THE HILLS GRAMMAR SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2003

MATHEMATICS

EXTENSION 2

Time Allowed: Three hours (plus 5 minutes reading time)

Teacher Responsible: Mr D Price

SPECIAL INSTRUCTIONS:

- This paper contains 8 questions. ALL questions to be attempted.
- ALL questions are of equal value.
- ALL necessary working should be shown in every question in the booklets provided.
- Start each question in a new booklet.
- A table of standard integrals is supplied at the back of this paper.
- Board approved calculators may be used.
- Hand up your paper in ONE bundle, together with this question paper.
- ALL HSC course outcomes are being assessed in this task. The Course Outcomes are listed on the back of this sheet.

Students are advised that this is a Trial Examination only and cannot in any way guarantee the content or the format of the Higher School Certificate Examination.

Question 2 (Start a NEW booklet) **Marks**

(a) Let $z = 2(\cos\frac{2\pi}{9} + i\sin\frac{2\pi}{9})$ **8**

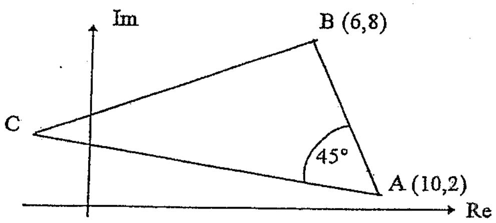
(i) Write down the modulus and argument of the complex numbers

$z, \bar{z}, z^2$, and $\frac{1}{z}$

(ii) Hence, or otherwise, clearly plot the points on an Argand diagram corresponding to $z, \bar{z}, z^2$, and $\frac{1}{z}$ labelling them A, B, C and D respectively.

(b) Find the two square roots of $-3 + 4i$ expressing each root in the form a + ib where a, b are real. **4**

(c) **3**



(NOT TO SCALE)

ΔABC is drawn on the Argand plane where angle BAC = 45°, A represents the complex number $10 + 2i$ and B represents $6 + 8i$.

If the length of side AC is twice the length of AB then find the complex number that point C represents.

Question 3 (Start a NEW booklet) **Marks**

(a) For the hyperbola $9x^2 - 16y^2 = 144$ find, **7**

(i) eccentricity

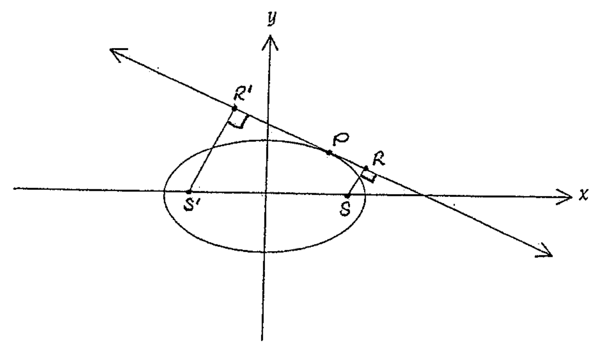
(ii) co-ordinates of the foci

(iii) equations of the directrices

(iv) equation of the asymptotes

Hence, sketch this hyperbola showing the foci, directrices and asymptotes.

(b) **8**



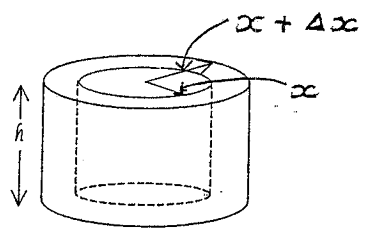
Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b > 0$.

(i) Show that the tangent to the ellipse at the point $P\ (a \cos\theta, b \sin\theta)$ has equation $b\,x \cos\theta + a\,y \sin\theta = a\,b$.

(ii) R and R' are the feet of the perpendiculars from the foci S and S' on to the tangent at P. Show that $SR\ .\ S'R' = b^2$.

