph of $f(x) = 2x^3 - 3x^2 - 36x + 26$ by showing the above information. **2**

(iv) For what values of x is the curve concave down and decreasing? **2**

(b) For the parabola $4x = 8y - y^2$.

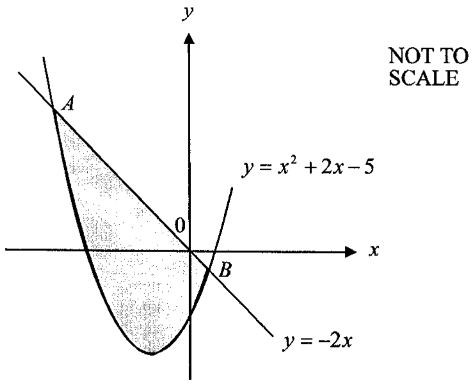
(i) Find the coordinates of the vertex. **2**

(ii) Find the coordinates of the focus. **1**

(iii) Sketch the curve clearly labelling the vertex and focus. **1**

Question 6 (12 marks) **Marks**

(a)



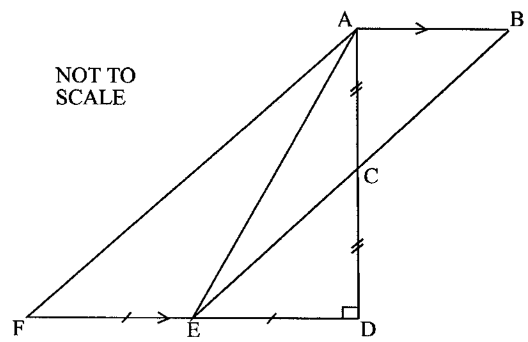
The diagram shows the graphs of $y = x^2 + 2x - 5$ and $y = -2x$. These two graphs intersect at point A and point B.

(i) Find the x values of the points of intersection A and B. **2**

(ii) Calculate the area of the shaded region. **3**

Question 6 (continued) **Marks**

(b)



In the diagram $AB \parallel FD$, ADF is a right-angled triangle, C is the midpoint of AD and E is the midpoint of FD.

(i) Explain why $\angle CED = \angle ABC$. **1**

(ii) Show that $\Delta CDE \equiv \Delta CAB$. **2**

(iii) Show that $AF = 2BC$. **2**

(iv) Show that $\angle ACB = \angle DAF$. **2**

Question 7 (12 marks) **Marks**

(a) In its first year of production, 4000 graphics calculators were sold by a company. Each year after that, sales were 15% more than the previous year's sales.

(i) Find the sales in the 10^{th} year of production. Express your answer to the nearest ten. **1**

(ii) Find the total sales in the first 10 years of production. Express your answer to the nearest ten. **2**

(iii) In which year did the total profit reach \$1 million if the company made \$10 profit on each sale? **2**

(b) The velocity, v, in metres per second, of a particle moving in a straight line is given by $v = 2 + 4\sin\frac{\pi}{2}t$ for $0 \le t \le 4$.

(i) Find the initial velocity and acceleration of the particle. Hence, determine whether the particle is slowing down or speeding up initially. Give reasons for your answer. **3**

(ii) Find the first time the particle is stationary. **2**

(iii) Find the distance travelled by the particle before it changed direction. **2**

Question 8 (12 marks) **Marks**

(a) Consider the geometric series

$$1 + \ln x + (\ln x)^2 + \dots$$

(i) Find the values of x for which the limiting sum of this series exists. **1**

(ii) Find an expression for the limiting sum of the series for those values of x for which it exists. **1**

(b) Tony invests \$150 at the beginning of each quarter in an investment account that pays interest at an annual rate of 8% compounding quarterly.
Let A_n be the amount in dollars in Tony's investment account at the end of the n^{th} quarter.

(i) Find A_1. **1**

(ii) Calculate, using a geometric series, the value of Tony's investment 20 years after his first investment of \$150 was made. **3**

(c) Sue and Charly play a game using 4 wooden blocks, one red, one white, one black and one green. The blocks are placed in a bucket.
Sue goes first and randomly selects a block then returns it to the bucket before randomly selecting a second block that she also returns to the bucket. Charly then repeats this process. A player wins the game if they select the red block at least once and the other player doesn't select the red block at all.
The game is a draw if neither player selects the red block at all or if both players select the red block at least once.

(i) What is the probability that Sue selects the red block at least once? **2**

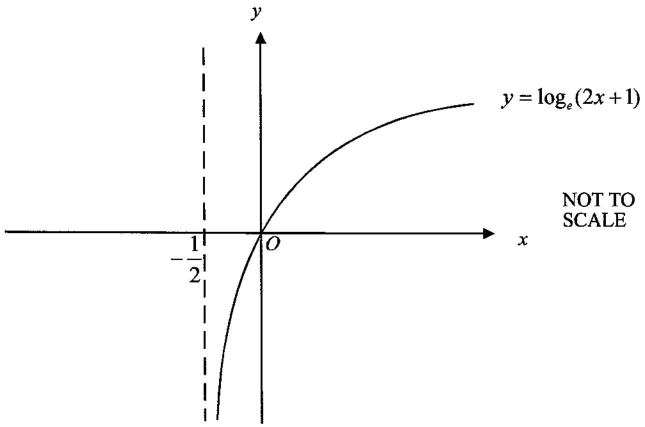
(ii) What is the probability that Sue wins a game? **1**

(iii) What is the probability that a game is a draw? **1**

(iv) If Sue now doesn't replace the first block before she chooses the second block, and then doesn't replace the second before Charly's turn, what is the probability that Charly wins the game? **2**

Question 9 (12 marks) **Marks**

(a) The graph of the function $y = \log_e(2x+1), x > -\frac{1}{2}$ is shown below.



