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HSC Mathematics Extension 1

Practice Paper 10

Question 1

(a) Differentiate $\sin^{-1}(x^2)$.

(b) Find, to the nearest degree, the acute angle formed by the lines $5x - y + 1 = 0$ and $x - 3y - 2 = 0$.

(c) If $\int_0^t \frac{1}{1+x^2}\,dx = 0{\cdot}9$, find t correct to two decimal places.

(d) Prove that, if $x^4 - x^3 + kx - 4$ has a factor of $(x+1)$, then it also has a factor of $(x-2)$.

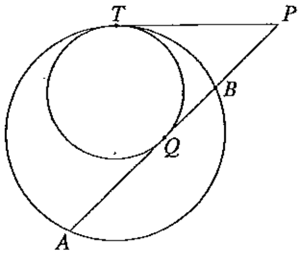
(e) Put $u = 1 - 2x$ to evaluate $\int_0^{\frac{1}{2}} 2x\sqrt{1-2x}\,dx$.

(f) Solve $\frac{x+6}{x^2} > 1$ and graph the solution on a number line.

Question 2

(a) Prove that $\frac{2}{\cot x + \tan x} = \sin 2x$.

(b)



PT is the common tangent to the circles which touch at T.
PA is the tangent to the smaller circle at Q.

(i) Explain why $PT^2 = PA \cdot PB$.

(ii) If $PT = t$, $QA = a$ and $QB = b$, prove that $t = \frac{ab}{a-b}$.

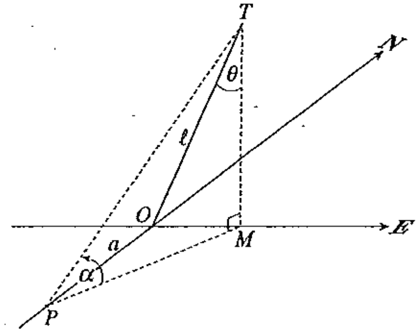
(c) Four couples sit at a round table for a séance. How many different seating arrangements are possible if

(i) there are no restrictions?

(ii) each person sits next to their partner?

Question 3

(a)



A pole, OT, of length ℓ m, stands on horizontal ground. The pole leans towards the east, making an angle of θ with the vertical.

From P, a m south of O, the elevation of T is α.

(i) Find expressions, in terms of ℓ and θ, for OM and MT.

(ii) Prove that $PM = \ell\cos\theta\cot\alpha$.

(iii) Prove that $\ell^2 = \frac{a^2}{\cos^2\theta\cot^2\alpha - \sin^2\theta}$.

(iv) Find the length of the pole, to the nearest m, if
$a = 25,\ \theta = 20°$ and $\alpha = 24°$.

(b) Two particles move on a number plane, A on the x axis and B on the y axis. They begin simultaneously and move so that, at time t, their positions are
$x = \sqrt{3}\cos t$ (for A) and $y = \sin t$ (for B).

(i) Show that, when their positions are the same on their respective axes, so also will be their accelerations.

(ii) Find, in general, the times at which they will have the same velocity, and their positions when this occurs.

(iii) At time t, the distance between the particles is z. Express z^2 as a function of either $\cos t$ or $\sin t$ (not both) and hence

1. find the times, $0 < t < 2\pi$, for which $z = \sqrt{2}$

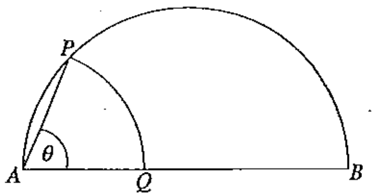
2. prove that the particles will never collide.

Question 4

(a) (i) Find the domain of the function $\sin^{-1}(1-x)$.

(ii) Sketch a graph of $2y = \sin^{-1}(1-x)$, indicating the scales on both axes.

(b)



AB is the diameter of a semicircle, radius r, $\angle PAB = \theta$ radians and PQ is a circular arc, centre A.

(i) Prove that $AP = 2r\cos\theta$. (Join PB.)

(ii) Prove that, if θ is variable, the arc PQ will have maximum length when $\theta\tan\theta = 1$.

(iii) Taking 1 as a first approximation to θ, use Newton's method to find an approximation to one decimal place.

(c) It is suspected that one of the functions

$$f(n) = \frac{n}{3}(n+1)(4n-1)$$

$$\text{or} \quad g(n) = \frac{n}{3}(n+2)(3n-1)$$

will sum the series $1\cdot2+3\cdot4+5\cdot6+\cdots+(2n-1)(2n)$ for all positive integers n.

