BOARD OF STUDIES
NEW SOUTH WALES

2009

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1–7
- All questions are of equal value

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Total marks – 84
Attempt Questions 1–7
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Factorise $8x^3 + 27$. **2**

(b) Let $f(x) = \ln(x-3)$. What is the domain of $f(x)$? **1**

(c) Find $\lim_{x \to 0} \dfrac{\sin 2x}{x}$. **1**

(d) Solve the inequality $\dfrac{x+3}{2x} > 1$. **3**

(e) Differentiate $x \cos^2 x$. **2**

(f) Using the substitution $u = x^3 + 1$, or otherwise, evaluate $\displaystyle\int_0^2 x^2 e^{x^3+1}\, dx$. **3**

Question 2 (12 marks) Use a SEPARATE writing booklet.

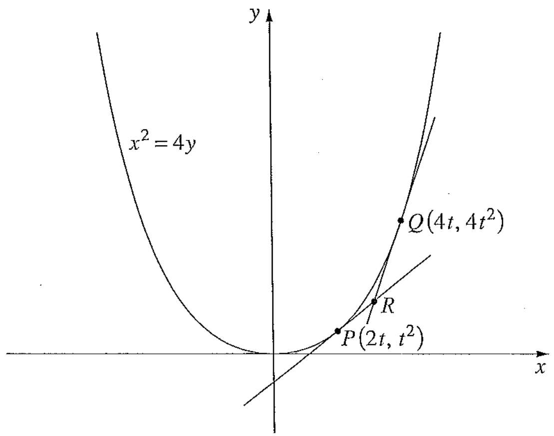
(a) The polynomial $p(x) = x^3 - ax + b$ has a remainder of 2 when divided by $(x-1)$ and a remainder of 5 when divided by $(x+2)$. **3**

Find the values of a and b.

(b) (i) Express $3\sin x + 4\cos x$ in the form $A\sin(x+\alpha)$ where $0 \le \alpha \le \dfrac{\pi}{2}$. **2**

(ii) Hence, or otherwise, solve $3\sin x + 4\cos x = 5$ for $0 \le x \le 2\pi$. Give your answer, or answers, correct to two decimal places. **2**

(c) The diagram shows points $P(2t, t^2)$ and $Q(4t, 4t^2)$ which move along the parabola $x^2 = 4y$. The tangents to the parabola at P and Q meet at R.



NOT TO SCALE

(i) Show that the equation of the tangent at P is $y = tx - t^2$. **2**

(ii) Write down the equation of the tangent at Q, and find the coordinates of the point R in terms of t. **2**

(iii) Find the Cartesian equation of the locus of R. **1**

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) Let $f(x) = \dfrac{3 + e^{2x}}{4}$.

(i) Find the range of $f(x)$. **1**

(ii) Find the inverse function $f^{-1}(x)$. **2**

(b) (i) On the same set of axes, sketch the graphs of $y = \cos 2x$ and $y = \dfrac{x+1}{2}$, for $-\pi \le x \le \pi$. **2**

(ii) Use your graph to determine how many solutions there are to the equation $2\cos 2x = x + 1$ for $-\pi \le x \le \pi$. **1**

(iii) One solution of the equation $2\cos 2x = x + 1$ is close to $x = 0.4$. Use one application of Newton's method to find another approximation to this solution. Give your answer correct to three decimal places. **3**

(c) (i) Prove that $\tan^2\theta = \dfrac{1 - \cos 2\theta}{1 + \cos 2\theta}$ provided that $\cos 2\theta \neq -1$. **2**

(ii) Hence find the exact value of $\tan\dfrac{\pi}{8}$. **1**

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) A test consists of five multiple-choice questions. Each question has four alternative answers. For each question only one of the alternative answers is correct.

Huong randomly selects an answer to each of the five questions.

(i) What is the probability that Huong selects three correct and two incorrect answers? **2**

(ii) What is the probability that Huong selects three or more correct answers? **2**

(iii) What is the probability that Huong selects at least one incorrect answer? **1**

(b) Consider the function $f(x) = \dfrac{x^4 + 3x^2}{x^4 + 3}$.

(i) Show that $f(x)$ is an even function. **1**

(ii) What is the equation of the horizontal asymptote to the graph $y = f(x)$? **1**

(iii) Find the x-coordinates of all stationary points for the graph $y = f(x)$. **3**

(iv) Sketch the graph $y = f(x)$. You are not required to find any points of inflexion. **2**

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) The equation of motion for a particle moving in simple harmonic motion is given by

$$\frac{d^2x}{dt^2} = -n^2x$$

where n is a positive constant, x is the displacement of the particle and t is time.