(i) Use the Trapezoidal rule with 5 function values to approximate $\int_0^2 \log_e(2x+1)dx$ and show that this approximation underestimates the value of the integral. **3**

(ii) Find $\int_0^{\ln 5} \frac{(e^y - 1)}{2} dy$ and hence find the exact value of $\int_0^2 \log_e(2x+1)dx$. **3**

Question 9 (continued) **Marks**

(b) The populations of two country towns Alphaville and Betatown are given respectively by

$P_A = 10\,000e^{k_1 t}$ for Alphaville and

$P_B = 6\,000e^{k_2 t}$ for Betatown

where k_1, k_2 are constants, and t is the time in years which have elapsed since January 1st, 1990.

(i) Given that at the beginning of 1990, the populations were 10 000 in Alphaville and 6 000 in Betatown, and they grew to 11 000 and 7 500 respectively at the beginning of 1992, find k_1 and k_2 correct to 3 decimal places. **2**

(ii) Find during which year the population of Betatown became larger than the population of Alphaville? **2**

(iii) Show that the rate of growth of population in Betatown is always larger than in Alphaville. **2**

Question 10 (12 marks) **Marks**

(a) For a particular plane, the cost per hour of fuel for a flight is proportional to the square of the speed of the plane. That is, $C = kv^2$, where C is the cost per hour of fuel, k is a constant and v is the speed of the plane.
If the speed of this plane is 500 km/h, then the cost of fuel per hour is \$250. Other running costs, such as maintenance, labour, and so on, amount to \$160 per hour of the flight.
Let A be the total cost in dollars of a flight at a speed of v km/h.

(i) On a flight of 1000 km, show that **3**

$$A = v + \frac{160000}{v}$$

(ii) What is the speed that this plane should travel at to minimize the cost of a 1000 km flight? **3**
Confirm that your answer does provide a minimum.

(b) Paul works out on a treadmill at his gym. He sets the treadmill to a hill climb where the gradient m increases at a constant rate throughout his jog; that is, $m = Kt$ where K is a constant. The speed, v km/h, that Paul jogs at, is inversely proportional to the gradient of the hill so that $v = \frac{k}{m}$ where k is a constant.

Let x be the distance in kilometres that Paul has jogged at time t hours.

(i) Explain why **2**

$$\frac{dx}{dt} = \frac{c}{t} \text{ where } c \text{ is a constant and } t > 0.$$

(ii) Half an hour after Paul starts jogging on the treadmill, his personal trainer starts to observe him. **4**
The personal trainer watches Paul until time $t = T$ hours, and then the personal trainer wanders off. During this time, the distance, x, that Paul has jogged, is 5 km.
From time $t = T$ hours until Paul finishes his jog 1 hour later, he jogs another 5 km. Find the total time that Paul spends jogging on the treadmill.

STANDARD INTEGRALS

$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \text{ if } n < 0$

$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$

$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$

$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, a > 0, \ -a < x < a$

$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \ x > a > 0$

$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$

NOTE : $\ln x = \log_e x, \quad x > 0$

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MATHEMATICS HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION 2005 SOLUTIONS

Question 1 (12 marks)

(a) $\sqrt{\dfrac{e^2-1\cdot 2}{2\pi}} = 0\cdot 992481...$

$= 0\cdot 9925$

2 marks	Correct answer
1 mark	Answer not expressed to 4 significant figures i.e. $0\cdot 992481...$

(b) $\dfrac{d}{dx}\left(3x^2(4x-1)\right)$

$= \dfrac{d}{dx}\left(12x^3-3x^2\right)$

$= 36x^2-6x$

2 marks	Correct answer
1 mark	Obtaining $36x^2$ term or $-6x$ term

(c) let $x = 0\cdot 181818...$

$100x = 18\cdot 181818...$

$99x = 18$

$x = \dfrac{18}{99}$

$= \dfrac{2}{11}$

So $0\cdot\dot{1}\dot{8} = \dfrac{2}{11}$

2 marks	Correct answer in simplest form
1 mark	Obtaining $\frac{18}{99}$

Question 1 (continued)

(d) $\dfrac{x-1}{2} - \dfrac{x}{3} = \dfrac{1}{2}$

$3(x-1)-2x = 3$

$3x-3-2x = 3$

So $x = 6$

2 marks	Correct answer
1 mark	Obtaining $3(x-1)-2x=3$

(e) $\left(2\sqrt{3}-1\right)^2 - 4\left(1-\sqrt{3}\right)$

$= 12-4\sqrt{3}+1-4+4\sqrt{3}$

$= 9$

2 marks	Correct answer
1 mark	Obtaining $12-4\sqrt{3}+1-4+4\sqrt{3}$

(f) $|3x-5| \le 4$

So $-4 \le 3x-5 \le 4$

$1 \le 3x \le 9$

So $\dfrac{1}{3} \le x \le 3$

2 marks	Correct answer
1 mark	Obtaining $-4 \le 3x-5 \le 4$

Question 2 (12 marks)

(a) $l = r\theta$

So $5 = r \times \frac{\pi}{8}$

$r = \frac{40}{\pi}$ cm

2 marks	Correct answer
1 mark	Correct formula with incorrect substitution or evaluation

(b) Method 1

$P(\text{exactly one 3 thrown})$

$= P$(not 3 on red die and 3 on black die) + P(3 on red die and not 3 on black die)

