HSC Mathematics Extension 1

Practice Paper 5

Question 1

(a) Solve $(2x-1)^2 \leq 25$.

(b) How many different letter sequences can be formed using the seven letters of the word PERMUTE.

(c) Find the exact value of $\displaystyle\int_0^1 \frac{x+2}{x^2+1}\,dx$.

(d) Show that, if $(x-2)$ is a factor of x^3+ax^2+bx+c, it is also a factor of

$$ax^3+bx^2+cx+16.$$

(e) $P\left(\frac{\pi}{4}, 1\right)$ is a point on $y = \tan x$ and O is the origin. Find, to the nearest minute, the acute angle between OP and the tangent to $y = \tan x$ at P.

Question 2

(a) For the points $A(-5, -1)$ and $B(1, 0)$:

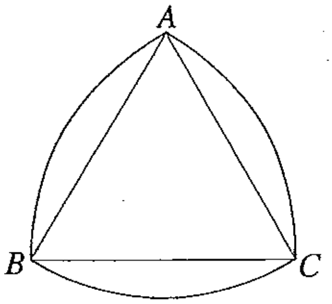
(i) Write down the coordinates of P, the point that divides AB in the ratio $k : 1$.

(ii) If P lies on $xy = 1$, show that $k^2+3k-4 = 0$.

(iii) Hence, or otherwise, find the coordinates of the points where the line AB meets $xy = 1$.

(b) Using the substitution $u = x^2$, evaluate $\displaystyle\int_{\sqrt{2}}^{2} \frac{x}{\sqrt{16-x^4}}\,dx$.

(c) ABC is an equilateral triangle, side $2r$. The circular arcs AB, BC and CA have centres at C, A and B.



Show that, for the figure bounded by the arcs:

(i) the perimeter is equal to that of a circle, radius r;

(ii) the area is approximately 90% of that of a circle, radius r.

Question 3

(a) (i) Evaluate $\int_0^{\frac{\pi}{4}} \sin^2 x \; dx$.

(ii) Show that $\int_0^{\frac{\pi}{4}} \tan x \; dx = \frac{1}{2}\ln 2$.

(iii) A function $f(x)$ is given by $f(x) = \sec x - \sin x$.

The graph of $y = f(x)$ for $0 \le x \le \frac{\pi}{4}$ is rotated about the x axis.

Find the volume of the solid generated.

[You may use the results of parts (i) and (ii).]

(b) (i) Show that the function $f(x) = x^2 - e^{-x}$ has a root between 0 and 1.

(ii) Determine whether this root lies closer to 0 or to 1.

(iii) Take 0·5 as an approximation to this root and use Newton's method to find this root correct to one decimal place.

Question 4

(a) (i) Sketch, on one number plane, graphs of $y = \sin^{-1} x$ and $y = \cos^{-1} x$, indicating their domains and ranges.

(ii) Verify that the graphs in part (i) intersect where $x = \frac{1}{\sqrt{2}}$.

(b) $f(x) = \left(\cos^{-1} x\right)\left(\sin^{-1} x\right)$

(i) Find $f'(x)$.

(ii) Prove that $f(x)$ has a maximum value and find this value.
[You may use any information from part (a).]

(c) Prove that $\frac{\sin 5\theta}{\sin\theta} - \frac{\cos 5\theta}{\cos\theta} = 4\cos 2\theta$.

(d) Figure 1 shows a piece of paper in the shape of an isosceles triangle.

$$AB = AC = a; \quad AM \perp BC; \quad \angle BAM = \theta.$$

Figure 2 shows the paper folded along AM until the two halves are perpendicular.

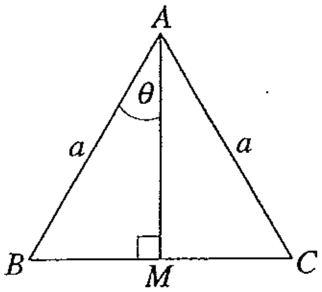


Figure 1

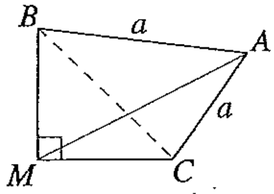


Figure 2

Show that, in Figure 2:

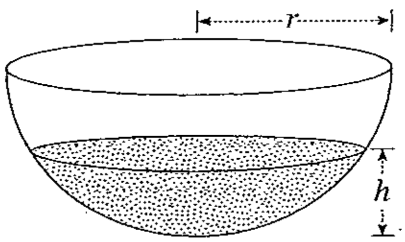
(i) $(BC)^2 = 2a^2 \sin^2\theta$;

(ii) if the angle formed by AB and AC is now ϕ, $\cos\phi = \cos^2\theta$.

