HSC Mathematics Extension 1

Practice Paper 4

Question 1

(a) Evaluate $\int_0^a \frac{1}{\sqrt{4a^2 - x^2}}\,dx, \quad a > 0.$

(b) Find the general solution to $\sin 2x = \sqrt{3}\cos 2x$.

(c) Find all the real values of a for which $ax^3 - 8x^2 - 9$ is divisible by $(x - a)$.

(d) If $\tan\theta = 2$, and $0 < \theta < \frac{\pi}{2}$, find the exact value of $\sin\left(\theta + \frac{\pi}{4}\right)$.

(e) The point (0, 4) divides the interval from (a, b) to (b, a) in the ratio 3 : 1. Find the values of a and b.

Question 2

(a) Find, in factorial and index form, the greatest coefficient in the expansion of $(3x + 2)^{100}$.

(b) Use the substitution $u = \sqrt{x}$ to evaluate $\int_0^3 \frac{1}{\sqrt{x}\,(1 + x)}\,dx$.

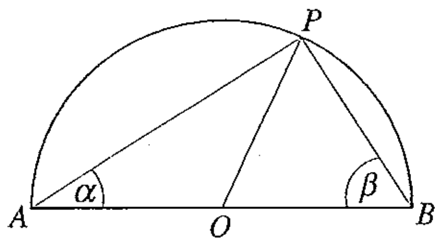
(c) The seven letters of the word GLENELG are arranged at random in a line.

(i) How many different letter sequences are possible?

(ii) What is the probability that the sequence is the same from right to left as from left to right?

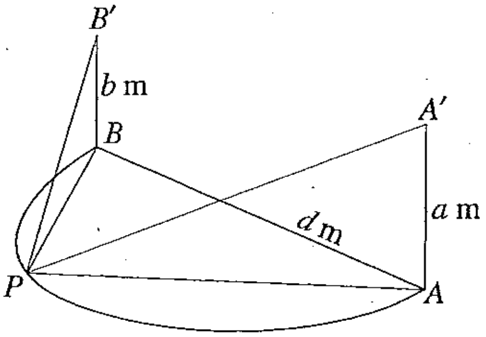
Question 3

(a) (i) AB is the diameter of a semicircle, centre O. Use the diagram to prove that an angle in a semicircle is a right angle.



(ii) APB is a horizontal semicircle, diameter d m.

At A and B are vertical posts of height a m and b m. From P, the angle of elevation of the tops of both posts is θ.



(1) Prove that $d^2 = \dfrac{a^2}{\tan^2\theta} + \dfrac{b^2}{\tan^2\theta}$.

(2) From B, the angle of elevation of A' is α and from A, the angle of elevation of B' is β.

Prove that $\tan^2\alpha + \tan^2\beta = \tan^2\theta$.

(b) A particle, moving in a straight line, is x m from an origin after t seconds, where $x = A + B\cos 2t$, A and B being constants.

(i) Prove that its acceleration is $-4(x-A)$ m s^{-2}.

(ii) Prove that its velocity, v m s^{-1}, is given by $v^2 = 4\left[B^2 - (x-A)^2\right]$.

(iii) At the origin, the acceleration is 12 m s^{-2} and the speed 8 m s^{-1}. Find the values of A and B^2.

(iv) Find the interval of the line on which the particle moves.

(v) What is its maximum speed?

Question 4

(a) (i) Sketch a graph of $y = x^3$ and using it, or otherwise, show that, if the equation $x^3 = mx + 1$ is to have a root between 0 and 1, then

1. m must be negative;
2. there will be no other real root.

(ii) For the case $m = -2$, take 0·5 as a first approximation to the root and use Newton's method to improve this to two decimal places.

(b) In a gambling game, a computer will take, at random, thirteen numbers from the integers from 1 to 40 inclusive. Before this occurs, you are invited to choose two numbers, hoping the computer will 'match' them among its thirteen.

(i) Show that the probability of having both your numbers 'matched' is 0·1.

(ii) Find the probability that the computer will match

1. neither of your numbers;

2. only one of your numbers.

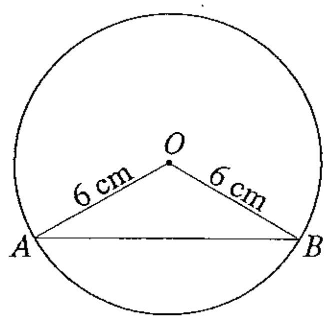
(iii) In a succession of 7 games, find, as decimals correct to 3 places, the probability that both your numbers will be matched

1. exactly once;

2. at least once.

