BOARD OF STUDIES
NEW SOUTH WALES

2009

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Total marks – 120
Attempt Questions 1–8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\displaystyle\int \frac{\ln x}{x}\,dx$. **2**

(b) Find $\displaystyle\int x e^{2x}\,dx$. **2**

(c) Find $\displaystyle\int \frac{x^2}{1+4x^2}\,dx$. **3**

(d) Evaluate $\displaystyle\int_2^5 \frac{x-6}{x^2+3x-4}\,dx$. **4**

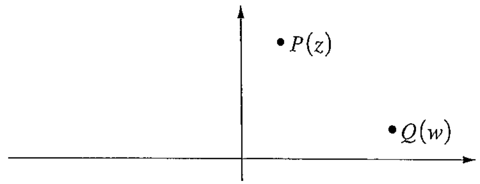
(e) Evaluate $\displaystyle\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{1+x^2}}\,dx$. **4**

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Write i^9 in the form $a+ib$ where a and b are real. **1**

(b) Write $\dfrac{-2+3i}{2+i}$ in the form $a+ib$ where a and b are real. **1**

(c) The points P and Q on the Argand diagram represent the complex numbers z and w respectively.



Copy the diagram into your writing booklet, and mark on it the following points:

(i) the point R representing iz **1**

(ii) the point S representing $\overline{w}$ **1**

(iii) the point T representing $z+w$. **1**

(d) Sketch the region in the complex plane where the inequalities $|z-1| \le 2$ and $-\frac{\pi}{4} \le \arg(z-1) \le \frac{\pi}{4}$ hold simultaneously. **2**

(e) (i) Find all the 5th roots of -1 in modulus-argument form. **2**

(ii) Sketch the 5th roots of -1 on an Argand diagram. **1**

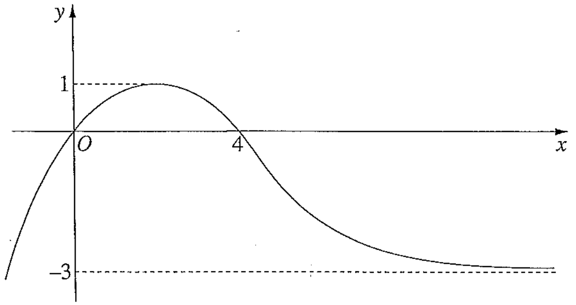
(f) (i) Find the square roots of $3+4i$. **3**

(ii) Hence, or otherwise, solve the equation **2**

$$z^2 + iz - 1 - i = 0.$$

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the graph $y = f(x)$.



Draw separate one-third page sketches of the graphs of the following:

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = [f(x)]^2$ **2**

(iii) $y = f(x^2)$. **2**

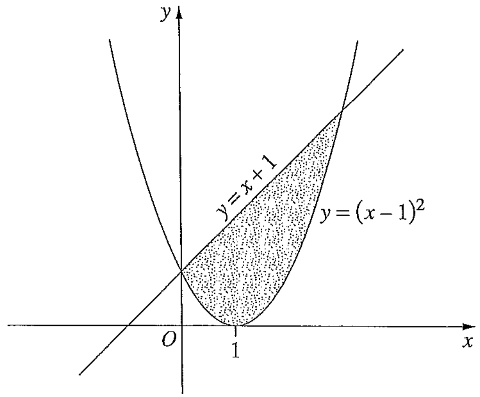
(b) Find the coordinates of the points where the tangent to the curve $x^2 + 2xy + 3y^2 = 18$ is horizontal. **3**

(c) Let $P(x) = x^3 + ax^2 + bx + 5$, where a and b are real numbers. **3**

Find the values of a and b given that $(x-1)^2$ is a factor of $P(x)$.

Question 3 continues on page 5

(d) The diagram shows the region enclosed by the curves $y = x + 1$ and $y = (x-1)^2$. **3**



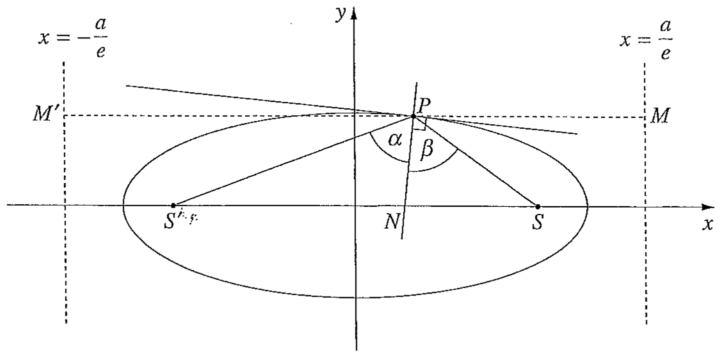
The region is rotated about the y-axis.

Use the method of cylindrical shells to find the volume of the solid formed.

End of Question 3

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) The ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ has foci $S(ae, 0)$ and $S'(-ae, 0)$ where e is the eccentricity, with corresponding directrices $x = \dfrac{a}{e}$ and $x = -\dfrac{a}{e}$. The point $P(x_0, y_0)$ is on the ellipse. The points where the horizontal line through P meets the directrices are M and M', as shown in the diagram.



(i) Show that the equation of the normal to the ellipse at the point P is **2**

$$y - y_0 = \frac{a^2 y_0}{b^2 x_0}(x - x_0).$$

(ii) The normal at P meets the x-axis at N. Show that N has coordinates $(e^2 x_0, 0)$. **2**

(iii) Using the focus-directrix definition of an ellipse, or otherwise, show that **2**

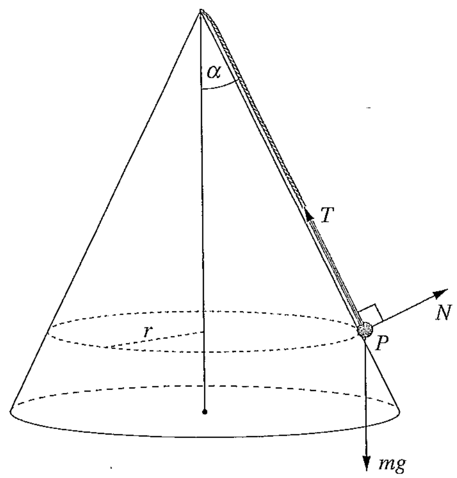
$$\frac{PS}{PS'} = \frac{NS}{NS'}.$$

(iv) Let $\alpha = \angle S'PN$ and $\beta = \angle NPS$. **2**

By applying the sine rule to $\triangle S'PN$ and to $\triangle NPS$, show that $\alpha = \beta$.

Question 4 continues on page 7

(b) A light string is attached to the vertex of a smooth vertical cone. A particle P of mass m is attached to the string as shown in the diagram. The particle remains in contact with the cone and rotates with constant angular velocity ω on a circle of radius r. The string and the surface of the cone make an angle of α with the vertical, as shown.



The forces acting on the particle are the tension, T, in the string, the normal reaction, N, to the cone and the gravitational force mg.