(i) Show that the square of the velocity of the particle is given by **3**

$$v^2 = n^2(a^2 - x^2)$$

where $v = \frac{dx}{dt}$ and a is the amplitude of the motion.

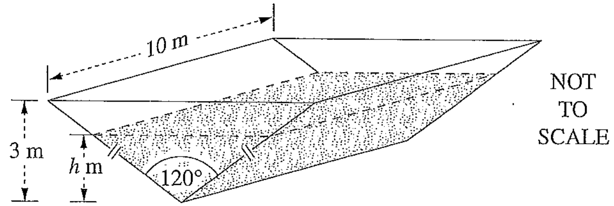
(ii) Find the maximum speed of the particle. **1**

(iii) Find the maximum acceleration of the particle. **1**

(iv) The particle is initially at the origin. Write down a formula for x as a function of t, and hence find the first time that the particle's speed is half its maximum speed. **2**

Question 5 continues on page 7

(b) The cross-section of a 10 metre long tank is an isosceles triangle, as shown in the diagram. The top of the tank is horizontal.



When the tank is full, the depth of water is 3 m. The depth of water at time t days is h metres.

(i) Find the volume, V, of water in the tank when the depth of water is h metres. **1**

(ii) Show that the area, A, of the top surface of the water is given by **1**

$$A = 20\sqrt{3}h.$$

(iii) The rate of evaporation of the water is given by **2**

$$\frac{dV}{dt} = -kA,$$

where k is a positive constant.

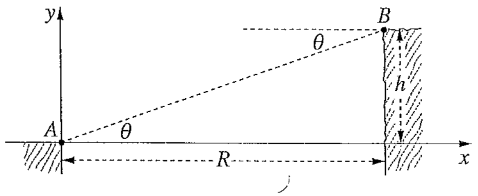
Find the rate at which the depth of water is changing at time t.

(iv) It takes 100 days for the depth to fall from 3 m to 2 m. Find the time taken for the depth to fall from 2 m to 1 m. **1**

End of Question 5

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) Two points, A and B, are on cliff tops on either side of a deep valley. Let h and R be the vertical and horizontal distances between A and B as shown in the diagram. The angle of elevation of B from A is θ, so that $\theta = \tan^{-1}\left(\frac{h}{R}\right)$.



At time $t = 0$, projectiles are fired simultaneously from A and B. The projectile from A is aimed at B, and has initial speed U at an angle θ above the horizontal. The projectile from B is aimed at A and has initial speed V at an angle θ below the horizontal.

The equations for the motion of the projectile from A are

$$x_1 = Ut\cos\theta \quad \text{and} \quad y_1 = Ut\sin\theta - \frac{1}{2}gt^2,$$

and the equations for the motion of the projectile from B are

$$x_2 = R - Vt\cos\theta \quad \text{and} \quad y_2 = h - Vt\sin\theta - \frac{1}{2}gt^2.$$

(Do NOT prove these equations.)

(i) Let T be the time at which $x_1 = x_2$. **1**

Show that $T = \dfrac{R}{(U+V)\cos\theta}$.

(ii) Show that the projectiles collide. **2**

(iii) If the projectiles collide on the line $x = \lambda R$, where $0 < \lambda < 1$, show that **1**

$$V = \left(\frac{1}{\lambda} - 1\right)U.$$

Question 6 continues on page 9

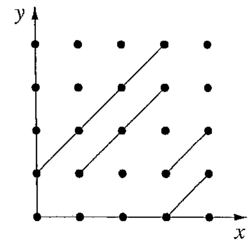
(b) (i) Sum the geometric series **3**

$$(1+x)^r + (1+x)^{r+1} + \cdots + (1+x)^n$$

and hence show that

$$\binom{r}{r} + \binom{r+1}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}.$$

(ii) Consider a square grid with n rows and n columns of equally spaced points.



The diagram illustrates such a grid. Several intervals of gradient 1, whose endpoints are a pair of points in the grid, are shown.