Question 5

(a) O is the centre of a circle, radius 6 cm.
$\angle AOB = \theta$ radians and
θ is increasing at a rate of
0·2 radians/second.



Describe the rate of change of the area of

(i) ΔAOB

(ii) the minor segment formed by AB

when $\angle AOB = 120°$.

(b) (i) Show that $\displaystyle\int_0^{\frac{\pi}{4}} \sin^2 x\, dx = \frac{\pi - 2}{8}$.

(ii) Find $\dfrac{d}{dx}(\sin x - x\cos x)$ and deduce that

$$\int_0^{\frac{\pi}{4}} x\sin x\, dx = \frac{\sqrt{2}(4-\pi)}{8}.$$

(iii) Show that, if $0 < x < \dfrac{\pi}{4}$, $\sin^2 x < x\sin x$.

(iv) Use the above results to prove that $\pi < 2\left(3-\sqrt{2}\right)$.

Question 6

(a) $f(x) = (1+x)(3-x)$.

(i) Sketch a graph of $y = f(x)$, using equal scales on both axes, and showing its stationary point and intersections with the coordinate axes.

(ii) A function $g(x)$ is to be defined by restricting the domain of $f(x)$ so that $g(x)$ has an inverse. Define $g(x)$ so that it has largest possible domain, given that this must include $x = 0$.

(iii) On your number plane in (i), sketch $y = x$ and $y = g^{-1}(x)$.

(iv) Find $g^{-1}(x)$.

(b) A particle is released from rest at an origin on a straight line.

When x m from the origin, its acceleration is $\dfrac{18}{(x-4)^2}$ m s^{-2} for $x < 4$.

(i) In which direction will it first move?

(ii) Find its velocity when it reaches $x = 2$.

Question 7

A rocket is fired from a point O on horizontal ground. Its initial speed, V m s^{-1}, is fixed, but the angle of elevation, θ, may be varied. You may assume that, after t seconds, its horizontal (x m) and vertical (y m) displacements from O are given by

$$x = Vt\cos\theta, \quad y = Vt\sin\theta - \frac{1}{2}gt^2.$$

(a) Prove that the rocket will strike the ground after a time of

$$\frac{2V\sin\theta}{g} \text{ seconds.}$$

(b) The rocket is aimed to strike a target on ground level, R m from O.

Prove that $\sin 2\theta = \dfrac{gR}{V^2}$, and, deduce that, for the rocket to succeed,

$$R \le \frac{V^2}{g}.$$

(c) Assuming $R < \dfrac{V^2}{g}$, using (b), or otherwise, show that there are two successful projection angles and that they are complementary.

(d) For the two projections in (c), let the maximum heights reached be h_1 metres and h_2 metres. Prove that $(h_1 + h_2)$ is independent of R, and equal to the height the rocket would reach if fired vertically.

Practice Paper 4

Question 1

a) $\int_0^a \frac{1}{\sqrt{4a^2-x^2}}\,dx$

$= \int_0^a \frac{1}{\sqrt{(2a)^2-(x)^2}}\,dx$

$= \left[\sin^{-1}\frac{x}{2a}\right]_0^a$

$= \sin^{-1}\frac{a}{2a} - \sin^{-1}0$

$= \sin^{-1}\left(\frac{1}{2}\right)$

$= \frac{\pi}{6}$

b) $\sin(2x) = \sqrt{3}\cos(2x)$

Divide both sides by $\cos(2x)$

$\tan(2x) = \sqrt{3}$.