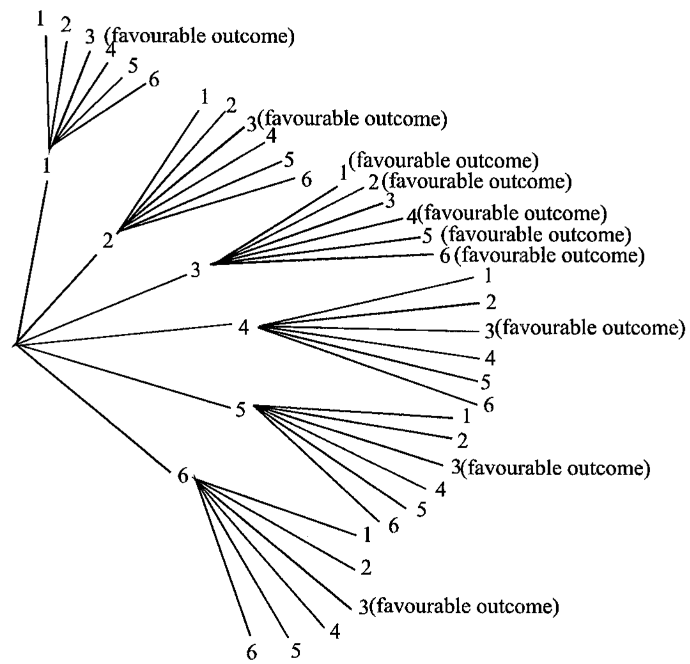
$= \frac{5}{6} \times \frac{1}{6} + \frac{1}{6} \times \frac{5}{6}$

$= \frac{5}{18}$

2 marks	Correct answer
1 mark	For a statement like P(not 3 on red die and 3 on black die) + P(3 on red die and not 3 on black die)

Question 2 (continued)

Method 2



There are 10 favourable outcomes; where exactly one 3 is thrown, out of 36 possible outcomes The probability is $\frac{10}{36}$ or $\frac{5}{18}$.

2 marks	Correct answer
1 mark	Correct diagram

Question 2 (continued)

(c) (i) Gradient of $AB = \dfrac{y_2 - y_1}{x_2 - x_1}$

$= \dfrac{4-3}{5-2}$

$= \dfrac{1}{3}$

Equation of AB is $y - 3 = \dfrac{1}{3}(x-2)$

$3y - 9 = x - 2$

$x - 3y + 7 = 0$

2 marks	Correct derivation
1 mark	Finding correct gradient OR correct method for finding equation with one mistake

(ii) $M = \left(\dfrac{2+5}{2}, \dfrac{3+4}{2}\right)$

$= \left(\dfrac{7}{2}, \dfrac{7}{2}\right)$

1 mark	Correct answer

(iii) The perpendicular bisector of AB passes through the midpoint, $\left(\dfrac{7}{2}, \dfrac{7}{2}\right)$

with a gradient of -3 (the negative reciprocal of $\dfrac{1}{3}$).

Its equation is $y - \dfrac{7}{2} = -3\left(x - \dfrac{7}{2}\right)$

$2y - 7 = -6\left(x - \dfrac{7}{2}\right)$

$2y - 7 = -6x + 21$

$6x + 2y - 28 = 0$

$3x + y - 14 = 0$

2 marks	Correct equation
1 mark	Correct method with incorrect gradient or point OR using correct gradient and point with an arithmetic mistake

Question 2 (continued)

(iv) C lies on the x-axis.
When $y = 0$, $3x - 14 = 0$

$x = \dfrac{14}{3}$

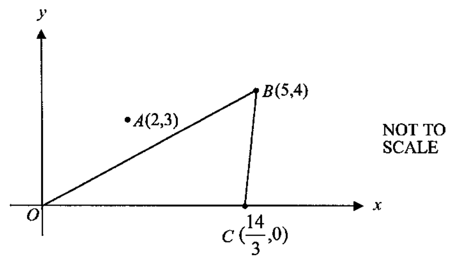
C is the point $\left(\dfrac{14}{3}, 0\right)$

1 mark	Correct answer

(v) Area of $\Delta BCO = \dfrac{1}{2} \times \text{base} \times \text{height}$

$= \dfrac{1}{2} \times \dfrac{14}{3} \times 4$

$= \dfrac{28}{3}$ square units



2 marks	Correct answer
1 mark	Correct method with arithmetic mistake or incorrect substitution

Question 3 (12 marks)

(a) (i) let $y = \left(e^{2x+1}\right)\sin x$

let $u = e^{2x+1}$ and $\frac{du}{dx} = 2e^{2x+1}$

$v = \sin x$ and $\frac{dv}{dx} = \cos x$

$$\frac{dy}{dx} = \left(2e^{2x+1}\right)\sin x + \left(e^{2x+1}\right)\cos x$$
$$= e^{2x+1}(\cos x + 2\sin x)$$

2 marks	Correct answer
1 mark	Obtaining the term $\left(e^{2x+1}\right)\cos x$ OR the term $\left(2e^{2x+1}\right)\sin x$

(ii) let $y = \frac{\tan 3x}{3x+2}$

let $u = \tan 3x$ so $\frac{du}{dx} = 3\sec^2 3x$

and $v = 3x + 2$ so $\frac{dv}{dx} = 3$

$$\frac{dy}{dx} = \frac{3\sec^2 3x(3x+2) - 3\tan 3x}{(3x+2)^2}$$
$$= \frac{3(3x+2)\sec^2 3x - 3\tan 3x}{(3x+2)^2}$$

2 marks	Correct answer
1 mark	Obtaining the term $3\sec^2 3x(3x+2)$ OR the term $3\tan 3x$

(b) (i) $\int \frac{3}{\sqrt{e^x}}\,dx = \int 3e^{-\frac{1}{2}x}\,dx$

$$= -6e^{-\frac{1}{2}x} + c$$

2 marks	Correct answer
1 mark	Obtaining $\int 3e^{-\frac{1}{2}x}\,dx$

Question 3 (cont'd)

(ii) $\int \frac{\cos 2x}{\sin 2x}\,dx = \frac{1}{2}\int \frac{2\cos 2x}{\sin 2x}\,dx$

$$= \frac{1}{2}\log_e(\sin 2x) + c$$

2 marks	Correct answer
1 mark	Omitting $\frac{1}{2}$ in the correct answer

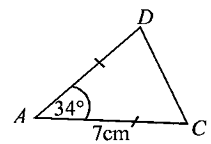
(c) (i) let θ = smallest angle

$= \angle BAC$

$$\cos\theta = \frac{5^2 + 7^2 - 4^2}{2 \times 5 \times 7}$$
$= 34^\circ$ (nearest degree)