(1) Explain why the number of such intervals on the line $y = x$ is equal to $\binom{n}{2}$. **1**

(2) Explain why the total number, S_n, of such intervals in the grid is given by **1**

$$S_n = \binom{2}{2} + \binom{3}{2} + \cdots + \binom{n-1}{2} + \binom{n}{2} + \binom{n-1}{2} + \cdots + \binom{3}{2} + \binom{2}{2}.$$

(iii) Using the result in part (i), show that **3**

$$S_n = \frac{n(n-1)(2n-1)}{6}.$$

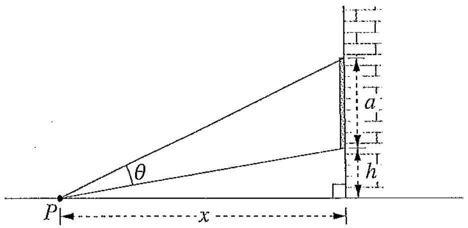
End of Question 6

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) (i) Use differentiation from first principles to show that $\frac{d}{dx}(x)=1$. **1**

(ii) Use mathematical induction and the product rule for differentiation to prove that $\frac{d}{dx}\left(x^n\right)=nx^{n-1}$ for all positive integers n. **2**

(b) A billboard of height a metres is mounted on the side of a building, with its bottom edge h metres above street level. The billboard subtends an angle θ at the point P, x metres from the building.



(i) Use the identity $\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$ to show that **2**

$$\theta=\tan^{-1}\left[\frac{ax}{x^2+h(a+h)}\right].$$

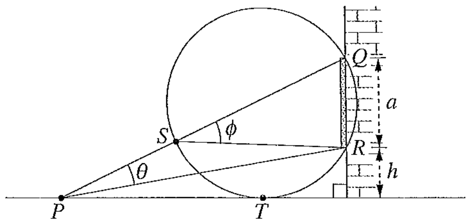
(ii) The maximum value of θ occurs when $\frac{d\theta}{dx}=0$ and x is positive. **3**

Find the value of x for which θ is a maximum.

Question 7 continues on page 11

(c) Consider the billboard in part (b). There is a unique circle that passes through the top and bottom of the billboard (points Q and R respectively) and is tangent to the street at T.

Let ϕ be the angle subtended by the billboard at S, the point where PQ intersects the circle.



Copy the diagram into your writing booklet.

(i) Show that $\theta<\phi$ when P and T are different points, and hence show that θ is a maximum when P and T are the same point. **3**

(ii) Using circle properties, find the distance of T from the building. **1**

End of paper

2009 Higher School Certificate Solutions

Mathematics Extension 1

Question 1

(a) $8x^3+27=(2x+3)(4x^2-6x+9)$.

(b) Domain: $x-3>0$

$x>3$.

(c) $\lim\limits_{x\to 0}\dfrac{\sin 2x}{x}$

$=\lim\limits_{x\to 0}\dfrac{2(\sin 2x)}{2x}$

$=2$ since $\lim\limits_{x\to 0}\dfrac{\sin 2x}{2x}=1$.

(d) *METHOD 1* Algebraic:

$\dfrac{x+3}{2x}>1$

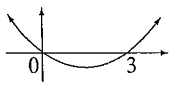
$4x^2\dfrac{(x+3)}{2x}>4x^2$

$2x(x+3)>4x^2$

$2x^2+6x>4x^2$

$0>2x^2-6x$

$0>2x(x-3)$.



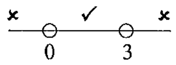
$\therefore 0<x<3$.

METHOD 2 Critical point:

$x\neq 0,\quad \dfrac{x+3}{2x}=1$

$x+3=2x$

$3=x$



$\dfrac{x+3}{2x}>1$: Test $x=-1$: $\quad \dfrac{2}{-2}<1$ ×

$x=1$: $\quad \dfrac{4}{2}>1$ ✓

$x=4$: $\quad \dfrac{7}{8}<1$ ×

$\therefore 0<x<3$.

(e) $f(x)=x\cos^2 x$

Using the product rule,

$f'(x)=x\,.\,2\cos x(-\sin x)+1\,.\cos^2 x$

$=-2x\sin x\cos x+\cos^2 x$.

(f) $\displaystyle\int_0^2 x^2e^{x^3+1}\,dx$

Let $\quad u=x^3+1,\quad \therefore e^{x^3+1}=e^u$

$du=3x^2dx,\quad \therefore x^2dx=\dfrac{du}{3}$

At $x=0,\quad u=1$

$x=2,\quad u=2^3+1=9$

$\dfrac{1}{3}\displaystyle\int_1^9 e^u\,du=\dfrac{1}{3}\Big[e^u\Big]_1^9$

$=\dfrac{1}{3}\left(e^9-e^1\right)$.