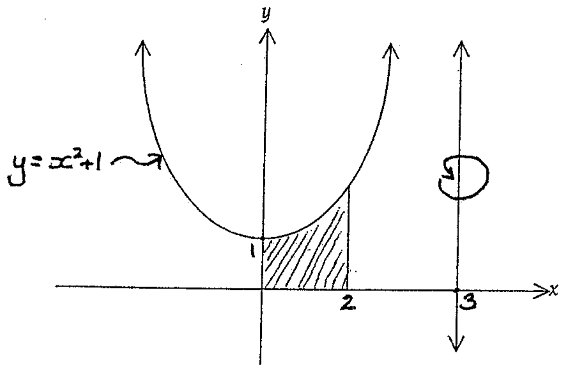
Question 4 (Start a NEW booklet) **Marks**

(a) (i) 8



Show that the volume of a cylindrical shell of height h cm with inner and outer radii x cm and $(x + \Delta x)$ cm respectively is $2\pi\, x\, h\, \Delta\, x$ cm^3 when terms involving $(\Delta x)^2$ can be neglected.

(ii)



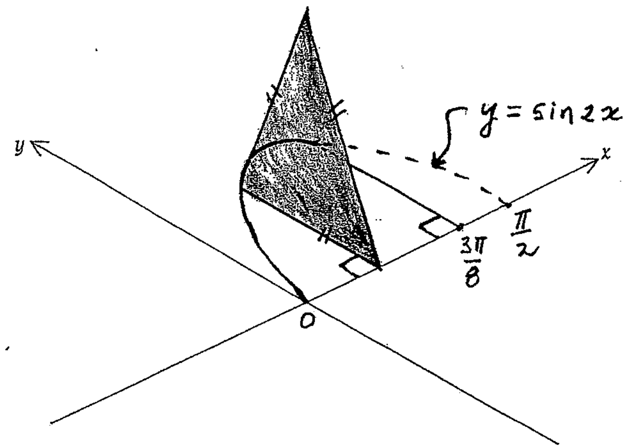
The region bounded by the curve $y = x^2 + 1$, the line $x = 2$ and the co-ordinate axes is rotated about the line $x = 3$ to form a solid of revolution.

Show that the volume V of the solid is given by $V = 2\pi \int_0^2 (3 - x)(x^2 + 1)\, dx$.

Hence find V.

Question 4 continued **Marks**

(b) 7



The base of a certain solid is the region between the x-axis and the curve $y = \sin 2x$ between $x = 0$ and $x = \frac{3\pi}{8}$.

Each plane section of the solid perpendicular to the x-axis is an equilateral triangle with one side in the base of the solid. An example of such a plane section is shown in the diagram above.

Show that the volume V of this solid is given by $V = \frac{\sqrt{3}}{4} \int_0^{\frac{3\pi}{8}} \sin^2 2x\, dx$.

Hence, evaluate V correct to two significant figures.

Question 6 (Start a NEW booklet) **Marks**

(a) A particle moving along the x-axis has position $x(t)$ at time t seconds. If $v(x)$ is the velocity of this particle at position x then show that **2**

$$\frac{d^2x}{dt^2} = v\frac{dv}{dx}$$

(b) A particle of mass 1 kg is moving along the x-axis and experiences a **resistive** force of magnitude kv^2 where $k > 0$ is a constant and v is the speed of the particle. **13**

(i) Find v as a function of time, t, given that the initial speed is 1 ms^{-1}.

(ii) Find v as a function of position, x, given that at $x = 0$ the speed is 1 ms^{-1}.

(iii) Suppose now that a variable **pushing** force of magnitude $\frac{c}{v}$ (where $c > 0$ is a constant) as well as the **resistive** force of magnitude kv^2 is acting on this particle.

(α) Show that the terminal velocity V is given by $V^3 = \frac{c}{k}$.

(β) Show that $\frac{dv}{dx} = c(\frac{1}{v^2} - \frac{v}{V^3})$.

(γ) Hence, find v as a function of x, given that $v = \frac{V}{2}$ when $x = 0$.