2 marks	Correct answer
1 mark	Substituting appropriate values into the cosine rule

(ii) In ΔACD,



$CD^2 = 7^2 + 7^2 - 2 \times 7 \times 7\cos 34^\circ$

$CD = 4\cdot 09$cm (to 2 decimal places)

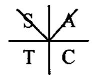
2 marks	Correct answer
1 mark	Substituting appropriate values into the cosine rule

Question 4 (12 marks)

(a) Check that your calculator is in radian mode

$3\sin x = 1 \qquad 0 < x \le 2\pi$

$\sin x = \frac{1}{3}$



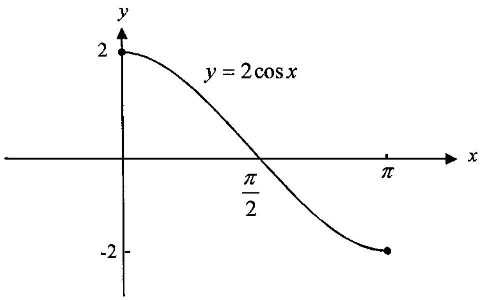
$x = \sin^{-1}\frac{1}{3}$

$= 0\cdot 3398..., \pi - 0\cdot 3398...$ for $0 < x \le 2\pi$

$x = 0\cdot 34, 2\cdot 80$ (each correct to 2 decimal places)

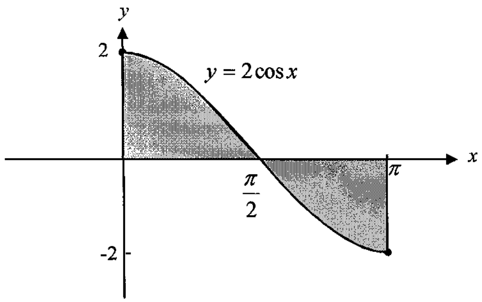
2 marks	Correct answers
1 mark	1 correct answer

(b) (i)



2 marks	Correct graph with intercepts and endpoints marked clearly
1 mark	Correctly shaped graph without intercepts and endpoints

(ii)



2 marks	Correct shading
1 mark	One area correctly shaded

Question 4 (continued)

(iii) Shaded area $= \int_0^{\frac{\pi}{2}} 2\cos x\,dx - \int_{\frac{\pi}{2}}^{\pi} 2\cos x\,dx$

$= 4\int_0^{\frac{\pi}{2}} \cos x\,dx$

The two shaded areas are equal because of the symmetrical nature of the cos curve.

$= 4[\sin x]_0^{\frac{\pi}{2}}$

$= 4\left(\sin\frac{\pi}{2} - \sin 0\right)$

$= 4$

Area required is 4 square units.

2 marks	Correct answer
1 mark	Obtaining a correct integral and terminals that describe the area required.

Question 4 (continued)

(c) Volume $= \pi \int_{e^{-1}}^{1} y^2\,dx$

$$= \pi \int_{e^{-1}}^{1} \frac{1}{2x}\,dx$$

$$= \frac{\pi}{2}\left[\log_e x\right]_{e^{-1}}^{1}$$

$$= \frac{\pi}{2}\left(\log_e 1 - \log_e e^{-1}\right)$$

$$= \frac{\pi}{2}(0 - -1)$$

$$= \frac{\pi}{2}$$

So the volume is $\frac{\pi}{2}$ cubic units.

4 marks	Correct answer
3 marks	Correct integration and substitution OR correct substitution into an incorrectly integrated function OR correct substitution with incorrect limits
2 marks	Evaluates $\pi \int_{e^{-1}}^{1} x^2\,dy$ correctly OR obtains $\pi \int_{e^{-1}}^{1} \frac{1}{2x}\,dx$
1 mark	Gives correct integrand OR Gives correct limits OR Gives $V = \pi \int_{e^{-1}}^{1} x^2\,dy$

Question 5 (12 marks)

(a) (i) $f(x) = 2x^3 - 3x^2 - 36x + 26$

$f'(x) = 6x^2 - 6x - 36$

$f''(x) = 12x - 6$

For stationary points we require $f'(x) = 0$ so

$6x^2 - 6x - 36 = 0$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = 3,\ x = -2$

$x = 3,\quad f(3) = 2(3)^3 - 3(3)^2 - 36(3) + 26$

$= -55$

$x = -2,\ f(-2) = 2(-2)^3 - 3(-2)^2 - 36(-2) + 26$

$= 70$

$(3,-55),\ (-2,70)$

Nature of stationary points:

$f''(3) > 0$ so we have a relative minimum at $(3,-55)$

$f''(-2) < 0$ so we have a relative maximum at $(-2,70)$

3 marks	Finding that a relative maximum occurs at $(-2,70)$ and a relative minimum occurs at $(3,-55)$
2 marks	Finding two correct stationary points
1 mark	Finding one correct stationary point

(ii) For a point of inflexion, firstly find when $f''(x) = 0$.

$12x - 6 = 0$, so $x = \frac{1}{2}$.

Secondly check that the sign of $f''(x)$ changes around the point where $x = \frac{1}{2}$. Now, $f''(0) = -6$ and $f''(1) = 6$ so we do have a point of inflexion at $x = \frac{1}{2}$.