Question 2

(a) $p(x)=x^3-ax+b$

$p(1)=2 \qquad p(-2)=5$

$\therefore 2=1-a+b \qquad \therefore 5=-8+2a+b$

$-a+b=1$ ① $\qquad 2a+b=13$ ②

② − ①: $3a=12,\quad \therefore a=4$

Substitute $a=4$ into ①.

$-4+b=1,\quad \therefore b=5$.

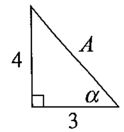
(b) (i) $A\sin(x+\alpha)=A\sin x\cos\alpha+A\cos x\sin\alpha$

Compare $3\sin x+4\cos x$.

$\therefore A\cos\alpha=3 \qquad A\sin\alpha=4$

$\dfrac{A\sin\alpha}{A\cos\alpha}=\dfrac{4}{3}$

$\therefore \tan\alpha=\dfrac{4}{3}$



$\therefore \alpha=\tan^{-1}\left(\dfrac{4}{3}\right) \qquad A^2=4^2+3^2$

$A=5$

$\therefore 3\sin x+4\cos x=5\sin\left[x+\tan^{-1}\left(\dfrac{4}{3}\right)\right]$.

(ii) $3\sin x+4\cos x=5$

$\therefore 5\sin\left[x+\tan^{-1}\left(\dfrac{4}{3}\right)\right]=5$

$\sin\left[x+\tan^{-1}\left(\dfrac{4}{3}\right)\right]=1$

$x+\tan^{-1}\left(\dfrac{4}{3}\right)=\dfrac{\pi}{2}$

$x=\dfrac{\pi}{2}-\tan^{-1}\left(\dfrac{4}{3}\right)$

$=\dfrac{\pi}{2}-0.9272\ldots$

$=0.64$ (to 2 decimal places).

(c) (i) $x^2=4y$

$\therefore y=\dfrac{x^2}{4}$

$\dfrac{dy}{dx}=\dfrac{1}{2}x$

At $P\left(2t,\ t^2\right)$, $\dfrac{dy}{dx}=\dfrac{2t}{2}=t$

$\therefore$ Tangent at P is

$y-t^2=t(x-2t)$

$y-t^2=tx-2t^2$

$y=tx-t^2$ ①

(ii) At $Q\left(4t, 4t^2\right)$, $\dfrac{dy}{dx}=\dfrac{4t}{2}=2t$

$\therefore$ Tangent at Q is

$y-4t^2=2t(x-4t)$

$y-4t^2=2tx-8t^2$

$y=2tx-4t^2$ ②

Solve tangents simultaneously.

② − ①: $0=tx-3t^2$

$tx=3t^2$

$x=3t$

Substitute $x=3t$ into ①.

$y=3t^2-t^2$

$=2t^2$

$\therefore R\left(3t, 2t^2\right)$.

(iii) At $R\left(3t,\ 2t^2\right)$,

$x=3t,\quad \therefore t=\dfrac{x}{3}$

$y=2t^2$

$\therefore$ Locus of R is

$y=2\left(\dfrac{x}{3}\right)^2$

$y=\dfrac{2x^2}{9}$.

Question 3

(a) (i) $f(x)=\dfrac{3+e^{2x}}{4}=\dfrac{3}{4}+\dfrac{e^{2x}}{4}$

Range of $\dfrac{e^{2x}}{4}>0$

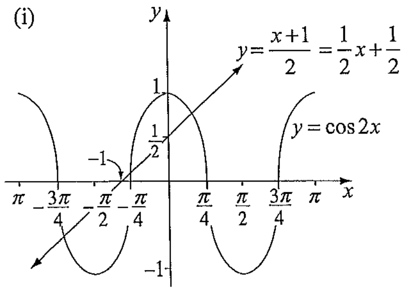
$\therefore$ Range of $f(x)>\dfrac{3}{4}$.

(ii) Let $y=\dfrac{3+e^{2x}}{4}$

Interchanging x and y,

$$x=\frac{3+e^{2y}}{4}$$
$$4x=3+e^{2y}$$
$$e^{2y}=4x-3$$
$$2y=\ln(4x-3)$$
$$y=\frac{1}{2}\ln(4x-3)$$
$$\therefore f^{-1}(x)=\frac{1}{2}\ln(4x-3).$$

(b) (i)



(ii) There are 3 solutions to the equation $2\cos x=x+1$ for $-\pi\le x\le\pi$ (since the two graphs intersect 3 times in the given domain).