(i) Resolve the forces on P in the horizontal and vertical directions. **2**

(ii) Show that $T = m\left(g\cos\alpha + r\omega^2 \sin\alpha\right)$ and find a similar expression for N. **2**

(iii) Show that if $T = N$ then **2**

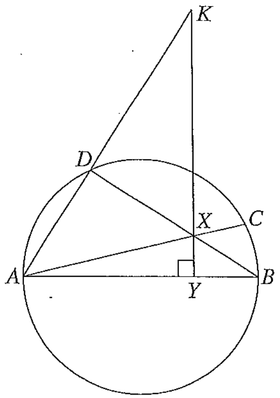
$$\omega^2 = \frac{g}{r}\left(\frac{\tan\alpha - 1}{\tan\alpha + 1}\right).$$

(iv) For which values of α can the particle rotate so that $T = N$? **1**

End of Question 4

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) In the diagram AB is the diameter of the circle. The chords AC and BD intersect at X. The point Y lies on AB such that XY is perpendicular to AB. The point K is the intersection of AD produced and YX produced.



Copy or trace the diagram into your writing booklet.

(i) Show that $\angle AKY = \angle ABD$. **2**

(ii) Show that $CKDX$ is a cyclic quadrilateral. **2**

(iii) Show that B, C and K are collinear. **2**

Question 5 continues on page 9

(b) For each integer $n \geq 0$, let

$$I_n = \int_0^1 x^{2n+1} e^{x^2}\, dx.$$

(i) Show that for $n \geq 1$ **2**

$$I_n = \frac{e}{2} - nI_{n-1}.$$

(ii) Hence, or otherwise, calculate I_2. **2**

(c) Let $f(x) = \dfrac{e^x - e^{-x}}{2} - x$.

(i) Show that $f''(x) > 0$ for all $x > 0$. **2**

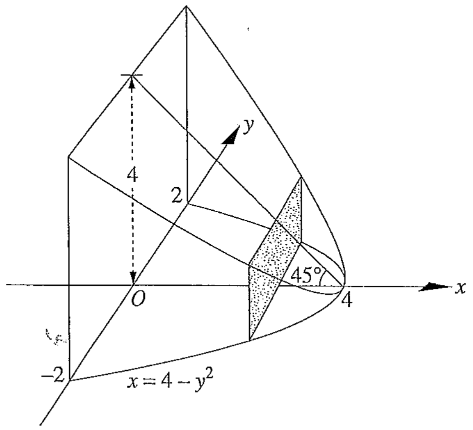
(ii) Hence, or otherwise, show that $f'(x) > 0$ for all $x > 0$. **2**

(iii) Hence, or otherwise, show that $\dfrac{e^x - e^{-x}}{2} > x$ for all $x > 0$. **1**

End of Question 5

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) The base of a solid is the region enclosed by the parabola $x = 4 - y^2$ and the y-axis. The top of the solid is formed by a plane inclined at 45° to the xy-plane. Each vertical cross-section of the solid parallel to the y-axis is a rectangle. A typical cross-section is shown shaded in the diagram. **3**



Find the volume of the solid.

(b) Let $P(x) = x^3 + qx^2 + qx + 1$, where q is real. One zero of $P(x)$ is -1.

(i) Show that if α is a zero of $P(x)$ then $\frac{1}{\alpha}$ is a zero of $P(x)$. **1**

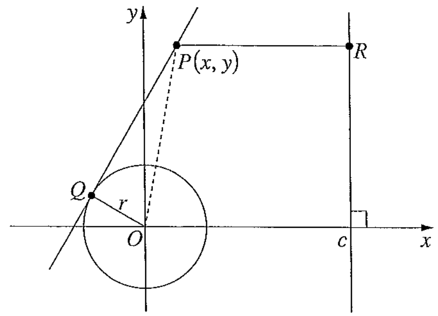
(ii) Suppose that α is a zero of $P(x)$ and α is not real.

(1) Show that $|\alpha| = 1$. **2**

(2) Show that $\text{Re}(\alpha) = \frac{1-q}{2}$. **2**

Question 6 continues on page 11

(c) The diagram shows a circle of radius r, centred at the origin, O. The line PQ is tangent to the circle at Q, the line PR is horizontal, and R lies on the line $x = c$.



(i) Find the length of PQ in terms of x, y and r. **1**

(ii) The point P moves such that $PQ = PR$. **2**

Show that the equation of the locus of P is

$$y^2 = r^2 + c^2 - 2cx.$$

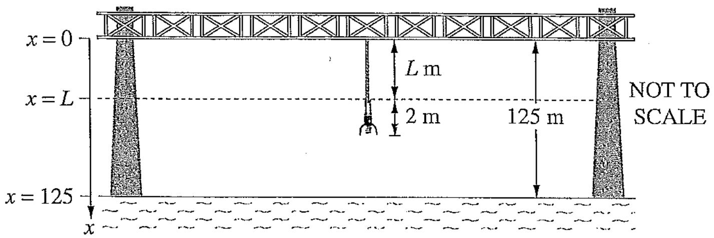
(iii) Find the focus, S, of the parabola in part (ii). **2**

(iv) Show that the difference between the length PS and the length PQ is independent of x. **2**

End of Question 6

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) A bungee jumper of height 2 m falls from a bridge which is 125 m above the surface of the water, as shown in the diagram. The jumper's feet are tied to an elastic cord of length L m. The displacement of the jumper's feet, measured downwards from the bridge, is x m.



The jumper's fall can be examined in two stages. In the first stage of the fall, where $0 \le x \le L$, the jumper falls a distance of L m subject to air resistance, and the cord does not provide resistance to the motion. In the second stage of the fall, where $x > L$, the cord stretches and provides additional resistance to the downward motion.

(i) The equation of motion for the jumper in the first stage of the fall is

$$\ddot{x} = g - rv$$

where g is the acceleration due to gravity, r is a positive constant, and v is the velocity of the jumper.

(1) Given that $x = 0$ and $v = 0$ initially, show that **3**

$$x = \frac{g}{r^2}\ln\left(\frac{g}{g - rv}\right) - \frac{v}{r}.$$

(2) Given that $g = 9.8 \text{ m s}^{-2}$ and $r = 0.2 \text{ s}^{-1}$, find the length, L, of the cord such that the jumper's velocity is 30 m s^{-1} when $x = L$. Give your answer to two significant figures. **1**

(ii) In the second stage of the fall, where $x > L$, the displacement x is given by **4**

$$x = e^{-\frac{t}{10}}\left(29\sin t - 10\cos t\right) + 92$$

where t is the time in seconds after the jumper's feet pass $x = L$.

Determine whether or not the jumper's head stays out of the water.

Question 7 continues on page 13

(b) Let $z = \cos\theta + i\sin\theta$.

(i) Show that $z^n + z^{-n} = 2\cos n\theta$, where n is a positive integer. **2**

(ii) Let m be a positive integer. Show that **3**

$$(2\cos\theta)^{2m} = 2\left[\cos 2m\theta + \binom{2m}{1}\cos(2m-2)\theta + \binom{2m}{2}\cos(2m-4)\theta + \cdots + \binom{2m}{m-1}\cos 2\theta\right] + \binom{2m}{m}.$$

(iii) Hence, or otherwise, prove that **2**

$$\int_0^{\frac{\pi}{2}} \cos^{2m}\theta \, d\theta = \frac{\pi}{2^{2m+1}}\binom{2m}{m}$$

where m is a positive integer.