State, with justification, which is the only one that could do so, and investigate, by induction, whether it does or not.

Question 5

(a) A particle is projected from P, on horizontal ground with speed V m s^{-1} at an elevation of θ radians, where $\theta < \frac{\pi}{4}$.

(i) Beginning with relevant accelerations, find expressions for its horizontal and vertical displacements from P after t seconds.

(ii) Prove that the time of flight, T s, is given by

$$T = \frac{2V\sin\theta}{g}.$$

(iii) It is found that, by increasing the projection angle by $\frac{\pi}{4}$ radians, the time of flight is doubled. Find θ correct to 2 decimal places.

(b) Box A contains five balls, two of which are red.

Box B contains five balls, one of which is red.

Players draw, at random, three balls from each box and win a prize if they draw the same number of red balls from both boxes (including zero).

(i) Show that the probability of winning a prize is 0·4.

(ii) If Peter plays the game four times, find the probability he wins

1. exactly one prize
2. more than one prize.

Question 6

(a) (i) Prove that $\int_0^{\frac{\pi}{4}} \sin^2 x\,dx = \frac{\pi}{8} - \frac{1}{4}$.

(ii) Prove that $\frac{d}{dx}\left(x\sin^2 x\right) - \sin^2 x = x\sin 2x$.

(iii) Hence, or otherwise, prove $\int_0^{\frac{\pi}{4}} x\sin 2x\,dx = \frac{1}{4}$.

(b) A particle moves in a straight line so that, when x m from an origin, its acceleration is $-9e^{-2x}$ m s^{-2}. Initially, it is at the origin where it is given a velocity of 3 m s^{-1}.

(i) Determine its velocity as a function of x, justifying any choice you may have to make.

(ii) Determine x as a function of t, where t is the number of seconds since it left the origin.

(iii) Find the particles velocity and acceleration 3 seconds after leaving the origin.

Question 7

(a) When $(1+ax)^5+(1+bx)^5$ is expanded in ascending powers of x, the expansion begins

$$2+30x+220x^2+\cdots$$

(i) Show that $a+b=6$ and $a^2+b^2=22$.

(ii) Deduce the value of ab.

(iii) Find the coefficient of x^3.

(b) $f(x)=\sqrt{4-\sqrt{x}}$

(i) Explain why the domain of $f(x)$ is $0\le x\le 16$.

(ii) Prove that $f(x)$ is a decreasing function and find its range.

(iii) Since $f(x)$ is monotonic, an inverse function exists. What is the domain and range of $f^{-1}(x)$?

(iv) Find $f^{-1}(x)$.

(v) By considering the graphs of $y=f(x)$ and $y=f^{-1}(x)$, prove that

$$\int_0^{16}\sqrt{4-\sqrt{x}}\ dx=17\tfrac{1}{15}.$$

Practice Paper 10

Question 1

(a) $f'(x)=\dfrac{2x}{\sqrt{1-x^4}}$

(b) $m_1=5,\ m_2=\frac{1}{3}$:

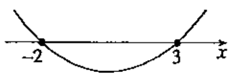
$$\tan\theta=\left|\frac{5-\frac{1}{3}}{1+\frac{5}{3}}\right|$$
$$\theta\doteqdot 60^\circ$$

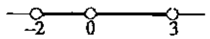
(c)
$$\int_0^t\frac{1}{1+x^2}\,dx=\left[\tan^{-1}x\right]_0^t=\tan^{-1}t$$
$$\tan^{-1}t=0\cdot9$$
$$t=\tan 0\cdot9\doteqdot 1\cdot26$$

(d)
$$(-1)^4-(-1)^3-k-4=0$$
$$k=-2$$
$$2^4-2^3-2(2)-4=16-8-4-4=0$$
$\therefore (x-2)$ is a factor

(e)
$$u=1-2x$$
$$du=-2\,dx$$
$$\int_0^{\frac{1}{2}}2x(1-2x)^{\frac{1}{2}}\,dx=-\frac{1}{2}\int_1^0(1-u)u^{\frac{1}{2}}\,du$$
$$=\frac{1}{2}\int_0^1\left(u^{\frac{1}{2}}-u^{\frac{3}{2}}\right)du$$
$$=\frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}}-\frac{2}{5}u^{\frac{5}{2}}\right]_0^1$$
$$=\frac{1}{2}\left(\frac{2}{3}-\frac{2}{5}\right)$$
$$=\frac{2}{15}$$

(f)
$$\frac{x+6}{x^2}>1$$
$$x+6>x^2\quad(x\ne0)$$
$$x^2-x-6<0$$
$$(x-3)(x+2)<0$$
$$-2<x<3,\ \ x\ne0$$





Question 2

(a)
$$\frac{2}{\cot x+\tan x}=\frac{2}{\frac{\cos x}{\sin x}+\frac{\sin x}{\cos x}}=\frac{2\sin x\cos x}{\cos^2x+\sin^2x}=\sin 2x$$

(b) (i) The square on the tangent is equal to the rectangle contained by the segments of the chord.