Question 5 (Start a NEW booklet) **Marks**

(a) (i) On the **same** diagram, sketch the graphs (**without** the use of calculus) of: **7**

$$y = \frac{1}{x^2+1} \quad \text{and} \quad y = \frac{x^2}{x^2+1}.$$

Mark on your diagram the co-ordinates of the intersection points of these two graphs.

(ii) On the diagram in (a)(i) above, shade the region R where

$\frac{1}{x^2+1} \geq \frac{x^2}{x^2+1}$. Find the area of R.

(b) Let the functions $g(x)$ and $f(x)$ be defined as follows: **6**

$$g(x) = \frac{1}{4}(4+x)(2-x)$$

$$f(x) = \begin{cases} g(x) & \text{for } x < 0 \\ g(-x) & \text{for } x \geq 0 \end{cases}$$

(i) By first sketching $y = g(x)$, make a neat sketch of $y = f(x)$ showing the co-ordinates of all critical points (i.e. where $f'(x) = 0$ or $f'(x)$ is undefined).

(ii) Hence, or otherwise, sketch the graph of $y = f'(x)$.

(c) By considering symmetry, draw a neat sketch of the graph $|x| - |y| = 1$. **2**

Question 7 (Start a NEW booklet) **Marks**

(a) (i) Let the polynomial $P(x) = (x-\alpha)^3 Q(x)$ where $Q(x)$ is also a polynomial and α is a real zero of $P(x)$. 5

Show that $P'(\alpha) = P''(\alpha) = 0$.

(ii) Hence, or otherwise, solve the equation

$$2x^4 + 9x^3 + 6x^2 - 20x - 24 = 0$$

given that it has a triple root (i.e. a root of multiplicity 3).

(b) Let α, β, γ be the roots of the equation $x^3 - 5x + 7 = 0$. 5

(i) Find $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$.

(ii) Find the equation whose roots are $2\alpha - 1$, $2\beta - 1$, $2\gamma - 1$.
Hence, or otherwise, find the value of

$$(2\alpha-1)(2\beta-1) + (2\beta-1)(2\gamma-1) + (2\alpha-1)(2\gamma-1).$$

(c) Let $P(x)$ be a polynomial with $P(-\frac{1}{2}) = 3$. When $P(x)$ is divided by $(x-4)$ the remainder is -1. 5

Find the polynomial $R(x)$ of minimum degree such that $P(x) = (x-4)(2x+1)Q(x) + R(x)$ where $Q(x)$ is a polynomial.

Question 8 (Start a NEW booklet) **Marks**

(a) Let $I_n = \int_0^1 (1-t^2)^{\frac{n-1}{2}}\, dt$ where $n = 0, 1, 2, \ldots$ 11

(i) Show that $t^2(1-t^2)^{\frac{n-3}{2}} = (1-t^2)^{\frac{n-3}{2}} - (1-t^2)^{\frac{n-1}{2}}$

(ii) Using integration by parts, show that $nI_n = (n-1)I_{n-2}$ for $n = 2, 3, 4, \ldots$

(iii) Let $J_n = nI_nI_{n-1}$ for $n = 1, 2, 3, \ldots$

By using the principle of mathematical induction, prove that

$$J_n = \frac{\pi}{2} \text{ for } n = 1, 2, 3, \ldots$$

(iv) Briefly explain why $0 < I_n < I_{n-1}$ for $n = 1, 2, 3, \ldots$

(v) Deduce that $\sqrt{\frac{\pi}{2(n+1)}} < I_n < \sqrt{\frac{\pi}{2n}}$ for $n = 1, 2, 3, \ldots$

(b) Show that $\sum_{r=1}^{r=n} \frac{r\binom{n}{r}}{\binom{n}{r-1}} = \frac{n}{2}(n+1)$ 4

END OF EXAMINATION

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$

E×T2 - Trial HSC 2003

1 (i) $\int_0^{\pi/2} \sin^2\theta\cos\theta\,d\theta$

$= \left[\frac{1}{3}\sin^3\theta\right]_0^{\pi/2}$

$= \frac{1}{3}$ (2)