Question 5

(a) Show that the equation of the circle, centre $(0, r)$ and radius r, may be written $x^2 = 2ry - y^2$.

(b) Water is entering a hemispherical bowl, radius r.



(i) Prove by integration that, when the depth of the water in the centre is h, the volume of water, V, in the bowl is given by

$$V = \pi\left(rh^2 - \frac{h^3}{3}\right).$$

[You may use the result of part (a).]

(ii) The water is entering the bowl at a constant rate of 1 L/min. If the radius of the bowl is 20 cm, find the rate at which the depth is increasing when:

1. the depth is 9 cm (correct to two decimal places);
2. the surface area of the water is 500 cm^2.

(c) A bag contains nine balls, of which two are red, three yellow and four green.

(i) If six balls are removed at random, find the probability that, of them, one is red, two are yellow and three are green.

(ii) The six balls are returned to the bag and then all nine balls are removed at random, one at a time.

Find the probability that the last three to be removed are red, yellow and green:

1. but not necessarily in that order;
2. in that order;
3. but not in that order.

Question 6

Two particles, A and B, move on an x axis, graduated in metres. Both particles are initially at rest at $x = 2$.

(a) Particle A is subject to forces that give it an acceleration of $-6x$ m s^{-2} when at position x.

(i) In which direction will A first move, and why?

(ii) If, at position x, A's velocity is v m s^{-1}, prove that $v^2 = 6\left(4 - x^2\right)$.

(iii) Describe A's motion.

(b) Particle B is subject to forces that give it, after t seconds, an acceleration of $(6 - 6t)$ m s^{-2}.

(i) In which direction will B first move, and why?

(ii) Find the velocity function for B.

(iii) Find the position function for B.

(iv) Where will B become stationary and what will be the subsequent motion?

(c) (i) Show that, when A and B first return to $x = 2$, they will have travelled the same distance.

(ii) Which particle will be first to return to $x = 2$?

(iii) Show that, when A reaches the origin for the third time, B will not be far away.

Question 7

(a) Tom plays Jerry a game in which, in any one game, Tom's chance of winning is 0·6 and drawn games are not possible.

(i) If they agree to play five games, find, in decimal form, the probability that Jerry wins by three games to two.

(ii) If they agree to play until one of them wins three games, find, in decimal form, the probability that Jerry wins by three games to two.

(b) Consider the function $f(x) = \dfrac{x}{\sqrt{1-x^2}}$.

(i) What is the domain of $f(x)$?

(ii) Show that $f'(x) = \dfrac{1}{\left(1-x^2\right)^{\frac{3}{2}}}$.

(iii) Using part (ii), prove that $f(x)$ is an increasing function.

(iv) Sketch a graph of $y = f(x)$, showing any asymptotes it may have. Use the same scale on both axes.

(v) Find the equation of the tangent to $y = f(x)$ at the origin and draw this line on your graph in part (iv).

(vi) Explain why $f(x)$ has an inverse function, $f^{-1}(x)$ and draw a graph of $y = f^{-1}(x)$ on your graph in part (iv).

(vii) Find $f^{-1}(x)$.

Practice Paper 5

Question 1

(a)
$$-5 \le 2x-1 \le 5$$
$$-4 \le 2x \le 6$$
$$-2 \le x \le 3$$

(b) Number of sequences $= \dfrac{7!}{2!} = 2520$

(c)
$$2^3 + a.2^2 + b.2 + c = 0$$
$$4a+2b+c+8 = 0$$

Remainder when $ax^3+bx^2+cx+16$ is divided by $(x-2)$ is $8a+4b+2c+16 = 2(4a+2b+c+8)$
$= 0.$

(d)
$$\int_0^1 \frac{x+2}{x^2+1}\,dx = \int_0^1 \left(\frac{x}{x^2+1}+\frac{2}{x^2+1}\right)dx$$
$$= \left[\frac{1}{2}\ln\left(x^2+1\right)+2\tan^{-1}x\right]_0^1$$
$$= \frac{1}{2}\ln 2 + 2\left(\frac{\pi}{4}\right) - 0$$
$$= \frac{\ln 2 + \pi}{2}$$

(e)
$$\frac{dy}{dx} = \sec^2 x$$
$$= \sec^2\frac{\pi}{4}$$
$$= 2$$

Gradient of $OP = \dfrac{4}{\pi}$

$$\tan\theta = \frac{2-\frac{4}{\pi}}{1+\frac{8}{\pi}}$$
$$= \frac{2\pi-4}{\pi+8}$$
$$\theta \doteq 11° 35'.$$

Question 2

(a) (i) $x = \dfrac{-5+k}{k+1}, \quad y = \dfrac{-1}{k+1}$

(ii)
$$\left(\frac{-5+k}{k+1}\right)\left(\frac{-1}{k+1}\right) = 1$$
$$5-k = (k+1)^2$$
$$k^2+3k-4 = 0$$

(iii)
$$(k+4)(k-1) = 0$$
$k=-4$ or $k=1$

$x=3,\ y=\frac{1}{3};\ x=-2,\ y=-\frac{1}{2}$

Points are $\left(3, \frac{1}{3}\right)$ and $\left(-2, -\frac{1}{2}\right)$.