End of Question 7

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Using the substitution $t = \tan\frac{\theta}{2}$, or otherwise, show that **2**

$$\cot\theta + \frac{1}{2}\tan\frac{\theta}{2} = \frac{1}{2}\cot\frac{\theta}{2}.$$

(ii) Use mathematical induction to prove that, for integers $n \geq 1$, **3**

$$\sum_{r=1}^{n} \frac{1}{2^{r-1}}\tan\frac{x}{2^r} = \frac{1}{2^{n-1}}\cot\frac{x}{2^n} - 2\cot x.$$

(iii) Show that **2**

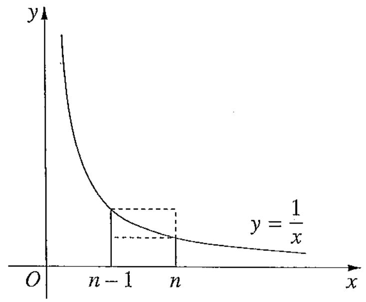
$$\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{2^{r-1}}\tan\frac{x}{2^r} = \frac{2}{x} - 2\cot x.$$

(iv) Hence find the exact value of **2**

$$\tan\frac{\pi}{4} + \frac{1}{2}\tan\frac{\pi}{8} + \frac{1}{4}\tan\frac{\pi}{16} + \cdots.$$

Question 8 continues on page 15

(b) **2**



Let n be a positive integer greater than 1.

The area of the region under the curve $y = \frac{1}{x}$ from $x = n-1$ to $x = n$ is between the areas of two rectangles, as shown in the diagram.

Show that

$$e^{-\frac{n}{n-1}} < \left(1 - \frac{1}{n}\right)^n < e^{-1}.$$

(c) A game is being played by n people, $A_1, A_2, \ldots, A_n$, sitting around a table. Each person has a card with their own name on it and all the cards are placed in a box in the middle of the table. Each person in turn, starting with A_1, draws a card at random from the box. If the person draws their own card, they win the game and the game ends. Otherwise, the card is returned to the box and the next person draws a card at random. The game continues until someone wins.

Let W be the probability that A_1 wins the game.

Let $p = \frac{1}{n}$ and $q = 1 - \frac{1}{n}$.

(i) Show that $W = p + q^n W$. **1**

(ii) Let m be a fixed positive integer and let W_m be the probability that A_1 wins in no more than m attempts. **3**

Use part (b) to show that, if n is large, $\frac{W_m}{W}$ is approximately equal to $1 - e^{-m}$.

End of paper

2009 Higher School Certificate Solutions
Mathematics Extension 2

Question 1

(a) Let $u = \ln x$, $\therefore du = \dfrac{1}{x}\,dx$

$$\int \frac{\ln x}{x}\,dx = \int u\,du$$
$$= \frac{u^2}{2} + c$$
$$= \frac{(\ln x)^2}{2} + c.$$

(b) Let $u = x$, $\quad \therefore du = dx$

Let $dv = e^{2x}\,dx$, $\quad \therefore v = \dfrac{e^{2x}}{2}$

$$\int u\left(\frac{dv}{dx}\right)dx = uv - \int v\left(\frac{du}{dx}\right)dx$$
$$\therefore \int xe^{2x}\,dx = \frac{xe^{2x}}{2} - \int \frac{e^{2x}}{2}\,dx$$
$$= \frac{xe^{2x}}{2} - \frac{e^{2x}}{4} + c.$$

(c)
$$\int \frac{x^2}{1+4x^2}\,dx = \frac{1}{4}\int \frac{4x^2}{1+4x^2}\,dx$$
$$= \frac{1}{4}\int\left(\frac{1+4x^2}{1+4x^2} - \frac{1}{1+4x^2}\right)dx$$
$$= \frac{1}{4}\int\left(1 - \frac{1}{1+4x^2}\right)dx$$
$$= \frac{1}{4}\int\left(1 - \frac{1}{4\left(\frac{1}{4}+x^2\right)}\right)dx$$
$$= \frac{1}{4}\left(x - \frac{1}{2}\tan^{-1}2x\right) + c.$$

(d)
$$\frac{x-6}{x^2+3x-4} = \frac{x-6}{(x+4)(x-1)}$$

Let $\dfrac{x-6}{(x+4)(x-1)} \equiv \dfrac{A}{x+4} + \dfrac{B}{x-1}$

$$\therefore x-6 \equiv A(x-1) + B(x+4)$$

METHOD 1 Substitution:

Let $x = 1$: $\quad -5 = 5B$, $\quad \therefore B = -1$

Let $x = -4$: $\quad -10 = -5A$, $\quad \therefore A = 2$

METHOD 2 Coefficients:

$$x - 6 \equiv Ax - A + Bx + 4B$$
$$\equiv (A+B)x + 4B - A$$

$\therefore A + B = 1 \quad$ ①

$4B - A = -6 \quad$ ②

① + ②

$5B = -5$

$\therefore B = -1 \quad$ ③

Substitute ③ into ①.

$A - 1 = 1$

$\therefore A = 2$

$$\therefore \int_2^5 \frac{x-6}{x^2+3x-4}\,dx$$
$$= \int_2^5 \left(\frac{2}{x+4} - \frac{1}{x-1}\right)dx$$
$$= \left[2\ln(x+4) - \ln(x-1)\right]_2^5$$
$$= (2\ln 9 - \ln 4) - (2\ln 6 - \ln 1)$$
$$= 2(\ln 9 - \ln 2 - \ln 6)$$
$$= 2\ln\left(\frac{9}{2\times 6}\right)$$
$$= 2\ln\left(\frac{3}{4}\right).$$

(e) Let $x = \tan\theta$, $\therefore dx = \sec^2\theta\,d\theta$

When $x = \sqrt{3}$, $\quad \theta = \dfrac{\pi}{3}$

When $x = 1$, $\quad \theta = \dfrac{\pi}{4}$

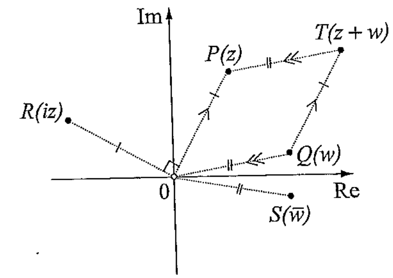
$$\therefore \int_1^{\sqrt{3}} \frac{dx}{x^2\sqrt{1+x^2}}$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec^2\theta\,d\theta}{\tan^2\theta\sqrt{1+\tan^2\theta}}$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{d\theta}{\cos^2\theta.\dfrac{\sin^2\theta}{\cos^2\theta}.\sqrt{\sec^2\theta}}$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos\theta}{\sin^2\theta}\,d\theta$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sin\theta}.\frac{\cos\theta}{\sin\theta}\,d\theta$$
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \operatorname{cosec}\theta\cot\theta\,d\theta$$
$$= \Big[-\operatorname{cosec}\theta\Big]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$
$$= \sqrt{2} - \frac{2}{\sqrt{3}}.$$

Question 2

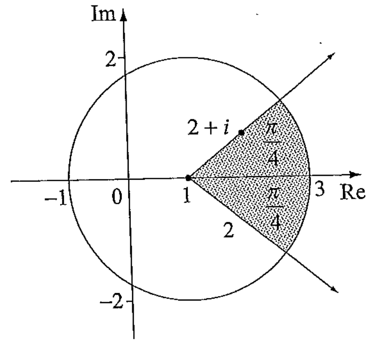
(a)
$$i^9 = i^4 \times i^4 \times i$$
$$= i \quad (\text{or } 0 + 1i)$$

(b)
$$\frac{-2+3i}{2+i} = \frac{-2+3i}{2+i}\cdot\frac{2-i}{2-i}$$
$$= \frac{-4+2i+6i+3}{2^2+1^2}$$
$$= \frac{-1+8i}{5} \quad \text{or} \quad -\frac{1}{5} + \frac{8i}{5}.$$