(ii) $\int_0^1 \frac{x}{\sqrt{x+1}}\,dx = I$

let $u^2 = x+1$; $u \geq 0$

$2u\,du = dx$

$I = \int_1^{\sqrt2} \frac{u^2-1}{\sqrt{u^2}}\,2u\,du$

$= 2\int_1^{\sqrt2}(u^2-1)\,du$

$= 2\left[\frac{1}{3}u^3 - u\right]_1^{\sqrt2}$

$= 2\left[\left(\frac{2}{3}\sqrt2 - \sqrt2\right) - \left(\frac{1}{3} - 1\right)\right]$

$= 2\left(-\frac{1}{3}\sqrt2 + \frac{2}{3}\right)$

$= \frac{2}{3}\left(2 - \sqrt2\right)$ (3)

(iii) $\int_0^1 \tan^{-1}x\,dx$

$= \int_0^1 \tan^{-1}x \frac{d}{dx}(x)\,dx$

$= \left[x\tan^{-1}x\right]_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx$

$= \frac{\pi}{4} - \frac{1}{2}\left[\ln|1+x^2|\right]_0^1$

$= \frac{\pi}{4} - \frac{1}{2}\left[\ln 2\right] = \frac{\pi}{4} - \frac{1}{2}\ln 2$ (2)

(b) (i) $\frac{3x+1}{(x+1)(x^2+1)} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$

$\Longrightarrow A(x^2+1) + (Bx+C)(x+1) \equiv 3x+1$

put $x=-1$

$2A = -2 \Longrightarrow A = -1$

put $x=0$

$A + C = 1$

$\therefore -1 + C = 1 \Longrightarrow C = 2$

coeff of x^2

$A + B = 0$

$-1 + B = 0 \Longrightarrow B = 1$

Hence result. (2)

(ii) $I = \int \frac{-1}{x+1}\,dx + \int \frac{x+2}{x^2+1}\,dx$

$= -\ln|x+1| + \int \frac{x}{x^2+1}\,dx$

$+ \int \frac{2\,dx}{x^2+1}$

$= -\ln|x+1| + \frac{1}{2}\ln|x^2+1|$

$+ 2\tan^{-1}x + c$ (2)

(c) $t = \tan\frac{x}{2}$

$dt = \frac{1}{2}\sec^2\frac{x}{2}\,dx = \frac{1}{2}(1+t^2)\,dx$

$\therefore dx = \frac{2\,dt}{1+t^2}$

$I = \int_0^1 \frac{1}{\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} + 1} \frac{2\,dt}{1+t^2}$

$= \int_0^1 \frac{2\,dt}{2t + 1 - t^2 + 1 + t^2}$

$= \int_0^1 \frac{2\,dt}{2t+2}$

$= \int_0^1 \frac{dt}{1+t}$

$= \left[\ln|1+t|\right]_0^1$

$= \ln 2$ (3)

2 (a) (i)

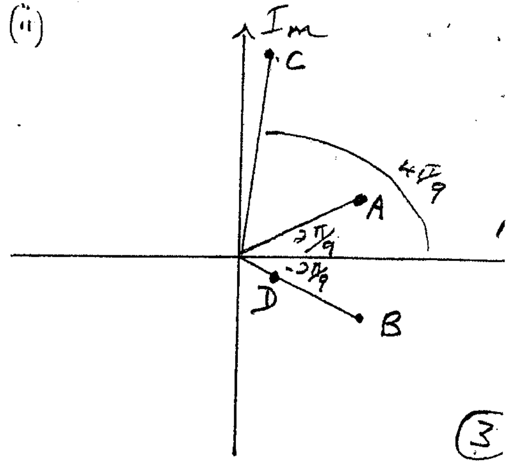
$z\begin{cases}|z| = 2\\ \operatorname{Arg} z = \frac{2\pi}{9}\end{cases}$