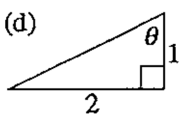
Take inverse tangent of both sides

$2x = \frac{\pi}{3} + \pi n$, where n is an integer

$x = \frac{\pi}{6} + \frac{\pi n}{2}$

(c)
$$a^4-8a^2-9 = 0$$
$$(a^2-9)(a^2+1) = 0$$
$$a^2 = 9 \quad (a^2 \neq -1)$$
$$\therefore\ a = 3 \text{ or } a = -3$$

(d)



$$\sin\left(\theta+\frac{\pi}{4}\right) = \sin\theta\cos\frac{\pi}{4}+\cos\theta\sin\frac{\pi}{4}$$
$$= \frac{2}{\sqrt{5}}\cdot\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{5}}\cdot\frac{1}{\sqrt{2}}$$
$$= \frac{3}{\sqrt{10}}$$

(e) $0 = \frac{a+3b}{3+1}$, $4 = \frac{b+3a}{3+1}$
$$a+3b = 0$$
$$3a+b = 16$$
$$-9b+b = 16$$
$$b = -2$$
$$a = 6$$

Question 2

(a) $\frac{T_{r+1}}{T_r} = \frac{100-r+1}{r}\cdot\frac{2}{3x}$
$$\frac{2(101-r)}{3r} > 1$$
$$202-2r > 3r$$
$$5r < 202$$
$$r = 40$$
$$\text{Coefficient} = {}^{100}C_{40}\,3^{60}.2^{40}$$
$$= \frac{100!\,3^{60}.2^{40}}{60!\,40!}$$

(b) $u = \sqrt{x}$, $\quad x=0,\ u=0$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$, $\quad x=3,\ u=\sqrt{3}$
$$\int_0^3 \frac{1}{\sqrt{x}(1+x)}\,dx = 2\int_0^{\sqrt{3}}\frac{1}{1+u^2}\,du$$
$$= 2\left[\tan^{-1}u\right]_0^{\sqrt{3}}$$
$$= 2\left(\tan^{-1}\sqrt{3}-\tan^{-1}0\right)$$
$$= \frac{2\pi}{3}$$

(c) (i) $\frac{7!}{2!\,2!\,2!} = 630$

(ii) For successful results, . . . N . . .
N must be in the centre.
The remaining pairs can be placed $3\times2\times1$ ways.
Probability $= \frac{6}{630} = \frac{1}{105}$

Question 3

(a) (i)
$$OA = OP \quad \text{(radii)}$$
$$\therefore\ \angle OPA = \alpha \quad \text{(isos. }\Delta)$$
Similarly, $\angle OPB = \beta$
$$2\alpha+2\beta = 180^\circ \quad (\angle \text{ sum } \Delta APB)$$
$$\alpha+\beta = 90^\circ$$
That is, $\angle APB = 90^\circ$

(ii) (1)
$$\frac{a}{PA} = \tan\theta$$
$$PA = \frac{a}{\tan\theta}$$

Similarly, $PB = \frac{b}{\tan\theta}$
$$\therefore\ d^2 = PA^2+PB^2 \quad (\angle APB = 90^\circ)$$
$$= \frac{a^2}{\tan^2\theta}+\frac{b^2}{\tan^2\theta}$$

(2) From (i), $d^2\tan^2\theta = a^2+b^2$
$$\tan^2\theta = \frac{a^2}{d^2}+\frac{b^2}{d^2}$$
$$= \tan^2\alpha+\tan^2\beta$$

(b) (i)
$$x = A+B\cos 2t$$
$$\frac{dx}{dt} = -2B\sin 2t$$
$$\frac{d^2x}{dt^2} = -4(B\cos 2t)$$
$$= -4(x-A)$$

(ii)
$$v^2 = 4B^2\sin^2 2t$$
$$= 4B^2\left(1-\cos^2 2t\right)$$
$$= 4\left(B^2-B^2\cos^2 2t\right)$$
$$= 4\left[B^2-(x-A)^2\right]$$

(iii)
$$12 = -4(0-A)$$
$$A = 3$$
$$64 = 4\left[B^2-(0-3)^2\right]$$
$$B^2 = 25$$

(iv) When $v=0$, $4\left[25-(x-3)^2\right]=0$

$$(x-3)^2 = 25$$

$\therefore$ $x-3=-5$ or $x-3=5$

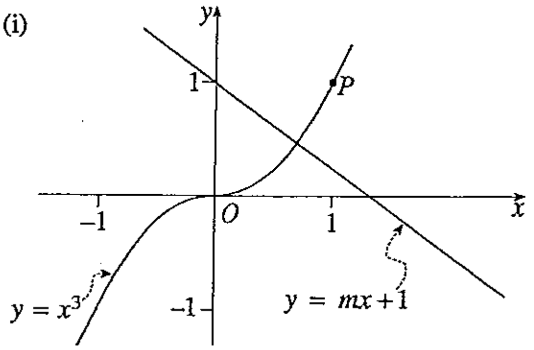
$x=-2$ or $x=8$

Hence $-2 \le x \le 8$.