(c) (i), (ii) and (iii)



(d)



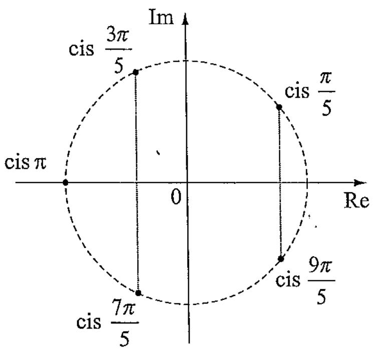
(e) (i) Let $z = \cos\theta + i\sin\theta$

$$z^5 = \cos 5\theta + i\sin 5\theta = -1$$
$$\therefore \cos 5\theta = -1$$
$$5\theta = \pi,\ 3\pi,\ 5\pi,\ 7\pi,\ 9\pi$$
$$\therefore \theta = \frac{\pi}{5},\ \frac{3\pi}{5},\ 1,\ \frac{7\pi}{5},\ \frac{9\pi}{5}$$

$\therefore$ Roots are:

$$z_1 = \text{cis}\,\frac{\pi}{5}$$
$$z_2 = \text{cis}\,\frac{3\pi}{5}$$
$$z_3 = \text{cis}\,\pi = -1$$
$$z_4 = \text{cis}\,\frac{7\pi}{5}$$
$$z^5 = \text{cis}\,\frac{9\pi}{5}.$$

(ii)



(f) (i) Let $\sqrt{3+4i} = a+ib$

(where a, b are real)

$$\therefore 3+4i = (a+ib)^2$$
$$= a^2 - b^2 + 2abi$$
$$\therefore a^2 - b^2 = 3, \quad 2ab = 4$$
$$b = \frac{2}{a}$$
$$\therefore a^2 - \frac{4}{a^2} = 3$$
$$a^4 - 3a^2 - 4 = 0$$
$$(a^2-4)(a^2+1) = 0$$
$$a^2 = 4, -1$$
$$\therefore a = \pm 2 \quad (a \text{ is real})$$
$$\therefore b = \pm 1$$
$$\therefore \sqrt{3+4i} = \pm(2+i).$$

(ii) $z^2 + iz - 1 - i = 0$

$$\therefore z = \frac{-i \pm \sqrt{i^2 - 4(-1-i)}}{2}$$
$$= \frac{-i \pm \sqrt{-1+4+4i}}{2}$$
$$= \frac{-i \pm \sqrt{3+4i}}{2}$$

From (i),

$$z = \frac{-i+(2+i)}{2}, \frac{-i-(2+i)}{2}$$
$$= 1, \ -1-i.$$

Question 3

(a) See next page.

(b)
$$x^2 + 2xy + 3y^2 = 18 \quad ①$$
$$\therefore 2x + 2\left(y + x\frac{dy}{dx}\right) + 6y\frac{dy}{dx} = 0$$
$$x + y + (x+3y)\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{-(x+y)}{x+3y}$$

Tangent is horizontal when $\frac{dy}{dx} = 0$,

i.e. $\frac{-(x+y)}{x+3y} = 0$

$$\therefore y = -x \quad ②$$

Substitute ② into ①.

$$x^2 + 2x(-x) + 3(-x)^2 = 18$$
$$2x^2 = 18$$
$$\therefore x = \pm 3$$

From ②,

$x = 3, \quad y = -3$

$x = -3 \quad y = 3$

$\therefore$ Tangent is horizontal at $(3,-3)$ and $(-3,3)$.

(c) $P(x) = x^3 + ax^2 + bx + 5$

$\therefore P'(x) = 3x^2 + 2ax + b$

If $(x-1)^2$ is a factor of $P(x)$, then $(x-1)$ is a factor of $P'(x)$.

$\therefore P(1) = 0$, i.e. $1 + a + b + 5 = 0$

$$a + b = -6 \quad ①$$

and

$P'(1) = 0$, i.e. $3 + 2a + b = 0$

$$2a + b = -3 \quad ②$$

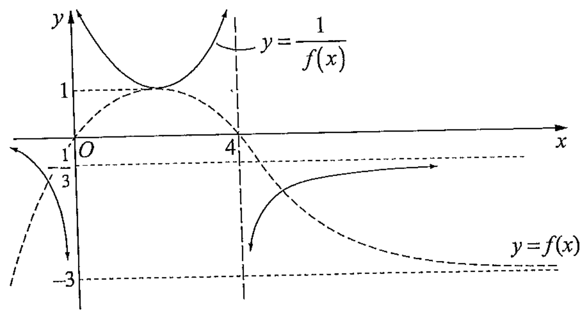
② − ①: $\therefore a = 3 \quad ③$

Substitute ③ into ①.

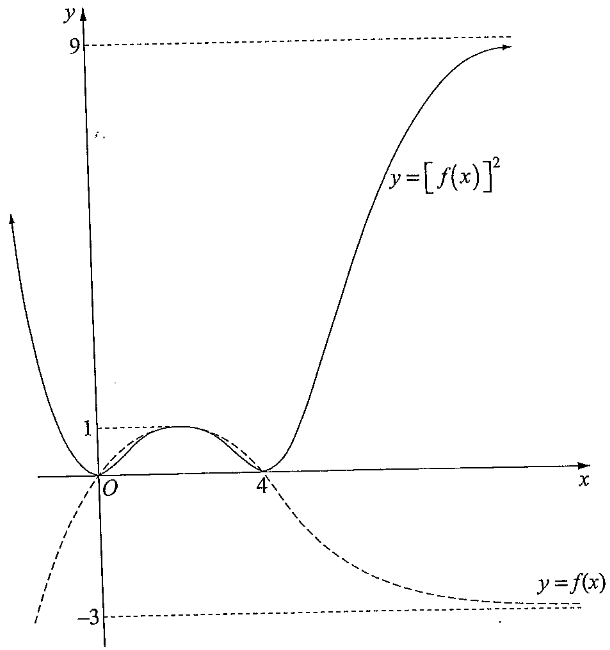
$3 + b = -6$, $\therefore b = -9$

$\therefore a = 3, \ b = -9.$

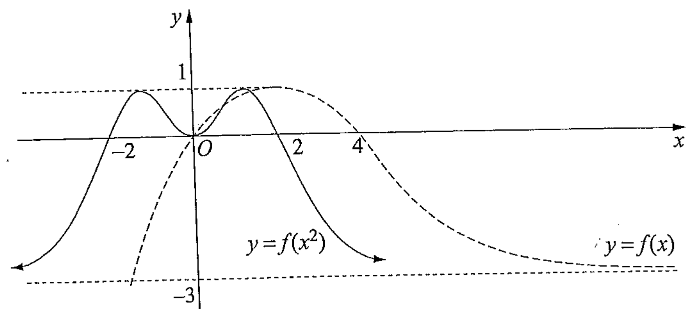
(a) (i)



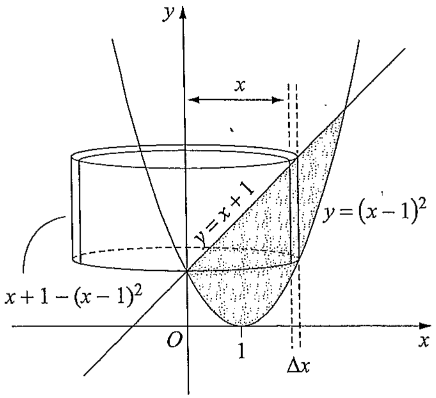
(ii)