$\bar z\begin{cases}|\bar z| = |z| = 2\\ \operatorname{Arg}\bar z = -\operatorname{Arg} z = -\frac{2\pi}{9}\end{cases}$

$z^2\begin{cases}|z^2| = |z|^2 = 4\\ \operatorname{Arg} z^2 = 2\operatorname{Arg} z = \frac{4\pi}{9}\end{cases}$

$\frac{1}{z}\begin{cases}\left|\frac{1}{z}\right| = \frac{1}{|z|} = \frac{1}{2}\\ \operatorname{Arg}\frac{1}{z} = -\operatorname{Arg} z = -\frac{2\pi}{9}\end{cases}$ (5)

(ii)



(3)

(b) $(-3+4i)^* = (a+ib)^2$

$= a^2 - b^2 + 2iab$

$\Longrightarrow$ $\boxed{\begin{aligned}a^2 - b^2 &= -3\\ ab &= 2\end{aligned}}$

also

$|-3+4i| = |a+ib|^2$

$\boxed{5 = a^2 + b^2}$

Hence $2a^2 = 2$

$\therefore a^2 = 1$

$a = \pm 1$

but $ab = 2$

$\therefore b = \pm 2$

$\therefore \sqrt{-3+4i} = \pm(1+2i)$ (4)

(c) $\vec{AC} = \frac{|\vec{AC}|}{|\vec{AB}|}\operatorname{cis}45^\circ\,\vec{AB}$

$= 2 \times \frac{1}{\sqrt2}(1+i)\,\vec{AB}$

$\vec{AC} = \sqrt{2}(1+i)\,\vec{AB}$

$\vec{AB} = (6+8i) - (10+2i)$

$= -4+6i$

$\therefore \vec{AC} = \sqrt{2}(1+i)(-4+6i)$

$\vec{OC} = \vec{OA} + \vec{AC}$

$= 10+2i + \sqrt{2}(1+i)(-4+6i)$

$= 10+2i + \sqrt{2}(-10+2i)$

$C \equiv -10(\sqrt{2}-1) + 2(\sqrt{2}+1)i$ (3)

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3 (a)

$9x^2 - 16y^2 = 144$

$\frac{x^2}{16} - \frac{y^2}{9} = 1$

(i) $\frac{b^2}{a^2} = e^2 - 1$

$\frac{9}{16} = e^2 - 1 \Rightarrow e^2 = \frac{25}{16}$ (1)

$\therefore e = \frac{5}{4}$

(ii) $S(\pm ae, 0)$ $a = 4$

$= S(\pm 5, 0)$ (1)

(iv) asymptotes

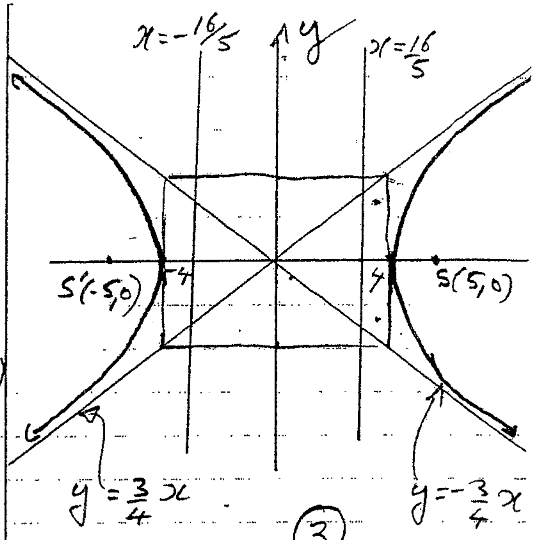
$y = \pm \frac{b}{a} x$

$y = \pm \frac{3}{4} x$ (1)

(iii) directrices

$x = \pm \frac{a}{e}$

$= \pm \frac{16}{5}$ (1)



(3)

(b)(i)

$x = a\cos\theta$

$y = b\sin\theta$

$\frac{dx}{d\theta} = -a\sin\theta$

$\frac{dy}{d\theta} = b\cos\theta$

$\frac{dy}{dx} = -\frac{b}{a}\frac{\cos\theta}{\sin\theta}$ ($\sin\theta \neq$ …)