(iii)
$$2\cos 2x=x+1$$
$$2\cos 2x-x-1=0$$

Let $f(x)=2\cos 2x-x-1$

$\therefore f'(x)=-4\sin 2x-1$

Newton's method:

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}$$
$$=0.4-\frac{2\cos(2\times0.4)-0.4-1}{-4\sin(2\times0.4)-1}$$
$$=0.4-\frac{-0.0065\ldots}{-3.8694\ldots}$$
$$=0.398 \text{ (to 3 decimal places)}$$

$\therefore 0.398$ is another approximation to this solution.

(c) (i)
$$\cos 2\theta=1-2\sin^2\theta$$
and $\cos 2\theta=2\cos^2\theta-1$

$$\therefore \text{RHS}=\frac{1-\cos 2\theta}{1+\cos 2\theta}$$
$$=\frac{1-(1-2\sin^2\theta)}{1+(2\cos^2\theta-1)}$$
$$=\frac{2\sin^2\theta}{2\cos^2\theta}$$
$$=\tan^2\theta$$
$$=\text{LHS}.$$

(ii) Let $\theta=\dfrac{\pi}{8}$

$$\therefore \tan^2\frac{\pi}{8}=\frac{1-\cos\frac{2\pi}{8}}{1+\cos\frac{2\pi}{8}}$$
$$=\frac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}$$
$$=\frac{1-\frac{1}{\sqrt2}}{1+\frac{1}{\sqrt2}}$$
$$=\frac{\sqrt2-1}{\sqrt2}\div\frac{\sqrt2+1}{\sqrt2}$$
$$=\frac{\sqrt2-1}{\sqrt2+1}$$

Rationalising,

$$\tan^2\frac{\pi}{8}=\frac{\sqrt2-1}{\sqrt2+1}\times\frac{\sqrt2-1}{\sqrt2-1}$$
$$=(\sqrt2-1)^2$$
$$\therefore \tan\frac{\pi}{8}=\sqrt2-1$$

$\left(\tan\dfrac{\pi}{8}>0 \text{ since } \dfrac{\pi}{8} \text{ lies in 1st quadrant}\right)$.

Question 4

(a) (i)
$$P(\text{correct})=\frac{1}{4}$$
$$\therefore P(3\text{ correct})={}^5C_3\left(\frac14\right)^3\left(\frac34\right)^2$$
$$=10\times\frac{1}{64}\times\frac{9}{16}$$
$$=\frac{45}{512}.$$

(ii) P(3 or more correct)

$=P(3\text{ correct})+P(4\text{ correct})+P(5\text{ correct})$

$$=\frac{45}{512}+{}^5C_4\left(\frac14\right)^4\left(\frac34\right)+{}^5C_5\left(\frac14\right)^5$$
$$=\frac{45}{512}+\frac{15}{1024}+\frac{1}{1024}$$
$$=\frac{53}{512}.$$

(iii) P(at least 1 incorrect)

$$=1-P(5\text{ correct})$$
$$=1-\frac{1}{1024}$$
$$=\frac{1023}{1024}.$$

(b) (i)
$$f(x)=\frac{x^4+3x^2}{x^4+3}$$
$$f(-x)=\frac{(-x)^4+3(-x)^2}{(-x)^4+3}$$
$$=\frac{x^4+3x^2}{x^4+3}$$
$$=f(x)$$

$\therefore f(x)$ is even.

(ii)
$$\lim_{x\to\infty}\frac{x^4+3x^2}{x^4+3}=\lim_{x\to\infty}\frac{1+\frac{3}{x^2}}{1+\frac{3}{x^4}}$$
$$=1$$

$\therefore$ Horizontal asymptote is $y=1$.