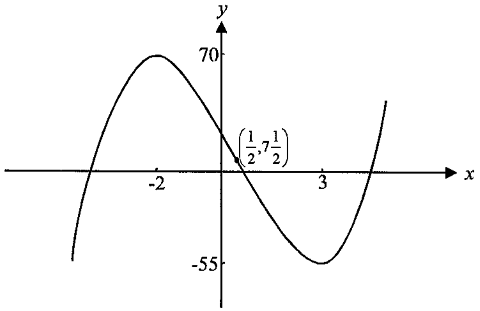
Now $f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 - 36\left(\frac{1}{2}\right) + 26 = 7\frac{1}{2}$

The point of inflexion is $\left(\frac{1}{2}, 7\frac{1}{2}\right)$.

1 mark	Finding $\left(\frac{1}{2}, 7\frac{1}{2}\right)$

Question 5 (cont'd)

(iii)



2 marks	Correct graph showing both stationary points and the point of inflexion
1 mark	Correct graph that omits any stationary point or inflexion point

(iv) For concave down, $f''(x) < 0$

$$12x - 6 < 0$$

$$x < \frac{1}{2}$$

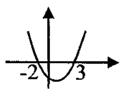
For decreasing function, $f'(x) < 0$

$$6x^2 - 6x - 36 < 0$$

$$x^2 - x - 6 < 0$$

$$(x-3)(x+2) < 0$$

$$-2 < x < 3$$



Hence the curve is concave down and decreasing for $-2 < x < \frac{1}{2}$

2 marks	Correct answer
1 mark	Finding correct values of x for which the curve is concave down OR decreasing

(b) (i)

$$8y - y^2 = 4x$$

$$y^2 - 8y = -4x$$

$$y^2 - 8y + 16 = -4x + 16$$

$$(y-4)^2 = -4(x-4)$$

So the vertex is $(4,4)$ since $(y-k)^2 = -4a(x-h)$.

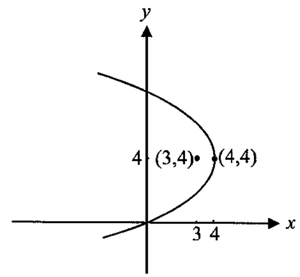
2 marks	Correct answer
1 mark	Obtaining $y^2 - 8y + 16 = -4x + 16$

Question 5 (cont'd)

(ii) The focus is $(3,4)$ since $a = 1$ and the parabola opens to the left.

1 mark	Correct answer

(iii)



1 mark	Correct sketch showing vertex and focus

Question 6 (12 marks)

(a) (i) $y = x^2 + 2x - 5$

$y = -2x$

At points of intersection

$-2x = x^2 + 2x - 5$

$0 = x^2 + 4x - 5$

$= (x+5)(x-1)$

$x = -5$ or $x = 1$

2 marks	Correct values of x
1 mark	Obtaining the equation $0 = x^2 + 4x - 5$

(ii) Area $= \int_{-5}^{1} \left(-2x - (x^2 + 2x - 5)\right)dx$

$= \int_{-5}^{1} (-x^2 - 4x + 5)dx$

$= \left[\frac{-x^3}{3} - \frac{4x^2}{2} + 5x\right]_{-5}^{1}$

$= \left(-\frac{1}{3} - 2 + 5\right) - \left(\frac{125}{3} - 50 - 25\right)$

$= \frac{8}{3} + \frac{100}{3}$

$= 36$ square units

3 marks	Correct answer
2 marks	Obtaining $\left[-\frac{x^3}{3} - 2x^2 + 5x\right]_{-5}^{1}$
1 mark	Obtaining $\int_{-5}^{1} (-x^2 - 4x + 5)dx$

Question 6 (continued)

(b) (i) $\angle CED = \angle ABC$ because alternate angles in parallel lines are equal.

1 mark	Correct answer

(ii) $\angle DCE = \angle ACB$ (vertically opposite angles are equal)

$\angle CED = \angle ABC$ (from part (i))

$AC = CD$ given

Since we have 2 corresponding pairs of angles equal and 1 corresponding pair of sides equal, Δ's CDE and CAB are congruent.

2 marks	Correctly reasoned conclusion
1 mark	No conclusion given OR insufficient reasons provided in argument

(iii) $AF = 2CE$ (A line joining the midpoints of two sides of a triangle is parallel to the third side and half its length.)

Since $CE = BC$, because $\Delta CDE \equiv \Delta CAB$ and CE and BC are corresponding sides, $AF = 2BC$.

2 marks	Correctly reasoned answer
1 mark	Showing $AF = 2CE$

(iv) Now $AB \parallel FE$ and $AB = FE$ since $\Delta CDE \equiv \Delta CAB$, AB and ED are corresponding sides and $ED = FE$.

Therefore $AF \parallel BE$.

So $\angle ACB = \angle DAF$ (alternate angles in parallel lines are equal).

2 marks	Correctly reasoned answer
1 mark	Showing $AF \parallel BE$

Question 7 (12 marks)

(a) (i) Given that $a = 4000$, $r = 1\cdot 15$

$t_{10} = 4000(1\cdot 15)^9$

$= 14\,070$ (nearest ten)

The sales in the 10th year of production were 14 070.

1 mark	Correct answer

(ii) $S_n = \frac{a(r^n - 1)}{r - 1}$

$S_{10} = \frac{4000(1\cdot 15^{10} - 1)}{1\cdot 15 - 1}$

$= \frac{4000(1\cdot 15^{10} - 1)}{0\cdot 15}$

$= 81210$ (nearest ten)

The total sales in the first 10 years of production were 81 210.

2 marks	Correct answer
1 mark	Obtaining $S_{10} = \frac{4000(1\cdot 15^{10} - 1)}{1\cdot 15 - 1}$

(iii) Total profit = \$1 000 000

Profit per calculator = \$10

Number of calculators = 100 000

$\frac{4000(1.15^n - 1)}{1\cdot 15 - 1} = 100000$

$1\cdot 15^n = 4\cdot 75$

$n = \frac{\log_{10} 4\cdot 75}{\log_{10} 1\cdot 15}$

$= 11\cdot 1485...$

During the 12th year of production the total profit reached 1 million.