$y - b\sin\theta = -\frac{b}{a}\frac{\cos\theta}{\sin\theta}(x - a\cos$ …

$ay\sin\theta - ab\sin^2\theta$

$= -bx\cos\theta + ab\cos^2$ …

$bx\cos\theta + ay\sin\theta = ab(\sin^2\theta +$ …

$\therefore bx\cos\theta + ay\sin\theta = ab$ (3)

(ii) $S = (ae, 0)$ $S' = (-ae,$ …

$SR = \frac{|bae\cos\theta + 0 - ab|}{\sqrt{b^2\cos^2\theta + \frac{b^2}{a^2}\sin^2\theta}}$

(β)

$\Sigma F = \frac{c}{v} - kv^2$

$= c\left(\frac{1}{v} - \frac{k}{c}v^2\right)$

$= c\left(\frac{1}{v} - \frac{v^2}{V^3}\right)$

By Newton's 2nd Law

$\Sigma F = m\frac{dv}{dt}$

$c\left(\frac{1}{v} - \frac{v^2}{V^3}\right) = 1 \times \frac{dv}{dt}$

But $\frac{dv}{dt} = v\frac{dv}{dx}$

$\therefore \frac{dv}{dx} = c\left(\frac{1}{v^2} - \frac{v}{V^3}\right)$ (3)

(γ)

$\int \frac{dv}{\left(\frac{1}{v^2} - \frac{v}{V^3}\right)} = c\int dx$

$LHS = V^3 \int \frac{v^2\,dv}{V^3 - v^3}$

(Note $v < V$)

$= -\frac{V^3}{3}\ln(V^3 - v^3)$

$RHS = cx + K$

$\Rightarrow -\frac{V^3}{3}\ln(V^3 - v^3) = cx + K$

$x = 0$ $v = \frac{V}{2}$

$\Rightarrow K = -\frac{V^3}{3}\ln\left(V^3 - \frac{V^3}{8}\right)$

$\therefore K = -\frac{V^3}{3}\ln\left(\frac{7V^3}{8}\right)$

$\therefore -\frac{V^3}{3}\ln(V^3 - v^3) = cx - \frac{V^3}{3}\ln$ …

$\therefore \frac{V^3}{3}\left[\ln\left(\frac{V^3 - v^3}{\frac{7}{8}V^3}\right)\right] = -c$ …

$\therefore \frac{V^3 - v^3}{\frac{7}{8}V^3} = e^{-\frac{3c}{V^3}x}$

$V^3 - v^3 = \frac{7}{8}V^3 e^{-\frac{3c}{V^3}x}$

$\therefore v^3 = V^3\left[1 - \frac{7}{8}e^{-\frac{3c}{V^3}}\right.$ …

or $v = V\left(1 - \frac{7}{8}e^{-\frac{3c}{V^3}x}\right.$ …

(3)

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7 (i)

$P(x) = (x-\alpha)^3 Q(x)$

$P'(x) = 3(x-\alpha)^2 Q(x) + (x-\alpha)^3 Q'(x)$

$P'(\alpha) = 0 + 0 = 0$

$P''(\alpha) = 6(x-\alpha)Q(x) + 3(x-\alpha)^2 Q'$ …

$+ 3(x-\alpha)^2 Q'(x) + (x-\alpha)^3 Q''$ …

$P''(\alpha) = 0 + 0 + 0 + 0$

$= 0$ (2)

(ii) let

$P(x) = 2x^4 + 9x^3 + 6x^2 - 20x - 24$

If α is the triple root then α is a root of $P''(x) = 0$

$P'(x) = 8x^3 + 27x^2 + 12x - 20$

$P''(x) = 24x^2 + 54x + 12$

$P''(x) = 0 \Rightarrow 4x^2 + 9x + 2 = 0$

$(4x+1)(x+2) = 0$

$x = -\frac{1}{4}, -2$

as $4 \nmid 2$ (from $2x^4$) then $-1/4$ is not a root of $P(x) = 0$ (or show $P(-1/4) \neq 0$)

so $x = -2$ is the triple root.