(ii) $PT=PQ$ (tangs. to a circle)
$\therefore\ PQ^2=PA.PB$ (part (i))
i.e. $t^2=(t+a)(t-b)$
$t^2=t^2+t(a-b)-ab$
$t=\dfrac{ab}{a-b}$

(c) (i) $7!=5040$

(ii) The couples may be arranged 3! ways.
Each couple can be arranged 2! ways.
Total $=3!\times(2!)^4$
$=96$

Question 3

(a) (i) $\dfrac{OM}{\ell}=\sin\theta;\quad \dfrac{MT}{\ell}=\cos\theta$
$OM=\ell\sin\theta;\quad MT=\ell\cos\theta$

(ii) $\dfrac{PM}{MT}=\cot\alpha$
$PM=MT\cot\alpha$
$=\ell\cos\theta\cot\alpha$

(iii) $PM^2-OM^2=a^2$ (Pythag.)
$$\ell^2\cos^2\theta\cot^2\alpha-\ell^2\sin^2\theta=a^2$$
$$\ell^2\left(\cos^2\theta\cot^2\alpha-\sin^2\theta\right)=a^2$$
$$\therefore\ \ell^2=\frac{a^2}{\cos^2\theta\cot^2\alpha-\sin^2\theta}$$

(iv)
$$\ell^2=\frac{25^2}{\cos^2 20^\circ\cot^2 24^\circ-\sin^2 20^\circ}$$
$$\ell\doteqdot 12$$

(b) (i) $x=\sqrt{3}\cos t\qquad y=\sin t$
$v_A=-\sqrt{3}\sin t\qquad v_B=\cos t$
$a_A=-\sqrt{3}\cos t\qquad a_B=-\sin t$
$=-x\qquad\qquad =-y$
When $x=y,\ a_A=a_B$

(ii) $-\sqrt{3}\sin t=\cos t$
$$\tan t=-\frac{1}{\sqrt{3}}$$
$$t=n\pi-\frac{\pi}{6}$$
For $n=1,\ t=\frac{5\pi}{6}$,
$x=\sqrt{3}\left(-\frac{\sqrt{3}}{2}\right),\quad y=0\cdot5$
$=-1\cdot5$

For $n=2,\ t=\frac{11\pi}{6}$,
$x=\sqrt{3}\left(\frac{\sqrt{3}}{2}\right),\quad y=-0\cdot5$
$=1\cdot5$
(Further n will repeat the above.)

(iii) $z^2=x^2+y^2$
$=3\cos^2t+\sin^2t$
$=3\cos^2t+1-\cos^2t$
$=2\cos^2t+1$

1. $2\cos^2t+1=2$
$\cos^2t=\frac{1}{2}$
$\cos t=\frac{1}{\sqrt{2}}$ or $\cos t=-\frac{1}{\sqrt{2}}$
$t=\frac{\pi}{4},\ \frac{7\pi}{4}\qquad t=\frac{3\pi}{4},\ \frac{5\pi}{4}$

2. To collide, $z=0$
$\therefore\ 2\cos^2t=-1$
which has no solutions.

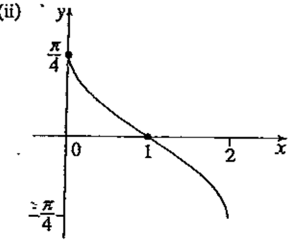
Question 4

(a) (i) $-1 \le 1-x \le 1$

$-2 \le -x \le 0$

$\therefore 2 \ge x \ge 0$

(ii)



(b) (i) $\frac{AP}{AB} = \cos\theta \quad (\angle APB = 90^\circ)$

$AP = 2r\cos\theta$

(ii) $\ell = AP \cdot \theta$

$= 2r\theta\cos\theta$

$\frac{d\ell}{d\theta} = 2r(\cos\theta - \theta\sin\theta)$

When $\cos\theta - \theta\sin\theta = 0$

$1 - \theta\tan\theta = 0$

$\theta\tan\theta = 1$

As $\theta \to 0,\ \ell \to 0$ and as $\theta \to \frac{\pi}{2},\ \ell \to 0$.