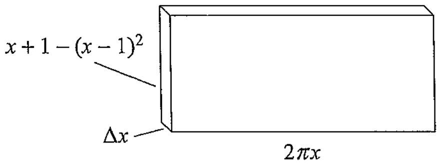


(iii)



(d)





$$\Delta V = 2\pi x\left[x+1-(x-1)^2\right]\Delta x$$
$$= 2\pi x\left(3x - x^2\right)\Delta x$$

Points of intersection:

Let $x+1=(x-1)^2$

$$x+1 = x^2-2x+1$$
$$x^2-3x=0$$
$$x(x-3)=0$$
$$\therefore x=0 \text{ or } x=3$$

$$\therefore V = \lim_{\Delta x\to 0}\sum_{x=0}^{3} 2\pi x\left(3x-x^2\right)\Delta x$$
$$= 2\pi\int_0^3\left(3x^2-x^3\right)dx$$
$$= 2\pi\left[x^3-\frac{x^4}{4}\right]_0^3$$
$$= 2\pi\left[\left(27-\frac{81}{4}\right)-0\right]$$
$$= \frac{27\pi}{2}$$

$\therefore$ Volume $= \dfrac{27\pi}{2}$ units^3.

Question 4

(a) (i)
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$
$$\therefore \frac{2x}{a^2}+\frac{2y}{b^2}\cdot\frac{dy}{dx}=0$$
$$\therefore \frac{dy}{dx} = -\frac{2x}{a^2}\div\frac{2y}{b^2}$$
$$= -\frac{xb^2}{ya^2}$$

At $P(x_0, y_0)$, gradient of normal is $\dfrac{y_0a^2}{x_0b^2}$.
$\therefore$Equation of normal is

$$y-y_0 = \frac{y_0a^2}{x_0b^2}(x-x_0).$$

(ii) For $N, y=0$

$$\therefore -y_0 = \frac{y_0a^2}{x_0b^2}(x-x_0) \text{ (from (i))}$$
$$-\frac{x_0b^2}{a^2} = x-x_0$$
$$x = x_0-\frac{x_0b^2}{a^2}$$
$$= x_0\left(1-\frac{b^2}{a^2}\right)$$

Now, $b^2 = a^2\left(1-e^2\right)$

$$\therefore e^2 = 1-\frac{b^2}{a^2}$$
$$\therefore x = x_0e^2$$

$\therefore N$ is $\left(e^2x_0, 0\right)$.

(iii) $\dfrac{PS}{PM}=e, \quad \dfrac{PS'}{PM'}=e$

$$\therefore \frac{PS}{PS'}=\frac{PM}{PM'}$$

Directrices are $x=\pm\dfrac{a}{e}$.

$$\therefore PM = \frac{a}{e}-x_0, \quad PM' = \frac{a}{e}+x_0$$

From (ii), N is $\left(e^2x_0, 0\right)$.

$$\therefore NS = ae-e^2x_0, \quad NS' = ae+e^2x_0$$

$$\therefore \frac{PM}{PM'} = \frac{\frac{a}{e}-x_0}{\frac{a}{e}+x_0}$$
$$= \frac{a-ex_0}{a+ex_0}$$
$$= \frac{ae-e^2x_0}{ae+e^2x_0}$$
$$= \frac{NS}{NS'}$$
$$\therefore \frac{PS}{PS'} = \frac{NS}{NS'}.$$

(iv) In ΔNPS, $\dfrac{PS}{\sin(PNS)} = \dfrac{NS}{\sin\beta}$ ①

In $\Delta S'PN$, $\dfrac{PS'}{\sin(PNS')} = \dfrac{NS'}{\sin\alpha}$ ②

From ①, $PS = \dfrac{NS\sin(PNS)}{\sin\beta}$

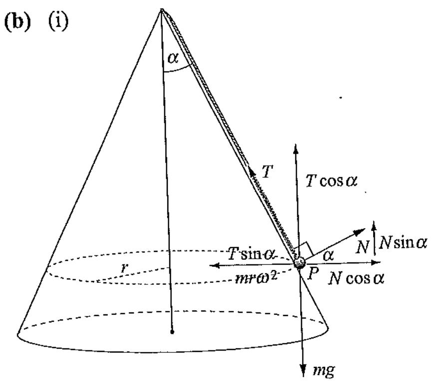
From ②, $PS' = \dfrac{NS'\sin(PNS')}{\sin\alpha}$

$$\therefore \frac{PS}{PS'} = \frac{NS\sin(PNS)}{\sin\beta}\cdot\frac{\sin\alpha}{NS'\sin(PNS')}$$
$$\therefore 1 = \frac{\sin(PNS).\sin\alpha}{\sin\beta.\sin(PNS')}$$
$$\left(\text{since } \frac{PS}{PS'}=\frac{NS}{NS'}, \text{ from (iii)}\right)$$
$$\therefore \frac{\sin(PNS)}{\sin\beta} = \frac{\sin(PNS')}{\sin\alpha}$$

Now, $\angle PNS$ and $\angle PNS'$ are supplementary.

$$\therefore \sin(PNS) = \sin(PNS')$$
$$\therefore \frac{1}{\sin\beta} = \frac{1}{\sin\alpha}$$
$$\sin\alpha = \sin\beta$$
$$\therefore \alpha=\beta \quad \text{(both are acute).}$$

(b) (i)



$$T\cos\alpha + N\sin\alpha = mg \quad ①$$
$$T\sin\alpha - N\cos\alpha = mr\omega^2 \quad ②$$

(ii) Multiply ① by $\cos\alpha$, ② by $\sin\alpha$.

$$\therefore T\cos^2\alpha + N\sin\alpha\cos\alpha = mg\cos\alpha$$
$$T\sin^2\alpha - N\sin\alpha\cos\alpha = mr\omega^2\sin\alpha$$

Adding the two new equations,

$$T\left(\cos^2\alpha+\sin^2\alpha\right)$$
$$= mg\cos\alpha + mr\omega^2\sin\alpha$$
$$\therefore T = m\left(g\cos\alpha + r\omega^2\sin\alpha\right)$$

Multiply ① by $\sin\alpha$, ② by $\cos\alpha$.

$$\therefore T\cos\alpha\sin\alpha + N\sin^2\alpha = mg\sin\alpha$$
$$T\cos\alpha\sin\alpha - N\cos^2\alpha = mr\omega^2\cos\alpha$$

Subtracting the two new equations,

$$N = mg\sin\alpha - mr\omega^2\cos\alpha$$
$$= m\left(g\sin\alpha - r\omega^2\cos\alpha\right).$$

(iii) If $T = N$,

$$m\left(g\cos\alpha + r\omega^2\sin\alpha\right) = m\left(g\sin\alpha - r\omega^2\cos\alpha\right)$$

$$\therefore g\cos\alpha + r\omega^2\sin\alpha = g\sin\alpha - r\omega^2\cos\alpha$$

$$r\omega^2(\sin\alpha + \cos\alpha) = g(\sin\alpha - \cos\alpha)$$

Dividing by $\cos\alpha$,

$$r\omega^2(\tan\alpha + 1) = g(\tan\alpha - 1)$$

$$\therefore \omega^2 = \frac{g(\tan\alpha - 1)}{r(\tan\alpha + 1)}.$$

(iv) For $\omega^2 > 0$,

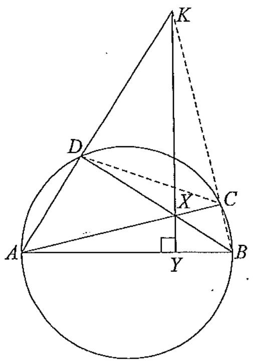
$g(\tan\alpha - 1) > 0$ (from (iii))

$\therefore \tan\alpha > 1$

$$\therefore \frac{\pi}{4} < \alpha < \frac{\pi}{2}.$$

Question 5

(a)



(i) In ΔAKY and ΔABD,

$\angle ADB = 90°$ (angle in a semi-circle)

$\angle KYA = 90°$ (given)

$\therefore \angle ADB = \angle KYA$

$\angle KAY = \angle DAB$ (common angle)

$\therefore \angle AKY = \angle ABD$ (angle sum of Δs).