(v) Maximum speed $= 10\text{ m s}^{-1}$ at $x=3$.

Question 4

(a) (i)



In order that the line $y = mx+1$ should meet $y = x^3$ between O and P,

(1) its gradient must be negative, and

(2) it cannot meet the curve again.

(ii)

$$x^3+2x-1 = 0$$
$$f(0\cdot5) = 0\cdot125$$
$$f'(x) = 3x^2+2$$
$$f'(0\cdot5) = 2\cdot75$$

Approximation $= 0\cdot5 - \dfrac{0\cdot125}{2\cdot75} = 0\cdot45$.

(b) (i) Number of pairs you can choose $= {}^{40}C_2$.

Number of successful pairs you can choose $= {}^{13}C_2$.

Probability $= \dfrac{{}^{13}C_2}{{}^{40}C_2} = 0\cdot1$

(ii) (1) Probability $= \dfrac{{}^{27}C_2}{{}^{40}C_2} = 0\cdot45$

(2) Probability $= 1-(0\cdot1+0\cdot45)$
$= 0\cdot45$

(iii) (1) Probability $= {}^{7}C_1(0\cdot1)(0\cdot9)^6$
$= 0\cdot372$

(2) Probability $= 1-(0\cdot9)^7$
$= 0\cdot521$

Question 5

(a) $\dfrac{d\theta}{dt} = 0\cdot2$

(i)

$$A = \frac{6^2 \sin\theta}{2} = 18\sin\theta$$

$$\frac{dA}{dt} = \frac{dA}{d\theta}\cdot\frac{d\theta}{dt} = (18\cos\theta)(0\cdot2)$$
$$= 3\cdot6\cos\theta$$
$$= 3\cdot6\left(-\tfrac{1}{2}\right)$$
$$= -1\cdot8$$

That is, A is decreasing at $1\cdot8\text{ cm}^2\text{ s}^{-1}$.

(ii)

$$A = \frac{36}{2}(\theta-\sin\theta) = 18(\theta-\sin\theta)$$
$$\frac{dA}{dt} = 18(1-\cos\theta)(0\cdot2)$$
$$= 3\cdot6\left(1+\tfrac{1}{2}\right)$$
$$= 5\cdot4$$

That is, A is increasing at $5\cdot4\text{ cm}^2\text{ s}^{-1}$.

(b) (i)

$$\int_0^{\frac{\pi}{4}} \sin^2 x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{4}}(1-\cos 2x)\,dx$$
$$= \frac{1}{2}\left[x-\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{2}\left(\frac{\pi}{4}-\frac{1}{2}-0\right)$$
$$= \frac{\pi-2}{8}$$

(ii)

$$\frac{d}{dx}(\sin x - x\cos x) = \cos x - (\cos x - x\sin x)$$
$$= x\sin x$$

$$\int_0^{\frac{\pi}{4}} x\sin x\,dx = \left[\sin x - x\cos x\right]_0^{\frac{\pi}{4}}$$
$$= \frac{1}{\sqrt{2}} - \frac{\pi}{4}\cdot\frac{1}{\sqrt{2}} - 0$$
$$= \frac{4-\pi}{4\sqrt{2}}$$
$$= \frac{\sqrt{2}(4-\pi)}{8}$$

(iii) For $0 < x < \dfrac{\pi}{4}$, $\sin x < x$.

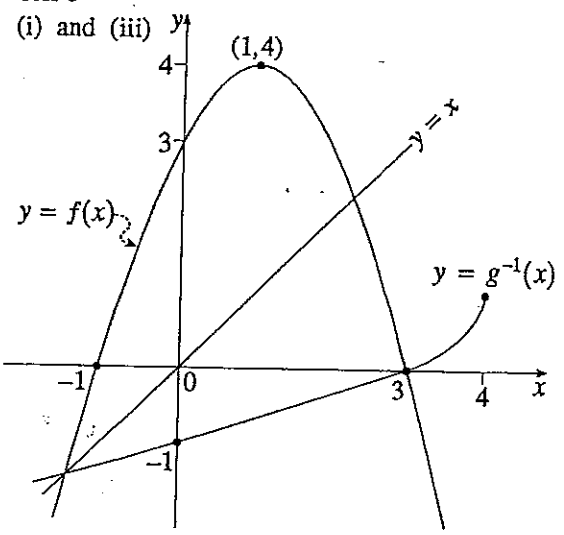
$\therefore$ $\sin^2 x < x\sin x$ $(\sin x > 0)$