Hence stationary point is a maximum.

(iii) $f(\theta) = \theta\tan\theta - 1$

$f'(\theta) = \tan\theta + \theta\sec^2\theta$

Approximation $= 1 - \frac{\tan 1 - 1}{\tan 1 + \sec^2 1}$

$\doteqdot 0\cdot 9$

(c) (i) $f(1) = \frac{1}{3}(2)(3) = 2 \qquad g(1) = \frac{1}{3}(3)(2) = 2$

$f(2) = \frac{2}{3}(3)(7) = 14 \qquad g(2) = \frac{2}{3}(4)(5) \ne 14$

Hence $f(n)$ could do so.

Let k be an integer for which the result is true.

i.e. $1\cdot 2 + \cdots + (2k-1)(2k) = \frac{k}{3}(k+1)(4k-1)$

Then $1\cdot 2 + \cdots + (2k-1)(2k) + (2k+1)(2k+2)$

$= \frac{k}{3}(k+1)(4k-1) + (2k+1)(2k+2)$

$= \frac{k+1}{3}\left[4k^2 - k + 6(2k+1)\right]$

$= \frac{k+1}{3}\left(4k^2 + 11k + 6\right)$

$= \frac{k+1}{3}(k+2)(4k+3)$

$= \frac{k+1}{3}\left[(k+1)+1\right]\left[4(k+1)-1\right]$

and the result is also true for the integer $(k+1)$. Since it is true for the integer 1, it is therefore true for 2 and therefore true for 3, etc.

Question 5

(a) (i) $x = Vt\cos\theta$

$y = Vt\sin\theta - \frac{1}{2}gt^2$ (Bookwork)

(ii) When $y = 0$, $Vt\sin\theta - \frac{1}{2}gt^2 = 0$

$V\sin\theta - \frac{1}{2}gt = 0 \qquad (t \ne 0)$

$T = \frac{2V\sin\theta}{g}$

(iii) $\frac{2V\sin\left(\theta + \frac{\pi}{4}\right)}{g} = \frac{4V\sin\theta}{g}$

$\sin\left(\theta + \frac{\pi}{4}\right) = 2\sin\theta$

$\sin\theta\cos\frac{\pi}{4} + \cos\theta\sin\frac{\pi}{4} = 2\sin\theta$

$\sin\theta\left(\frac{1}{\sqrt{2}}\right) + \cos\theta\left(\frac{1}{\sqrt{2}}\right) = 2\sin\theta$

$\sin\theta + \cos\theta = 2\sqrt{2}\sin\theta$

$\tan\theta + 1 = 2\sqrt{2}\tan\theta$

$\tan\theta\left(2\sqrt{2} - 1\right) = 1$

$\tan\theta = \frac{1}{2\sqrt{2} - 1}$

$\theta \doteqdot 0\cdot 50$

(b) (i) $\boxed{\begin{matrix}2r\\3\end{matrix}} \qquad \boxed{\begin{matrix}1r\\4\end{matrix}}$

${}^5C_3 = 10$

P(no red and no red) $= \frac{{}^3C_3}{10} \times \frac{{}^4C_3}{10} = 0\cdot 04$

P(1 red and 1 red) $= \frac{{}^2C_1 \times {}^3C_2}{10} \times \frac{{}^1C_1 \times {}^4C_2}{10}$

$= 0\cdot 36$

P(same number) $= 0\cdot 04 + 0\cdot 36 = 0\cdot 4$

(ii) 1. Prob. $= {}^4C_1(0\cdot 4)(0\cdot 6)^3 = 0\cdot 3456$

2. P(n0 prize) $= (0\cdot 6)^4$

P(more than one prize)

$= 1 - \left(0\cdot 3456 + 0\cdot 6^4\right)$

$= 0\cdot 5248$

Question 6

(a) (i) $\frac{1}{2}\int_0^{\frac{\pi}{4}}(1 - \cos 2x)\ dx = \frac{1}{2}\left[x - \frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{2}\left[\left(\frac{\pi}{4} - \frac{1}{2}\right) - 0\right]$