(ii) From (i), $\angle AKY = \angle ABD$

$\angle ABD = \angle DCA$ (angles in the same segment on chord AD)

$\therefore \angle DKX = \angle DCX$

$\therefore \angle DKX$ and $\angle DCX$ are on chord DX

$\therefore CKDX$ is a cyclic quadrilateral.

(iii) $\angle KDX = 90°$ (angles on a straight line are supplementary)

$\therefore \angle KDX + \angle KCX = 180°$

(opposite angles of cyclic quadrilateral $CKDX$ are supplementary)

$\therefore \angle KCX = 90°$

Now, $\angle ACB = 90°$ (angle in a semi-circle)

$\therefore \angle KCB =$ straight angle

$\therefore B, C, K$ are collinear.

(b) (i)

$$I_n = \int_0^1 x^{2n+1}e^{x^2}\,dx$$

$$= \frac{1}{2}\int_0^1 x^{2n}.2xe^{x^2}\,dx$$

Let $u = x^{2n}$, $\quad \therefore du = 2nx^{2n-1}\,dx$

Let $dv = 2xe^{x^2}\,dx$, $\quad \therefore v = e^{x^2}$

$$\therefore I_n = \frac{1}{2}\left(\left[x^{2n}e^{x^2}\right]_0^1 - \int_0^1 e^{x^2}.2nx^{2n-1}\,dx\right)$$

$$= \frac{1}{2}\left(1^{2n}e^1 - 0\right) - n\int_0^1 x^{2n-1}e^{x^2}\,dx$$

$$= \frac{e}{2} - nI_{n-1}.$$

(ii)

$$I_2 = \frac{e}{2} - 2I_1$$

$$I_1 = \frac{e}{2} - I_0$$

$$I_0 = \int_0^1 xe^{x^2}\,dx$$

$$= \left[\frac{e^{x^2}}{2}\right]_0^1$$

$$= \frac{e}{2} - \frac{1}{2}$$

$$= \frac{1}{2}(e-1)$$

$$\therefore I_1 = \frac{e}{2} - \frac{1}{2}(e-1) = \frac{1}{2}$$

$$\therefore I_2 = \frac{e}{2} - 2.\frac{1}{2} = \frac{e}{2} - 1.$$

(c) (i)

$$f(x) = \frac{e^x - e^{-x}}{2} - x$$

$$f'(x) = \frac{e^x + e^{-x}}{2} - 1$$

$$f''(x) = \frac{e^x - e^{-x}}{2}$$

For $x > 0$, $e^x > 1$ and $0 < e^{-x} < 1$

$$\therefore f''(x) = \frac{e^x - e^{-x}}{2} > 0.$$

(ii) *METHOD 1*

$$f'(x) = \frac{e^x + e^{-x}}{2} - 1$$

$$= \frac{e^x + e^{-x} - 2}{2}$$

$$= \frac{e^{2x} - 2e^x + 1}{2e^x} \quad (e^x > 0)$$

$$= \frac{\left(e^x - 1\right)^2}{2e^x}$$

> 0 for $x > 0$.

METHOD 2

$$f'(0) = \frac{e^0 + e^0}{2} - 1$$

$$= 0$$

From (i), $f''(x) > 0$

$\therefore f'(x)$ is increasing.

$\therefore f'(x) > 0$ for $x > 0$.

(iii)

$$f(0) = \frac{e^0 - e^0}{2} - 0$$

$$= 0$$

From (ii), $f'(x) > 0$ for $x > 0$

$\therefore f(x)$ is increasing.

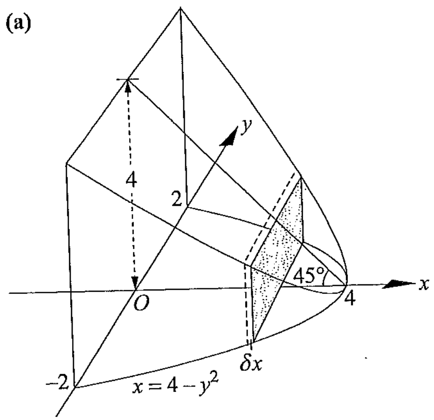
$\therefore f(x) > 0$ for $x > 0$,

i.e. $\dfrac{e^x - e^{-x}}{2} - x > 0$

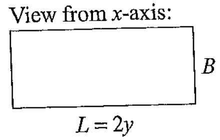
$\dfrac{e^x - e^{-x}}{2} > x$ for $x > 0$.

Question 6

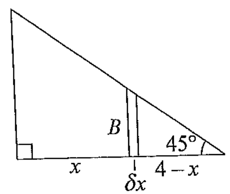
(a)



View from x-axis:



View from y-axis:



Now, $\delta V = LB\,\delta x$

$\doteq 2y(4-x)\,\delta x$

Since $x = 4 - y^2$,

$y = \sqrt{4-x} \quad (y > 0)$

$$\therefore \delta V = 2\sqrt{4-x}\,(4-x)\,\delta x$$

$$= 2(4-x)^{\frac{3}{2}}\,\delta x$$

$$\therefore V = \lim_{\delta x \to 0}\sum_{x=0}^{4} 2(4-x)^{\frac{3}{2}}\,\delta x$$

$$= 2\int_0^4 (4-x)^{\frac{3}{2}}\,dx$$

$$= 2\left[-\frac{2}{5}(4-x)^{\frac{5}{2}}\right]_0^4$$

$$= -\frac{4}{5}\left[(4-x)^{\frac{5}{2}}\right]_0^4$$

$$= -\frac{4}{5}(0) + \frac{4}{5}(4)^{\frac{5}{2}}$$

$$= \frac{128}{5}$$

$\therefore$ Volume $= \dfrac{128}{5}$ units^3.

(b) $P(x) = x^3 + qx^2 + qx + 1$

(i) Let the roots be -1, α, β.

Product $= -\dfrac{d}{a}$

$-1.\alpha.\beta = -1$

$\therefore\ \beta = \dfrac{1}{\alpha}$

$\therefore$ If α is a zero of $P(x)$, so is $\dfrac{1}{\alpha}$.