2 marks	Correct answer
1 mark	Obtaining $\frac{4000(1\cdot 15^n - 1)}{1\cdot 15 - 1} = 100000$

Question 7 (cont'd)

(b) (i) $v = 2 + 4\sin\frac{\pi}{2}t, \quad 0 \le t \le 4$

$a = \frac{dv}{dt} = 2\pi\cos\frac{\pi}{2}t$

When $t = 0$,

$v = 2 + 4\sin 0$

$= 2\,\text{m/s}$

When $t = 0$,

$a = 2\pi\cos 0$

$= 2\pi\,\text{m/s}^2$

The initial velocity is 2 m/s and initial acceleration is $2\pi\,\text{m/s}^2$.

Since $v > 0$ and $a > 0$, the particle is speeding up initially.

3 marks	Finding the initial velocity and acceleration and explaining why the particle is speeding up initially
2 marks	Finding the initial velocity and acceleration
1 mark	Find the initial velocity OR the initial acceleration

(ii) Find the lowest value of t when $v = 0$.

When $v = 0$,

$2 + 4\sin\frac{\pi}{2}t = 0$

$\sin\frac{\pi}{2}t = -\frac{1}{2}$

$\frac{\pi}{2}t = \frac{7\pi}{6}$

$t = \frac{7}{3}$

$t = 2\frac{1}{3}$

The time is $2\frac{1}{3}$ seconds when the particle is first stationary.

2 marks	Correct answer
1 mark	Obtaining $\sin\frac{\pi}{2}t = -\frac{1}{2}$

Question 7 (continued)

(iii) $d = \int_0^{2\frac{1}{3}} \left(2 + 4\sin\frac{\pi}{2}t\right) dt$

$= \left[2t - \frac{8}{\pi}\cos\frac{\pi}{2}t\right]_0^{2\frac{1}{3}}$

$= \left(2\left(2\frac{1}{3}\right) - \frac{8}{\pi}\cos\left(\frac{\pi}{2}\right)\left(2\frac{1}{3}\right)\right) - \left(-\frac{8}{\pi}\cos 0\right)$

$d = \frac{14}{3} - \frac{8}{\pi}\cos\frac{7\pi}{6} + \frac{8}{\pi}$

$= \frac{14}{3} - \frac{8}{\pi}\left(-\frac{\sqrt{3}}{2}\right) + \frac{8}{\pi}$

$= \frac{28\pi + 24\sqrt{3} + 48}{6\pi}$

$= \frac{14\pi + 12\sqrt{3} + 24}{3\pi}$

The distance travelled is $\frac{14\pi + 12\sqrt{3} + 24}{3\pi}$ metres before it changed direction.

2 marks	Correct answer
1 mark	Obtaining $d = \left[2t - \frac{8}{\pi}\cos\frac{\pi}{2}t\right]_0^{2\frac{1}{3}}$

Question 8 (12 marks)

(a) (i) The limiting sum of the series exists if $-1 < r < 1$.

We require $-1 < \ln x < 1$

So $e^{-1} < e^{\ln x} < e^1$

So $\frac{1}{e} < x < e$

1 mark	Correct answer

(ii) $S_\infty = \frac{a}{1-r}$

$= \frac{1}{1 - \ln x}$

1 mark	Correct answer

(b) (i) $A_1 = \$150 \times 1\cdot02$

$= \$153$

1 mark	Correct answer

(ii) The first amount of \$150 that Tony invests will be worth $\$150 \times 1\cdot02^{80}$.

The second amount of \$150 that Tony invests will be worth $\$150 \times 1\cdot02^{79}$ since this second amount was invested for one quarter less than the first \$150.

The last, or 80th amount of \$150 that Tony invests will be worth $\$150 \times 1\cdot02^{1}$ since it will only have been invested for one quarter.

In total, after 20 years (or 80 quarters), the value of Tony's investment is given by

$A_{80} = 150 \times 1\cdot02^{80} + 150 \times 1\cdot02^{79} + \ldots + 150 \times 1\cdot02^{1}$.

This is a geometric series with $a = 150 \times 1\cdot02$ and $r = 1\cdot02$.

So $A_{80} = \frac{a(r^n - 1)}{r - 1}$

$= \frac{150 \times 1\cdot02 \times (1\cdot02^{80} - 1)}{0\cdot02}$

$= 29\,647\cdot11$

The value of Tony's investment after twenty years is \$29 647.11.

3 marks	Correct answer
2 marks	Recognizing the geometric series with $a = 150 \times 1\cdot02$ and $r = 1\cdot02$
1 mark	Recognizing that the first \$150 invested is worth $\$150 \times 1\cdot02^{80}$ at the end of the investment

Question 8 (cont'd)

(c) (i) Method 1

$$P(N,R)+P(R,N)+P(R,R)$$ R = red and N = not red

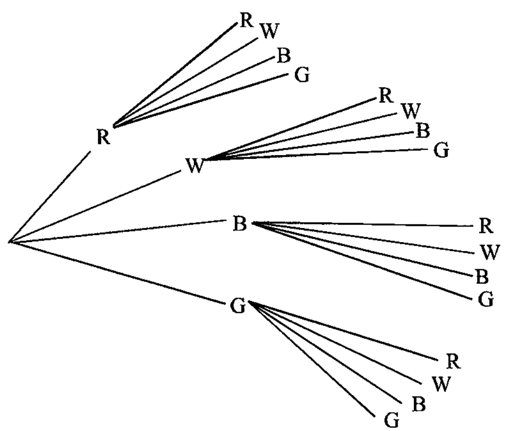
$$=\frac{3}{4}\times\frac{1}{4}+\frac{1}{4}\times\frac{3}{4}+\frac{1}{4}\times\frac{1}{4}$$

$$=\frac{6}{16}+\frac{1}{16}$$

$$=\frac{7}{16}$$

2 marks	Correct answer
1 mark	Giving the first line or its equivalent

Method 2



16 possible outcomes
7 are favourable (i.e. a red block is chosen on the first draw or on the second or on both)

$$\Pr(\text{Sue selects red block at least once})=\frac{7}{16}$$

2 marks	Correct answer
1 mark	Showing the tree diagram correctly

Question 8 (cont'd)