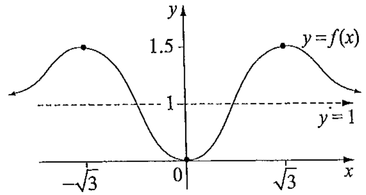
(iii) $f'(x)$
$$=\frac{(x^4+3)(4x^3+6x)-(x^4+3x^2)(4x^3)}{(x^4+3)^2}$$
$$=\frac{4x^7+6x^5+12x^3+18x-4x^7-12x^5}{(x^4+3)^2}$$
$$=\frac{-6x^5+12x^3+18x}{(x^4+3)^2}$$

Stationary points when $f'(x)=0$.

$$\therefore -6x^5+12x^3+18x=0$$
$$-6x(x^4-2x^2-3)=0$$
$$-6x(x^2-3)(x^2+1)=0$$
$$\therefore x=0 \text{ or } x=\pm\sqrt3.$$

(iv)

x	$-\sqrt3$	0	$\sqrt3$
y	1.5	0	1.5



Question 5

(a) (i)
$$\frac{d^2x}{dt^2}=-n^2x$$
$$\frac{d}{dx}\left(\frac12 v^2\right)=-n^2x$$
$$\frac12 v^2=\int -n^2x\,dx$$
$$=-\frac{n^2x^2}{2}+c_1$$
$$\therefore v^2=-n^2x^2+c_2$$

When $x=a$, $v=0$

$$\therefore 0=-n^2a^2+c_2$$
$$\therefore c_2=n^2a^2$$
$$\therefore v^2=-n^2x^2+n^2a^2$$
$$v^2=n^2(a^2-x^2).$$

(ii) Maximum speed occurs when

$\frac{d^2x}{dt^2}=0$, i.e. when $x=0$.

$\therefore v^2=n^2a^2$ (from (i))

$\therefore v=na$ is maximum

(since speed has no direction).

(iii) Maximum acceleration occurs when $v=0$, i.e. when $x=a$.

$\therefore \frac{d^2x}{dt^2}=-n^2a$ is maximum.

(iv) Since particle is initially at origin,

$x=a\sin nt$

$\therefore \frac{dx}{dt}=an\cos nt$

At half maximum speed,

$\frac{1}{2}na=an\cos nt$ (from (ii))

$\cos nt=\frac{1}{2}$

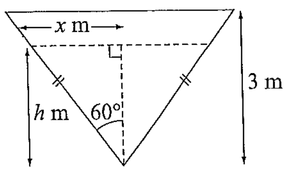
$nt=\frac{\pi}{3}$ (1st quadrant only for first time)

$\therefore t=\frac{\pi}{3n}$

$\therefore$ The first time the particle's speed is half its maximum speed is at

$t=\frac{\pi}{3n}$.

(b) (i)



$x=h\tan 60°$

$x=h\sqrt{3}$ m

Volume, V, is a triangular prism.

$\therefore V=\left(\frac{1}{2}\times 2h\sqrt{3}\times h\right)\times 10$

$=10\sqrt{3}\,h^2$ m^3.

(ii) Area, A, is a rectangle.

$\therefore A=2h\sqrt{3}\times 10$

$=20\sqrt{3}h$ m^2.

(iii) We require $\frac{dh}{dt}$.

$\frac{dh}{dt}=\frac{dh}{dV}\times\frac{dV}{dt}$

From (i), $V=10\sqrt{3}h^2$

$\therefore \frac{dV}{dh}=20\sqrt{3}h$

$\frac{dh}{dV}=\frac{1}{20\sqrt{3}h}$

Substituting,

$\frac{dh}{dt}=\frac{1}{20\sqrt{3}h}\times -kA$

From (ii), $A=20\sqrt{3}h$

$\therefore \frac{dh}{dt}=\frac{1}{20\sqrt{3}h}\times -k\,.\,20\sqrt{3}h$

$=-k$.

(iv) $\frac{dh}{dt}=-k$

$h=\int -k\,dt$

$\therefore h=-kt+c$

When $t=0$, $h=3$

$\therefore c=3$

$\therefore h=-kt+3$

When $t=100$, $h=2$

$\therefore 2=-100k+3$

$100k=1$

$k=\frac{1}{100}$

$\therefore h=\frac{-t}{100}+3$

When $h=1$,

$1=\frac{-t}{100}+3$

$\frac{t}{100}=2$

$t=200$ days

$\therefore$ It takes a further 100 days to fall from 2 m to 1 m.

Question 6

(a) (i) When $x_1=x_2$,

$UT\cos\theta=R-VT\cos\theta$

$T(U+V)\cos\theta=R$

$\therefore T=\frac{R}{(U+V)\cos\theta}$.

(ii) Projectiles collide when $y_1=y_2$.