(b)
$u=x^2, \quad x=\sqrt{2}, \quad u=2$

$\dfrac{du}{dx}=2x, \quad x=2, \quad u=4$

$$\int_{\sqrt{2}}^{2}\frac{x}{\sqrt{16-x^4}}\,dx = \frac{1}{2}\int_2^4 \frac{1}{\sqrt{16-u^2}}\,du$$
$$= \frac{1}{2}\left[\sin^{-1}\frac{u}{4}\right]_2^4$$
$$= \frac{1}{2}\left(\sin^{-1}1-\sin^{-1}\frac{1}{2}\right)$$
$$= \frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi}{6}\right)$$
$$= \frac{\pi}{6}$$

(c) (i) Length of one arc $= 2r\left(\dfrac{\pi}{3}\right)$.

Perimeter $= 3\times 2r\left(\dfrac{\pi}{3}\right) = 2\pi r.$

(ii) Area of one segment
$$= \frac{(2r)^2}{2}\left(\frac{\pi}{3}-\sin\frac{\pi}{3}\right)$$
$$= r^2\left(\frac{2\pi}{3}-\sqrt{3}\right).$$

Area of triangle
$$= \frac{(2r)^2\sin\frac{\pi}{3}}{2}$$
$$= r^2\sqrt{3}.$$

Total area
$$= 3r^2\left(\frac{2\pi}{3}-\sqrt{3}\right)+r^2\sqrt{3}$$
$$= r^2\left(2\pi-2\sqrt{3}\right).$$

Percentage of circle
$$= \frac{r^2\left(2\pi-2\sqrt{3}\right)}{\pi r^2}\times 100$$
$$= 89\cdot 7\%.$$

Question 3

(a) (i)
$$\int_0^{\frac{\pi}{4}}\sin^2 x\,dx = \frac{1}{2}\int_0^{\frac{\pi}{4}}(1-\cos 2x)\,dx$$
$$= \frac{1}{2}\left[x-\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$$
$$= \frac{\pi}{8}-\frac{1}{4}$$

(ii)
$$\int_0^{\frac{\pi}{4}}\frac{\sin x}{\cos x}\,dx = \left[-\ln(\cos x)\right]_0^{\frac{\pi}{4}}$$
$$= \left[\ln(\sec x)\right]_0^{\frac{\pi}{4}}$$
$$= \ln\sqrt{2}-\ln 1$$
$$= \frac{1}{2}\ln 2$$

(iii)
$$V = \pi\int_0^{\frac{\pi}{4}}(\sec x-\sin x)^2\,dx$$
$$= \pi\int_0^{\frac{\pi}{4}}\left(\sec^2 x-2\tan x+\sin^2 x\right)dx$$
$$= \pi\left[\tan x\right]_0^{\frac{\pi}{4}}+\pi\left(-\ln 2+\frac{\pi}{8}-\frac{1}{4}\right)$$
$$= \pi(1-0)+\pi\left(-\ln 2+\frac{\pi}{8}-\frac{1}{4}\right)$$
$$= \pi\left(\frac{3}{4}+\frac{\pi}{8}-\ln 2\right) \text{ cubic units.}$$

(b) (i) $f(0) = 0-1, \quad f(1) = 1-e^{-1}$
$< 0 \qquad\qquad > 0$

(ii)
$$f(0\cdot 5) = (0\cdot 5)^2 - e^{-0\cdot 5}$$
$$= -0\cdot 3\ldots$$

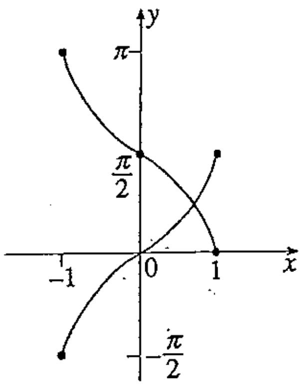
Hence the root lies between $0\cdot 5$ and 1, that is, closer to 1 than 0.