$\therefore P(x) = (x+2)^3 (2x - 3)$

$\therefore x = -2, -2, -2, \frac{3}{2}$

or $x = -2, \frac{3}{2}$ (3)

(b)

(i)

method 1

$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$= \frac{-5}{-7} = 5/7$

method 2 subst. $y = \frac{1}{x}$

which gives

$7y^3 - 5y^2 + 1 = 0$

roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$

$\Rightarrow \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{--5}{7}$

$= \frac{5}{7}$ (2)

(ii)

Substitute $y = 2x - 1$

$\Rightarrow x = \frac{1}{2}(y+1)$

$\left[\frac{1}{2}(y+1)\right]^3 - 5\left[\frac{1}{2}(y+1)\right] + 7 =$

$(y+1)^3 - 20(y+1) + 56 = 0$

$y^3 + 3y^2 - 17y + 37 = 0$

The roots of $2\alpha - 1$, $2\beta - 1$, $2\gamma - 1$ (2)

$\therefore \sum (2\alpha - 1)(2\beta - 1) = -17$ (1)

(c)

~~$R(x)$~~

as $(2x+1)$ nor $(x-4)$ are factors of $P(x) \Rightarrow$

$R(x) \neq 0$ and $R(x)$ is at most linear for min deg.

Let $R(x) = mx + n$

$P(-\frac{1}{2}) = 3 = R(-\frac{1}{2}) = -\frac{1}{2}m +$

$\therefore$ $\boxed{-m + 2n = 6}$

$P(4) = -1 = R(4) = 4m + n$

$\boxed{4m + n = -1}$

Solving equations :—

$m = -\frac{8}{9}$, $n = \frac{23}{9}$

$\therefore R(x) = \frac{1}{9}(-8x+23)$ (5)

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8 (a)

(i) $RHS = (1-t^2)^{\frac{n-3}{2}} - (1-t^2)^{\frac{n-1}{2}}$

$= (1-t^2)^{\frac{n-3}{2}}\left[1-(1-t^2)\right]$

$= t^2(1-t^2)^{\frac{n-3}{2}}$

$= LHS.$ (1)

(ii) $I_n = \int_0^1 (1-t^2)^{\frac{n-1}{2}} dt$

$= \int_0^1 (1-t^2)^{\frac{n-1}{2}} \frac{d(t)}{dt} dt$ $n = 0,1,2 \ldots$

$= \left[t(1-t^2)^{\frac{n-1}{2}}\right]_0^1 - \frac{n-1}{2}\int_0^1 t(1-t^2)^{\frac{n-3}{2}} \times(-2t)\,dt.$

$= (n-1)\int_0^1 t^2(1-t^2)^{n-3/2}$

using (a)(i)

$= (n-1)\left[\int_0^1 (1-t^2)^{\frac{n-3}{2}} - \int_0^1 (1-t^2)^{\frac{n-1}{2}}\right]$

$I_n = (n-1)I_{n-2} - (n-1)I_n$

$nI_n = (n-1)I_{n-2}$

$n = 2,3,4,\ldots$

(as I_{n-2} defined). (3)

(iii)