$Ut\sin\theta-\frac{1}{2}gt^2=h-Vt\sin\theta-\frac{1}{2}gt^2$

$Ut\sin\theta+Vt\sin\theta=h$

$(U+V)t\sin\theta=h$

But $h=R\tan\theta$

$\therefore (U+V)t\sin\theta=R\tan\theta$

$\therefore t=\frac{R\tan\theta}{(U+V)\sin\theta}$

$=\frac{R}{(U+V)\cos\theta}$

$=T$

When $t=T$, the projectiles have the same coordinates, so they collide.

(iii) Let $x_1=\lambda R$, $0<\lambda<1$

$\therefore UT\cos\theta=\lambda R$

From (i), $T=\frac{R}{(U+V)\cos\theta}$

$\therefore \frac{UR\cos\theta}{(U+V)\cos\theta}=\lambda R$

$\frac{U}{U+V}=\lambda$

$U=\lambda(U+V)$

$=\lambda U+\lambda V$

$\therefore V=\frac{U-\lambda U}{\lambda}$

$=\frac{U(1-\lambda)}{\lambda}$

$=\left(\frac{1}{\lambda}-1\right)U$.

(b) (i) $(1+x)^r+(1+x)^{r+1}+\ldots+(1+x)^n$

$\therefore a=(1+x)^r$

$r=(1+x)$

$(n-r+1)$ terms

$\therefore S=\frac{(1+x)^r\left[(1+x)^{n-r+1}-1\right]}{1+x-1}$

$=\frac{(1+x)^{n+1}-(1+x)^r}{x}$

The coefficient of x^r on LHS

$=\binom{r}{r}+\binom{r+1}{r}+\ldots+\binom{n}{r}$

The coefficient of x^r on RHS

$=\binom{n+1}{r+1}$ from $(1+x)^{n+1}$

since $\frac{1}{x}\left[\binom{n+1}{r+1}x^{r+1}\right]=\binom{n+1}{r+1}x^r$

and $(1+x)^r$ does not contain x^{r+1}.

(ii) (1) The line $y=x$ passes through n points along the diagonal.

$\therefore$ An interval is formed by choosing two points.

$\therefore$ Number of intervals is $\binom{n}{2}$.

(2) The lines parallel to line $y=x$ that lie above it, go through $(n-1)$, $(n-2)$, …, (2) points, so we can form

$\binom{n-1}{2},\binom{n-2}{2},\ldots,\binom{2}{2}$ intervals.

This also occurs for lines parallel to and below line $y=x$.

$\therefore$ Total number of intervals is

$S_n=\binom{2}{2}+\binom{3}{2}+\ldots+\binom{n-1}{2}+\binom{n}{2}$

$+\binom{n-1}{2}+\ldots+\binom{3}{2}+\binom{2}{2}$.

(iii) Let $r = 2$ in part (i)

$$\therefore \binom{2}{2} + \binom{3}{2} + \ldots + \binom{n-1}{2} = \binom{n}{3}$$

$$\therefore S_n = \binom{n}{3} + \binom{n}{2} + \binom{n}{3}$$

$$= 2 \times \frac{n!}{3!(n-3)!} + \frac{n!}{2!(n-2)!}$$

$$= \frac{n(n-1)(n-2)}{3} + \frac{n(n-1)}{2}$$

$$= \frac{2n(n-1)(n-2) + 3n(n-1)}{6}$$

$$= \frac{n(n-1)[2(n-2)+3]}{6}$$

$$= \frac{n(n-1)(2n-1)}{6}.$$

Question 7

(a) (i) Let $f(x) = x$

Now $f'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{x+h-x}{h}$$

$$= \lim_{h \to 0}$$

$$= 1.$$

(ii) Required to prove: $\dfrac{d}{dx}\left(x^n\right) = nx^{n-1}$

for all positive integers n.

For $n = 1$,

LHS $= \dfrac{d}{dx}(x) = 1$ (from (i))

RHS $= 1$

$\therefore$ True for $n = 1$.

Let the result be true for some $n = k$,

i.e. $\dfrac{d}{dx}\left(x^k\right) = kx^{k-1}$

Required to prove:

$$\frac{d}{dx}\left(x^{k+1}\right) = (k+1)x^k$$

for $n = k + 1$.

$$\text{LHS} = \frac{d}{dx}\left(x^{k+1}\right) = \frac{d}{dx}\left(x^k . x\right)$$

Using the product rule,

$$\text{LHS} = x^k . \frac{d}{dx}(x) + x . \frac{d}{dx}\left(x^k\right)$$

$$= x^k + x . kx^{k-1}$$

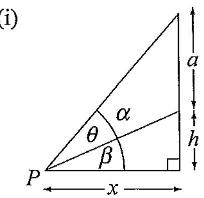
$$= x^k + k . x^k$$

$$= x^k(k+1) \text{ as required}$$

$\therefore$ True for $n = k + 1$.