(ii) Sue wins if Sue selects the red block at least once AND Charly doesn't select the red block at all.

$$P(\text{Sue wins})=P(\text{Sue selects the red block at least once})\times P(\text{Charly doesn't select the red block at all})$$

$$=\frac{7}{16}\times\frac{9}{16} \quad \text{(using part(i))}$$

$$=\frac{63}{256}$$

1 mark	Correct answer

(iii) A draw occurs if neither player selects the red block at all OR if both players select the red ball at least once.

$$P(\text{draw})=P(\text{Sue doesn't select the red block at all})\times P(\text{Charly doesn't select the red block at all})+P(\text{Sue selects red block at least once})\times P(\text{Charly selects red block at least once})$$

$$=\frac{9}{16}\times\frac{9}{16}+\frac{7}{16}\times\frac{7}{16}$$

$$=\frac{65}{128}$$

1 mark	Correct answer

Check your answers to parts (i), (ii) and (iii) here. The only possible outcomes for the game are that Sue wins or Charly wins or that the game is a draw.

Check that $\frac{63}{256}+\frac{63}{256}+\frac{130}{256}=1$.

(iv) For Charly to win the game, Sue cannot select the red at all.

$$P(\text{not red, not red})$$

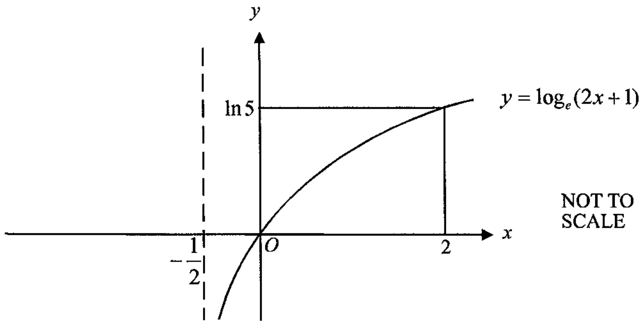
$$=\frac{3}{4}\times\frac{2}{3}$$

$$=\frac{1}{2}$$

2 marks	Correct answer
1 mark	Explaining the possible outcomes but making a mistake with the probabilities

Question 9 (12 marks)

(a) (i)



x	0	0.5	1	1.5	2
$y = \log_e(2x+1)$	0	ln2	ln3	ln4	ln5

$$\int_0^2 \log_e(2x+1)\,dx \approx \frac{0\cdot 5}{2}\left[0 + 2(\ln 2 + \ln 3 + \ln 4) + \ln 5\right]$$
$$= 1\cdot 991 \text{ (3 decimal places)}$$

$$y = \log_e(2x+1)$$
$$\frac{dy}{dx} = \frac{2}{2x+1} = 2(2x+1)^{-1}$$
$$\frac{d^2y}{dx^2} = -2(2x+1)^{-2}(2)$$
$$= \frac{-4}{(2x+1)^2}$$

For $x > -\frac{1}{2}$, $\frac{d^2y}{dx^2} < 0$ so $y = \log_e(2x+1)$ is concave down, hence the approximation is underestimated.

3 marks	Obtaining correct approximation and explaining why it is underestimated
2 marks	Obtaining correct approximation
1 mark	Indicating the 5 correct function values to be used

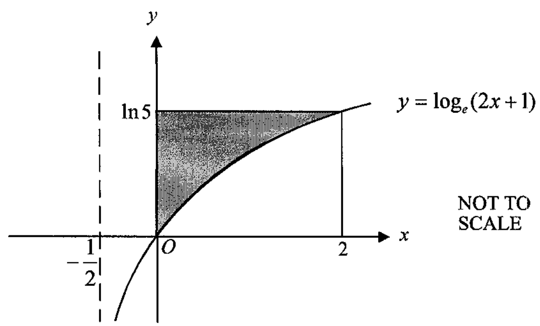
Question 9 (cont'd)

(ii) $$\int_0^{\ln 5} \frac{(e^y - 1)}{2}\,dy = \left[\frac{e^y}{2} - \frac{y}{2}\right]_0^{\ln 5}$$
$$= \left(\frac{e^{\ln 5}}{2} - \frac{\ln 5}{2}\right) - \left(\frac{1}{2}\right)$$
$$= \frac{5}{2} - \frac{\ln 5}{2} - \frac{1}{2}$$
$$= 2 - \frac{1}{2}\ln 5$$

From the diagram below, we see that $\int_0^2 \log_e(2x+1)dx$ is equal to the area of the rectangle with width 2 units and height $\log_e 5$ units minus the shaded area in the diagram. This shaded area is given by $\int_0^{\ln 5} \frac{e^y - 1}{2}dy$ because

$$y = \log_e(2x+1)$$
so, $$e^y = 2x+1$$
$$x = \frac{e^y - 1}{2}$$

So the area can be described by $\int_0^{\ln 5} \frac{e^y - 1}{2}dy$.