$$\int_0^{\frac{\pi}{4}} \sin^2 x\,dx < \int_0^{\frac{\pi}{4}} x\sin x\,dx$$
$$\frac{\pi-2}{8} < \frac{\sqrt{2}(4-\pi)}{8}$$

$$\pi-2 < 4\sqrt{2}-\pi\sqrt{2}$$
$$\pi\sqrt{2}+\pi < 4\sqrt{2}+2$$
$$\pi\left(\sqrt{2}+1\right) < 2\left(2\sqrt{2}+1\right)$$
$$\pi < \frac{2\left(2\sqrt{2}+1\right)\left(\sqrt{2}-1\right)}{\left(\sqrt{2}+1\right)\left(\sqrt{2}-1\right)}$$
$$\pi < 2\left(3-\sqrt{2}\right)$$

Question 6

(a) (i) and (iii)



(ii) $g(x) = (1+x)(3-x)$, $\quad x \le 1$

(iv)

$$x = (1+y)(3-y), \quad y \le 1$$
$$x = 3+2y-y^2, \quad y \le 1$$
$$y^2-2y+1 = 3-x+1, \quad y \le 1$$
$$(y-1)^2 = 4-x, \quad y \le 1$$

$\therefore$ $y-1 = -\sqrt{4-x}$, since $y-1 \le 0$

$$g^{-1}(x) = 1-\sqrt{4-x}$$

(b) (i) The positive direction, since it is initially at rest and the acceleration (and hence the force) is positive.

(ii)

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 18(x-4)^{-2}$$

$$\frac{1}{2}v^2 = -18(x-4)^{-1} + c$$

At $x = 0$, $v = 0$

$$0 = \frac{9}{2} + c$$

$$\frac{1}{2}v^2 = -\frac{18}{x-4} - \frac{9}{2}$$

$$v^2 = -\frac{36}{x-4} - 9$$

At $x = 2$, $v^2 = 9$

$\therefore$ $v = 3 \text{ m s}^{-1}$ $(v > 0)$

Question 7

(a) Bookwork

(b) When $t = \dfrac{2V\sin\theta}{g}$,

$$x = V\cos\theta \,.\, \frac{2V\sin\theta}{g}$$

That is, $R = \dfrac{V^2 \sin 2\theta}{g}$

$$\sin 2\theta = \frac{gR}{V^2}$$

Since $\sin 2\theta \le 1$, $\dfrac{gR}{V^2} \le 1$

$$R \le \frac{V^2}{g}$$

(c)

$$2\theta = \sin^{-1}\left(\frac{gR}{V^2}\right)$$

$$= \alpha, \text{ say,} \qquad \left(0 < \alpha < \frac{\pi}{2}\right)$$

or $2\theta = \pi - \alpha$

That is, $\theta = \left(\dfrac{\alpha}{2}\right)$ or $\theta = \dfrac{\pi}{2} - \left(\dfrac{\alpha}{2}\right)$

(d) Maximum height when $\dfrac{dy}{dt} = 0$.

That is, $V\sin\theta - gt = 0$

$$t = \frac{V\sin\theta}{g}.$$

$$y = \frac{V^2\sin^2\theta}{g} - \frac{g}{2}\cdot\frac{V^2\sin^2\theta}{g^2}$$

$$= \frac{V^2\sin^2\theta}{2g}$$

Let the projection angles be θ_1 and θ_2.

$$h_1 + h_2 = \frac{V^2\sin^2\theta_1}{2g} + \frac{V^2\sin^2\theta_2}{2g}$$

$$= \frac{V^2}{2g}\left(\sin^2\theta_1 + \sin^2\theta_2\right)$$

$$= \frac{V^2}{2g}\left(\sin^2\theta_1 + \cos^2\theta_1\right)$$

$$= \frac{V^2}{2g}.$$

When fired vertically,

$$\text{maximum height} = \frac{V^2\sin^2\frac{\pi}{2}}{2g}$$

$$= \frac{V^2}{2g}.$$