$= \frac{\pi}{8} - \frac{1}{4}$

(ii) $\frac{d}{dx}\left(x\sin^2 x\right) = \sin^2 x + x\cdot 2\sin x\cos x$

$= \sin^2 x + x\sin 2x$

(iii) $x\sin 2x = \frac{d}{dx}\left(x\sin^2 x\right) - \sin^2 x$

$\int_0^{\frac{\pi}{4}} x\sin 2x\ dx = \left[x\sin^2 x\right]_0^{\frac{\pi}{4}} - \left(\frac{\pi}{8} - \frac{1}{4}\right)$

$= \frac{\pi}{4}\left(\frac{1}{2}\right) - 0 - \frac{\pi}{8} + \frac{1}{4}$

$= \frac{1}{4}$

(b) (i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -9e^{-2x}$

$\frac{1}{2}v^2 = -9\int e^{-2x}\ dx$

$= \frac{9}{2}e^{-2x} + c$

$v = 3$ at $x = 0$,

$\frac{9}{2} = \frac{9}{2} + c$

$c = 0$

$\therefore \quad v^2 = 9e^{-2x}$

Since $v^2 \ne 0$ for any x, the particle must maintain its original direction of motion, which was positive.

Hence $v = 3e^{-x}$.

(ii) $\frac{dx}{dt} = 3e^{-x}$

$\frac{dt}{dx} = \frac{1}{3}e^x$

$t = \frac{1}{3}\int e^x\ dx$

$= \frac{1}{3}e^x + c$

When $t = 0$, $x = 0$: $\quad 0 = \frac{1}{3} + c$

$t = \frac{1}{3}e^x - \frac{1}{3}$

$\therefore\ e^x = 3t + 1$

$x = \ln(3t+1)$

(iii) $v = \frac{dx}{dt}$

$= \frac{3}{3t+1}$

$= \frac{3}{9+1}$

$= 0\cdot 3\ \text{m s}^{-1}$

$v = 3(3t+1)^{-1}$

$a = \frac{dv}{dt}$

$= -3(3t+1)^{-2}(3)$

$= -\frac{9}{(3t+1)^2}$

$= -\frac{9}{(9+1)^2}$

$= -0\cdot 09\ \text{m s}^{-2}$

Question 7

(a) (i) $(1+ax)^5 + (1+bx)^5$

$= \left(1 + 5ax + 10a^2x^2 + \cdots\right)$

$+ \left(1 + 5bx + 10b^2x^2 + \cdots\right)$

$= 2 + 5(a+b)x + 10\left(a^2+b^2\right)x^2 + \cdots$

$\therefore\ 5(a+b) = 30$ and $10\left(a^2+b^2\right) = 220$

$\therefore\ a + b = 6 \qquad a^2 + b^2 = 22$

(ii) $(a+b)^2 = a^2 + b^2 + 2ab$

$36 = 22 + 2ab$

$ab = 7$

(iii) Coeff. of $x^3 = 10\left(a^3 + b^3\right)$

$= 10(a+b)\left(a^2 + b^2 - ab\right)$

$= 10(6)(22-7)$

$= 900$

(b) (i) For $\sqrt{x}$ to exist, $x \ge 0$.

For $\sqrt{4 - \sqrt{x}}$ to exist, $4 - \sqrt{x} \ge 0$

$4 \ge \sqrt{x}$

$x \le 16$.

$\therefore\ 0 \le x \le 16$

(ii) $f(x) = \left(4 - x^{\frac{1}{2}}\right)^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}\left(4 - x^{\frac{1}{2}}\right)^{-\frac{1}{2}}\left(-\frac{1}{2}x^{-\frac{1}{2}}\right)$

$= -\frac{1}{4\sqrt{x}\sqrt{4-\sqrt{x}}}$

< 0

$f(0) = 2,\ f(16) = 0$

$\therefore\ 0 \le f(x) \le 2$

(iii) Domain: $0 \le x \le 2$

Range: $0 \le f^{-1}(x) \le 16$

(iv) $x = \sqrt{4 - \sqrt{y}}$

$\therefore x^2 = 4 - \sqrt{y} \qquad 0 \le x \le 2$

$\sqrt{y} = 4 - x^2 \qquad 0 \le x \le 2$

$y = \left(4 - x^2\right)^2 \qquad 0 \le x \le 2$

(v) $\int_0^{16}\sqrt{4 - \sqrt{x}}\ dx = \int_0^2\left(4 - x^2\right)^2\ dx$

$= \int_0^2\left(16 - 8x^2 + x^4\right)\ dx$

$= \left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_0^2$

$= \left(32 - \frac{64}{3} + \frac{32}{5}\right)$

$= 17\frac{1}{15}$