(ii) (1) If q is real and α complex, then $\bar{\alpha}$ is also a root.

i.e. $\dfrac{1}{\alpha} = \bar{\alpha}$

$1 = \bar{\alpha}\alpha$

$\therefore |\alpha|^2 = 1$

$\therefore |\alpha| = 1.$

(2) Sum of roots $= -\dfrac{b}{a}$

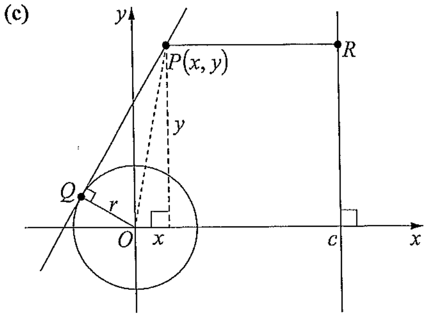
$\therefore\ -1 + \alpha + \bar{\alpha} = -q$

$\alpha + \bar{\alpha} = 1 - q$

$\therefore 2\operatorname{Re}(\alpha) = 1 - q$

$\therefore \operatorname{Re}(\alpha) = \dfrac{1-q}{2}.$

(c)



(i) $OP^2 = x^2 + y^2$ by Pythagoras

and $PQ^2 = OP^2 - OQ^2$ by Pythagoras

$= x^2 + y^2 - r^2$

$\therefore PQ = \sqrt{x^2 + y^2 - r^2}$ $(PQ > 0)$.

(ii) R is (c, y).

$\therefore\ PR = c - x$

If $PQ = PR$,

then $PQ^2 = PR^2$

$\therefore x^2 + y^2 - r^2 = c^2 - 2cx + x^2$

$\therefore y^2 = r^2 + c^2 - 2cx$

which is the locus of P.

(iii) Write $y^2 = r^2 + c^2 - 2cx$ in the form

$(y-k)^2 = -4a(x-h)$

$$\therefore y^2 = -2c\left[x - \left(\frac{c}{2} + \frac{r^2}{2c}\right)\right]$$

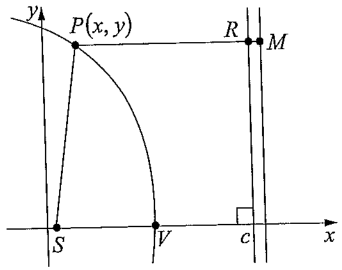
$\therefore$ Vertex is $\left(\dfrac{c}{2} + \dfrac{r^2}{2c}, 0\right)$

Now, $4a = -2c$

$a = -\dfrac{c}{2}$

$\therefore$ Focus, S, is $\left(\dfrac{r^2}{2c}, 0\right)$.

(iv)



Directrix is $x = c + \dfrac{r^2}{2c}$

$PS = PM$

(definition of parabola)

$= c + \dfrac{r^2}{2c} - x$

Now, $PQ = PR$ (from (ii))

$= c - x$

$\therefore PS - PQ = c + \dfrac{r^2}{2c} - x - (c - x)$

$= \dfrac{r^2}{2c}$

which is independent of x.

Question 7

(a) (i) (1) $\ddot{x} = g - rv$

$v\dfrac{dv}{dx} = g - rv$

$\dfrac{dx}{dv} = \dfrac{v}{g - rv}$

$$\therefore \int dx = \int \frac{v}{g - rv}\, dv$$

$$= \frac{1}{r}\int\left(\frac{rv - g + g}{g - rv}\right) dv$$

$$= \frac{1}{r}\int\left(\frac{g}{g - rv} - \left[\frac{g - rv}{g - rv}\right]\right) dv$$

$$= \frac{1}{r}\int\left(\frac{g}{g - rv} - 1\right) dv$$

$$= \frac{1}{r}\int\left(\frac{g}{-r}\cdot\frac{-r}{g - rv} - 1\right) dv$$

$$\therefore x = \frac{1}{r}\left(-\frac{g}{r}\ln(g - rv) - v\right) + c$$

When $x = 0$, $v = 0$

$\therefore\ c = \dfrac{g}{r^2}\ln g$

$\therefore\ x = -\dfrac{g}{r^2}\ln(g - rv) - \dfrac{v}{r} + \dfrac{g}{r^2}\ln g$

$= \dfrac{g}{r^2}\ln\left(\dfrac{g}{g - rv}\right) - \dfrac{v}{r}.$

(2) Given $g = 9.8$, $r = 0.2$

Let $v = 30$, $x = L$

From (1),

$$L = \frac{9.8}{(0.2)^2}\ln\left(\frac{9.8}{9.8 - 0.2 \times 30}\right) - \frac{30}{0.2}$$

$= 82$ m (to 2 s.f.).

(ii) $x = e^{-\frac{t}{10}}(29\sin t - 10\cos t) + 92$ ①

$x' = e^{-\frac{t}{10}}(29\cos t + 10\sin t)$

$+ (29\sin t - 10\cos t)\left(-\dfrac{1}{10}e^{-\frac{t}{10}}\right)$

$= e^{-\frac{t}{10}}(30\cos t + 7.1\sin t)$

Maximum x will occur the first time $x' = 0$,

i.e. $e^{-\frac{t}{10}}(30\cos t + 7.1\sin t) = 0$

$\therefore 30\cos t + 7.1\sin t = 0$

$\tan t = -\dfrac{300}{71}$

$t = \tan^{-1}\left(-\dfrac{300}{71}\right)$

$\doteqdot 1.8$ seconds

Substitute into ①.

$x = e^{-\frac{1.8}{10}}(29\sin 1.8 - 10\cos 1.8) + 92$

$\doteqdot 117.5$ m

$\therefore$ Distance from bridge to top of head is 119.5 m.

$\therefore$ The jumper's head will stay out of the water.

(b) $z = \cos\theta + i\sin\theta$

(i) $z^n = \cos n\theta + i\sin n\theta$

(by DeMoivre's Theorem) ①

$z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$

(by DeMoivre's Theorem)

$= \cos n\theta - i\sin n\theta$ ②

① + ②:

$z^n + z^{-n} = 2\cos n\theta, \quad n > 0.$

(ii) $(2\cos\theta)^{2m}$

$$= \left(z + z^{-1}\right)^{2m}$$

$$= \binom{2m}{0} z^{2m} + \binom{2m}{1} z^{2m-1} z^{-1} + \ldots + \binom{2m}{2m-1} z^1 z^{1-2m} + \binom{2m}{2m} z^{-2m}$$

$$= \left(z^{2m} + z^{-2m}\right) + \binom{2m}{1}\left(z^{2m-2} + z^{2-2m}\right) + \ldots + \binom{2m}{m-1}\left(z + z^{-1}\right) + \binom{2m}{m}$$

$$= 2\left[\cos 2m\theta + \binom{2m}{1}\cos(2m-2)\theta + \binom{2m}{2}\cos(2m-4)\theta + \ldots + \binom{2m}{m-1}\cos 2\theta\right] + \binom{2m}{m}.$$

(iii) From (ii),

$$\cos^{2m}\theta = \frac{1}{2^{2m}}\left\{2\left[\cos 2m\theta + \binom{2m}{1}\cos(2m-2)\theta + \ldots + \binom{2m}{m-1}\cos 2\theta\right] + \binom{2m}{m}\right\}$$

Now, $\displaystyle\int_0^{\frac{\pi}{2}} \cos^{2m}\theta \, d\theta$

$$= \frac{1}{2^{2m}}\int_0^{\frac{\pi}{2}} \left\{2\left[\cos 2m\theta + \binom{2m}{1}\cos(2m-2)\theta + \ldots + \binom{2m}{m-1}\cos 2\theta\right] + \binom{2m}{m}\right\} d\theta$$

$$= \frac{1}{2^{2m}}\left[2\left[\frac{1}{2m}\sin 2m\theta + \frac{1}{2m-2}\cdot\binom{2m}{1}\sin(2m-2)\theta + \ldots + \frac{1}{2}\binom{2m}{m-1}\sin 2\theta\right] + \binom{2m}{m}\theta\right]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2^{2m}}\left[2(0+0+\ldots+0) + \binom{2m}{m}\cdot\frac{\pi}{2}\right] - \frac{1}{2^{2m}}\left[2(0+0+\ldots+0) + \binom{2m}{m}.0\right]$$

$$= \frac{\pi}{2^{2m+1}}\cdot\binom{2m}{m}.$$

Question 8

(a) (i) If $t = \tan\dfrac{\theta}{2}$,

$$\tan\theta = \frac{2t}{1-t^2}$$

$$\therefore \cot\theta + \frac{1}{2}\tan\frac{\theta}{2} = \frac{1-t^2}{2t} + \frac{t}{2}$$

$$= \frac{1-t^2+t^2}{2t}$$

$$= \frac{1}{2t}$$

$$= \frac{1}{2}\cot\frac{\theta}{2}.$$

(ii) Step 1: Prove true for $n = 1$.