Hence if the result is true for $n = k$, it is true for $n = k + 1$. It is true for $n = 1$, so by the principle of mathematical induction it is true for all positive integers $n \geq 1$.

(b) (i)



$$\tan\beta = \frac{h}{x}$$

$$\tan\alpha = \frac{a+h}{x}$$

$$\theta = \alpha - \beta$$

$$\therefore \tan\theta = \tan(\alpha - \beta)$$

$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

$$= \frac{\frac{a+h}{x} - \frac{h}{x}}{1 + \frac{a+h}{x} . \frac{h}{x}}$$

$$= \frac{a}{x} \div \left(\frac{x^2}{x^2} + \frac{h(a+h)}{x^2}\right)$$

$$= \frac{ax}{x^2 + h(a+h)}$$

$$\therefore \theta = \tan^{-1}\left(\frac{ax}{x^2 + h(a+h)}\right).$$

(ii) Let $y = \dfrac{ax}{x^2 + h(a+h)}$

Now, $\theta = \tan^{-1} y$

$$\frac{d\theta}{dy} = \frac{1}{1+y^2}$$

$$= \frac{1}{1 + \left(\frac{ax}{x^2 + h(a+h)}\right)^2}$$

$$= 1 \div \frac{\left[x^2 + h(a+h)\right]^2 + a^2x^2}{\left[x^2 + h(a+h)\right]^2}$$

$$= \frac{\left[x^2 + h(a+h)\right]^2}{\left[x^2 + h(a+h)\right]^2 + a^2x^2}$$

Now, using the quotient rule,

$$\frac{dy}{dx} = \frac{\left[x^2 + h(a+h)\right] . a - ax . 2x}{\left[x^2 + h(a+h)\right]^2}$$

$$= \frac{ax^2 + ah(a+h) - 2ax^2}{\left[x^2 + h(a+h)\right]^2}$$

$$= \frac{ah(a+h) - ax^2}{\left[x^2 + h(a+h)\right]^2}$$

$$\therefore \frac{d\theta}{dx} = \frac{d\theta}{dy} \times \frac{dy}{dx}$$

$$= \frac{\left[x^2 + h(a+h)\right]^2}{\left[x^2 + h(a+h)\right]^2 + a^2x^2} \times \frac{ah(a+h) - ax^2}{\left[x^2 + h(a+h)\right]^2}$$

$$= \frac{ah(a+h) - ax^2}{\left[x^2 + h(a+h)\right]^2 + a^2x^2}$$

When $\dfrac{d\theta}{dx} = 0$,

$$ah(a+h) - ax^2 = 0$$

$$ax^2 = ah(a+h)$$

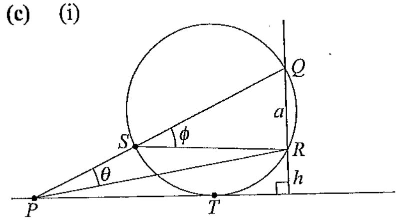
$$x^2 = h(a+h)$$

$$\therefore x = \pm\sqrt{h(a+h)} \text{ m}$$

Since x is positive, maximum is

$$x = \sqrt{h(a+h)} \text{ m.}$$

(c) (i)



$\phi = \theta + \angle SRP$ (exterior angle of ΔSRP is sum of interior opposite angles)

$\therefore \theta < \phi$ since $\angle SPR > 0$.

As P approaches T, S also approaches T.

Now, $\angle QTR = \phi$ (angles in same segment on same arc QR)

But $\angle QPR = \theta$

$\therefore$ When P and T (and S) are coincident, $\theta = \phi$.

$\therefore \theta$ is a maximum when P and T are the same point since there is no triangle for ϕ to be the exterior angle.

(ii) Let d be the distance of T from the building.

$\therefore d^2 = h . (a+h)$ since the square the tangent equals the product of t intercepts

$\therefore d = \sqrt{h(a+h)}$.

End of Mathematics Extension 1 solutions