So,
$$\int_0^2 \log_e(2x+1)dx = 2\ln 5 - \int_0^{\ln 5} \frac{(e^y - 1)}{2}dy$$
$$= 2\ln 5 - \left(2 - \frac{1}{2}\ln 5\right)$$
$$= \frac{5}{2}\ln 5 - 2$$

3 marks	Correct answer
2 marks	Showing $\int_0^{\ln 5} \frac{(e^y - 1)}{2} dy = 2 - \frac{1}{2}\ln 5$
1 mark	Showing $\int_0^{\ln 5} \frac{(e^y - 1)}{2} dy = \left[\frac{e^y}{2} - \frac{y}{2}\right]_0^{\ln 5}$

(b) (i)

$$P_A = 10\,000e^{k_1 t}$$
$$t = 2,\ P_A = 11000$$
$$11000 = 10\,000e^{2k_1}$$
$$e^{2k_1} = \frac{11}{10}$$
$$2k_1 = \ln\frac{11}{10}$$
$$k_1 = \frac{1}{2}\ln\frac{11}{10}$$
$$= 0\cdot 048 \text{ (3 decimal places)}$$

$$P_B = 6000e^{k_2 t}$$
$$t = 2, P_B = 7500$$
$$7500 = 6000e^{2k_2}$$
$$e^{2k_2} = \frac{75}{60}$$
$$2k_2 = \ln\frac{5}{4}$$
$$k_2 = \frac{1}{2}\ln\frac{5}{4}$$
$$= 0\cdot 112 \text{ (3 decimal places)}$$

2 marks	Obtaining k_1 and k_2 correctly
1 mark	Obtaining k_1 OR k_2 correctly

Question 9 (cont'd)

(ii) If $P_A = P_B$

$$10\,000e^{0\cdot 048t} = 6\,000e^{0\cdot 112t}$$
$$\frac{e^{0\cdot 112t}}{e^{0\cdot 048t}} = \frac{10\,000}{6\,000}$$
$$e^{0\cdot 064t} = \frac{5}{3}$$
$$0\cdot 064t = \ln\frac{5}{3}$$
$$t = \frac{\ln\frac{5}{3}}{0\cdot 064}$$
$$= 7\cdot 98 \quad \text{(2 decimal places)}$$

During 1997, the population of Betatown became larger than the population of Alphaville.

2 marks	Obtaining correct year
1 mark	Obtaining $t = 7\cdot 98$

(iii)

$$P_A = 10\,000e^{0\cdot 048t} \qquad P_B = 6\,000e^{0\cdot 112t}$$
$$\frac{dP_A}{dt} = 480e^{0\cdot 048t} \qquad \frac{dP_B}{dt} = 672e^{0\cdot 112t}$$

Now $e^{0\cdot 112t} \geq e^{0\cdot 048t}$

and $672 > 480$

so $\frac{dP_B}{dt} > \frac{dP_A}{dt}$

2 marks	Correct explanation
1 mark	Finding $\frac{dP_A}{dt}$ and $\frac{dP_B}{dt}$ correctly

Question 10 (12 marks)

(a) (i) $C = kv^2$

So, $250 = k \times 500^2$

$k = \frac{1}{1000}$

So the cost per hour for fuel is $C = \frac{v^2}{1000}$.

The other costs per hour are 160.

The total cost per hour is $\frac{v^2}{1000} + 160$.

The number of hours required to make a 1 000 km trip at v km/h is

$\frac{1\,000\text{km}}{v\text{km/h}}$

$= \frac{1\,000}{v}$ hours

So the total cost for the 1 000 km flight is A where

$A = \frac{1000}{v}\left(\frac{v^2}{1000} + 160\right)$

$= v + \frac{160\,000}{v}$

as required

3 marks	Correct derivation
2 marks	Any two of the points below
1 mark	Finding k OR finding the total cost per hour of the flight OR finding the number of hours required to make a 1 000 km trip at v km/h

Question 10 (cont'd)

(ii) $A = v + \frac{160\,000}{v}$

$\frac{dA}{dv} = 1 - \frac{160\,000}{v^2}$

Min/max occurs when $\frac{dA}{dv} = 0$

So $1 = \frac{160\,000}{v^2}$

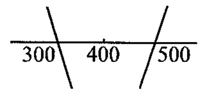
$v^2 = 160\,000$

$v = 400$

Check for minimum.

Let $v = 300$, $\frac{dA}{dv} = -ve$

Let $v = 500$, $\frac{dA}{dv} = +ve$



A minimum exists at $v = 400$.

For the cost of 1 000 km flight to be minimized, the speed of the plane should be 400 km/h.

3 marks	Correct answer including a check for a minimum
2 marks	Correct answer with no check for a minimum
1 mark	Obtaining correctly $\frac{dA}{dv}$

(b) (i) $v = \frac{k}{m}$ (given)

Now, the gradient increases at a constant rate throughout the jog.

So, $m = Kt$ where K is a constant.

So $v = \frac{k}{Kt}$

$= \frac{c}{t}$ where c is a constant $(c = \frac{k}{K})$

$\frac{dx}{dt} = \frac{c}{t}$ as required

2 marks	Correct derivation
1 mark	Substituting $m = Kt$ (K is constant)

Question 10 (cont'd)

(ii) Now $x = \int_{\frac{1}{2}}^{T} \frac{c}{t}\,dt$

So, $5 = c[\ln t]_{\frac{1}{2}}^{T}$

$= c\ln\frac{T}{\frac{1}{2}}$

$= c\ln 2T \qquad -(A)$

Also $x = \int_{T}^{T+1} \frac{c}{t}\,dt$

So $5 = c[\ln t]_{T}^{T+1}$

$= c\ln\frac{T+1}{T} \qquad -(B)$

So, equating (A) and (B),

$c\ln 2T = c\ln\frac{T+1}{T}$

$\ln 2T = \ln\frac{T+1}{T} \qquad c \neq 0$

$2T = \frac{T+1}{T}$

$2T^2 - T - 1 = 0$

$(2T+1)(T-1) = 0$

$T = -\frac{1}{2},\ T = 1$

So $T = 1$ since $T > 0$ $(t > 0)$

So Paul jogs for 2 hours in total.

4 marks	Correct answer
3 marks	Obtaining $2T^2 - T - 1 = 0$
2 marks	Obtaining equation (A) and (B)
1 mark	Obtaining equation (A) or (B)