$\text{LHS} = \dfrac{1}{2^0}\tan\dfrac{x}{2} = \tan\dfrac{x}{2}$

$\text{RHS} = \dfrac{1}{2^0}\cot\dfrac{x}{2} - 2\cot x$

$= \cot\dfrac{x}{2} - 2\cot x$

$= 2\left(\cot x + \dfrac{1}{2}\tan\dfrac{x}{2}\right) - 2\cot x$ (from (i))

$= 2\cot x + \tan\dfrac{x}{2} - 2\cot x$

$= \tan\dfrac{x}{2}$

$= \text{LHS}$

$\therefore$ Result is true for $n = 1$.

Step 2: Let k be a value up to which the result is true, i.e.

$$\sum_{r=1}^{k}\frac{1}{2^{r-1}}\tan\frac{x}{2^r} = \frac{1}{2^{k-1}}\cot\frac{x}{2^k} - 2\cot x$$

Step 3: Prove true for $n = k+1$,
i.e. required to prove:

$$\sum_{r=1}^{k+1}\frac{1}{2^{r-1}}\tan\frac{x}{2^r} = \frac{1}{2^k}\cot\frac{x}{2^{k+1}} - 2\cot x$$

$\text{LHS} = \dfrac{1}{2^{k-1}}\cot\dfrac{x}{2^k} - 2\cot x + \dfrac{1}{2^k}\tan\dfrac{x}{2^{k+1}}$

Let $\dfrac{x}{2^k} = \theta$, then

$\text{LHS} = \dfrac{1}{2^{k-1}}\cot\theta - 2\cot x + \dfrac{1}{2^k}\tan\dfrac{\theta}{2}$

$= \dfrac{1}{2^{k-1}}\left(\cot\theta + \dfrac{1}{2}\tan\dfrac{\theta}{2}\right) - 2\cot x$

$= \dfrac{1}{2^{k-1}}\cdot\dfrac{1}{2}\cot\dfrac{\theta}{2} - 2\cot x$ (from (i))

$= \dfrac{1}{2^k}\cot\dfrac{\theta}{2} - 2\cot x$

$= \dfrac{1}{2^k}\cot\dfrac{x}{2^{k+1}} - 2\cot x$

$= \text{RHS}$

$\therefore$ If the result is true for $n = k$, it is true for $n = k+1$. If the result is true for $n = 1$, then by the principle of mathematical induction it is true for all integers $n \geq 1$.

(iii) $\displaystyle\sum_{r=1}^{n}\frac{1}{2^{r-1}}\tan\frac{x}{2^r} = \frac{1}{2^{n-1}}\cot\frac{x}{2^n} - 2\cot x$

Let $\dfrac{x}{2^n} = \theta$, then

$\dfrac{1}{2^{n-1}}\cot\dfrac{x}{2^n} = \dfrac{2\theta}{x}\cot\theta$

$= \dfrac{2}{x}\cdot\dfrac{\theta}{\tan\theta}$

As $n \to \infty$, $\dfrac{x}{2^n} \to 0$, i.e. $\theta \to 0$

Since $\displaystyle\lim_{\theta\to 0}\left(\frac{\theta}{\tan\theta}\right) = 1$,

$\dfrac{1}{2^{n-1}}\cot\dfrac{x}{2^n} \to \dfrac{2}{x}$ as $n \to \infty$

$$\therefore \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{2^{r-1}}\tan\frac{x}{2^r} = \frac{2}{x} - 2\cot x.$$

(iv) If $x = \dfrac{\pi}{2}$,

$\dfrac{1}{2^0}\tan\dfrac{1}{2}\cdot\dfrac{\pi}{2} + \dfrac{1}{2^1}\tan\dfrac{1}{4}\cdot\dfrac{\pi}{2} + \dfrac{1}{2^2}\tan\dfrac{1}{8}\cdot\dfrac{\pi}{2}\ldots$

$= \dfrac{2}{\left(\dfrac{\pi}{2}\right)} - 2\cot\dfrac{\pi}{2}$ (from (iii))

$= \dfrac{4}{\pi} - 0$

$\therefore \tan\dfrac{\pi}{4} + \dfrac{1}{2}\tan\dfrac{\pi}{8} + \dfrac{1}{4}\tan\dfrac{\pi}{16} + \ldots$

$= \dfrac{4}{\pi}.$

(b) Area of inner rectangle $= \dfrac{1}{n}$

Area of outer rectangle $= \dfrac{1}{n-1}$

$$\therefore \frac{1}{n} < \int_{n-1}^{n} \frac{dx}{x} < \frac{1}{n-1}$$

$$\frac{1}{n} < [\ln x]_{n-1}^{n} < \frac{1}{n-1}$$

$$\frac{1}{n} < \ln\left(\frac{n}{n-1}\right) < \frac{1}{n-1}$$

$$e^{\frac{1}{n}} < \frac{n}{n-1} < e^{\frac{1}{n-1}}$$

$$e < \left(\frac{n}{n-1}\right)^n < e^{\frac{n}{n-1}}$$

$$e^{-\frac{n}{n-1}} < \left(\frac{n-1}{n}\right)^n < e^{-1}$$

$$\therefore e^{-\frac{n}{n-1}} < \left(1-\frac{1}{n}\right)^n < e^{-1}.$$

(c) (i) $P(A_1 \text{ wins 1st round}) = p$

For A to win a second time, there must be n unsuccessful draws.

$\therefore P(A_1 \text{ wins 2nd round}) = q^n p$

$P(A_1 \text{ wins 3rd round}) = q^{2n} p$, etc.

$\therefore P(A_1 \text{ wins}) = p + q^n p + q^{2n} p + \ldots$

$$\therefore W = \frac{p}{1-q^n}$$

$\therefore W - Wq^n = p$

$\therefore W = p + q^n W.$

(ii) $W_m = p + q^n p + q^{2n} p + \ldots + q^{(m-1)n} p$

$$= p\frac{\left(1-q^{nm}\right)}{1-q^n}$$

$$\therefore \frac{W_m}{W} = \frac{p\left(1-q^{nm}\right)}{1-q^n}\cdot\frac{1-q^n}{p}$$

$$= 1-q^{nm}$$

where $q = 1-\dfrac{1}{n}$

$$\therefore \frac{W_m}{W} = 1-\left(1-\frac{1}{n}\right)^{nm}$$

and from part (b), as $n \to \infty$

$$\left(1-\frac{1}{n}\right)^n \to e^{-1}$$

$$\therefore \frac{W_m}{W} \doteqdot 1-\left(e^{-1}\right)^m$$

$$= 1-e^{-m}.$$

End Mathematics Extension 2 solutions