$J_n = nI_nI_{n-1}$

Let $P(n)$ be the proposition that $J_n = \frac{\pi}{2}$ $n = 1,2,3,\ldots$

$P(1)$: $J_1 = 1 \times I_1 \times I_0$

$I_1 = \int_0^1 dt = 1$

$I_0 = \int_0^1 \frac{dt}{\sqrt{1-t^2}}$

This is an improper integral

$= \lim_{x\to 1^-} \int_0^x \frac{dt}{\sqrt{1-t^2}}$

$= \lim_{x\to 1^-} (\sin^{-1}x).$

$= \pi/2$

$\therefore J_1 = \frac{\pi}{2}$

To Prove: $P(n)$ true $\Longrightarrow$ $P(n+1)$ true for $n = 1,2,3,\ldots$

$P(n)$: $J_n = \pi/2$

$P(n+1)$: $J_{n+1} = \pi/2$

$J_{n+1} = (n+1)I_{n+1}I_n$

By (a)(ii) as $n+1 = 2,3,4,\ldots$

$(n+1)I_{n+1} = nI_{n-1}$

$\therefore J_{n+1} = nI_nI_{n-1}$

$= \pi/2$

by induction hypothesis

Hence by math Induction

$J_n = \pi/2$ for $n = 1,2,3,\ldots$ (3)

(i.e. $P(n)$ true for $n = 1,2,3,\ldots$)

(iv) $I_n = \int_0^1 (1-t^2)^{\frac{n-1}{2}} dt.$

This is clearly > 0

$I_{n-1} = \int_0^1 (1-t^2)^{\frac{n-2}{2}} dt$

$I_n - I_{n-1} = \int_0^1 \left[(1-t^2)^{\frac{n-1}{2}} - (1-t^2)^{\frac{n-2}{2}}\right] dt$

$= \int_0^1 (1-t^2)^{\frac{n-2}{2}}\left[(1-t^2)^{\frac{1}{2}} - 1\right] dt$

$= \int_0^1 (1-t^2)^{\frac{n-2}{2}}\left[\sqrt{1-t^2} - 1\right] dt$

For $0 \le t \le 1$

$0 \le \sqrt{1-t^2} \le 1$

$-1 \le \sqrt{1-t^2} - 1 \le 0$

Hence $I_n - I_{n-1} < 0$

$\Rightarrow 0 < I_n < I_{n-1}$ (2)

(v) taking (2)

$0 < I_n < I_{n-1}$

$n = 1,2,3,\ldots$

$\times I_n$ and $I_n > 0 \Longrightarrow$

$0 < I_n^2 < I_{n-1}I_n$

$\Rightarrow 0 < I_n^2 < \frac{\pi}{2n}$

$\Rightarrow 0 < I_n < \sqrt{\frac{\pi}{2n}}$

~~Problem~~

Now $0 < I_n < I_{n-1}$ $n = 1, 2, 3, 4 —$

$\Rightarrow$ $0 < I_{n+1} < I_n$ $n = 0, 1, 2, \ldots$

ie, $n = 1, 2, 3, \ldots$

$\times I_n^2$ $0 < I_{n+1} I_n < I_n^2$

$0 < \frac{\pi}{2(n+1)} < I_n^2$

$\Rightarrow$ $I_n > \sqrt{\frac{\pi}{2(n+1)}}$

$n = 1, 2, \ldots$

Hence

$\sqrt{\frac{\pi}{2(n+1)}} < I_n < \sqrt{\frac{\pi}{2n}}$

$n = 1, 2, 3, \ldots$

(b) $\frac{\binom{n}{r}}{\binom{n}{r-1}} = \frac{n!}{r!(n-r)!} \times \frac{(r-1)!(n-(r-1))!}{n!}$

Now $r! = r(r-1)!$

$(n-(r-1))! = (n-(r-1))(n-r)!$

$\frac{\binom{n}{r}}{\binom{n}{r-1}} = \frac{n-r+1}{r}$

$\therefore r \frac{\binom{n}{r}}{\binom{n}{r-1}} = (n-r+1)$

method 1

$\sum_{r=1}^{n} (n-r+1)$

$= n\sum n - \sum r + \sum 1$

$= n^2 - \frac{n}{2}(n+1) + n$

$= n(n+1) - \frac{n}{2}(n+1)$

$= \frac{1}{2} n(n+1)$

(4)

method 2

$\sum_{r=1}^{r=n} n-r+1$ let $k = n-(r-1)$

$= \sum_{n}^{1} k$

$= \sum_{1}^{n} k$

$= \frac{n}{2}(n+1)$