(iii) $f'(x) = 2x+e^{-x}$

Approximation
$$= 0\cdot 5 - \frac{0\cdot 25 - e^{-0\cdot 5}}{1+e^{-0\cdot 5}}$$
$$= 0\cdot 7$$

Question 4

(a) (i)



(ii) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4} = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$

(b) (i) $f'(x)$

$$= \cos^{-1}x\left(\frac{1}{\sqrt{1-x^2}}\right) + \sin^{-1}x\left(-\frac{1}{\sqrt{1-x^2}}\right)$$

$$= \frac{\cos^{-1}x - \sin^{-1}x}{\sqrt{1-x^2}}$$

(ii) When $f'(x) = 0$,

$$\cos^{-1}x = \sin^{-1}x$$

$$x = \frac{1}{\sqrt{2}}, \text{ from part (ii).}$$

From part (i),

for $x < \frac{1}{\sqrt{2}}$, $\cos^{-1}x > \sin^{-1}x$,
$\cos^{-1}x - \sin^{-1}x > 0$.

For $x > \frac{1}{\sqrt{2}}$, $\cos^{-1}x < \sin^{-1}x$,
$\cos^{-1}x - \sin^{-1}x < 0$.

Hence the stationary point is a maximum.

$$f\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}\cdot\frac{\pi}{4} = \frac{\pi^2}{16}$$

(c) $$\text{LHS} = \frac{\sin 5\theta\cos\theta - \cos 5\theta\sin\theta}{\sin\theta\cos\theta}$$

$$= \frac{\sin 4\theta}{\sin\theta\cos\theta}$$

$$= \frac{2\sin 2\theta\cos 2\theta}{\frac{1}{2}\sin 2\theta}$$

$$= 4\cos 2\theta$$

$$= \text{RHS}.$$

(d) (i) $$\frac{BM}{a} = \sin\theta$$

$$BM = a\sin\theta$$

$$MC = a\sin\theta$$

$$(BC)^2 = (BM)^2 + (MC)^2$$

$$= 2a^2\sin^2\theta.$$

(ii) $$\cos\phi = \frac{a^2 + a^2 - 2a^2\sin^2\theta}{2(a)(a)}$$

$$= \frac{2a^2 - 2a^2\sin^2\theta}{2a^2}$$

$$= \frac{2a^2(1-\sin^2\theta)}{2a^2}$$

$$= \cos^2\theta$$

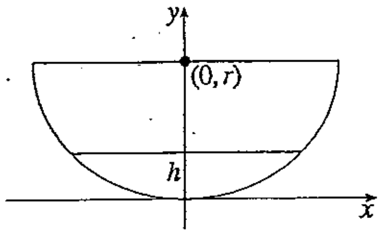
Question 5

(a) $$x^2 + (y-r)^2 = r^2$$

$$x^2 + y^2 - 2ry + r^2 = r^2$$

$$x^2 = 2ry - y^2$$

(b) (i)



$$V = \pi\int_0^h x^2\,dy$$

$$= \pi\int_0^h \left(2ry - y^2\right)dy$$

$$= \pi\left[ry^2 - \frac{y^3}{3}\right]_0^h$$

$$= \pi\left(rh^2 - \frac{h^3}{3}\right)$$

(ii) $$\frac{dV}{dt} = 1000$$

$$\frac{dh}{dt} = \frac{dh}{dV}\cdot\frac{dV}{dt} = \frac{1000}{\pi\left(2rh - h^2\right)}$$

(1) $$\frac{dh}{dt} = \frac{1000}{\pi(360-81)}$$

$$= 1{\cdot}14 \text{ cm/min}$$

(2) $\pi x^2 = 500$, i.e. $\pi\left(2rh - h^2\right) = 500$

$$\frac{dh}{dt} = \frac{1000}{500}$$

$$= 2 \text{ cm/min}$$

(c) (i) $$\text{Probability} = \frac{{}^2C_1 \times {}^3C_2 \times {}^4C_3}{{}^9C_6}$$

$$= \frac{24}{84}$$

$$= \frac{2}{7}$$

(ii) (1) This requires part (i) to occur, i.e., probability $= \frac{2}{7}$.

(2) This requires part (i), then red, then yellow, then green.
Probability $= \frac{2}{7}\times\left(\frac{1}{3}\times\frac{1}{2}\times\frac{1}{1}\right) = \frac{1}{21}$.

(3) Probability $= \frac{2}{7}\times\left(1 - \frac{1}{6}\right) = \frac{5}{21}$.

Question 6

(a) (i) Negative, since it is at rest at $x = 2$, where acceleration is negative.

(ii)
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -6x$$
$$\frac{1}{2}v^2 = -3x^2 + c$$
At $x = 2$, $v = 0$
$$0 = -12 + c$$
$$v^2 = 6\left(4 - x^2\right)$$

(iii) SHM; amplitude 2, period $\frac{2\pi}{\sqrt{6}}$.

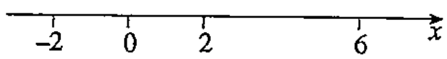
(b) (i) Positive, since it is initially at rest, at which time its acceleration is positive.

(ii)
$$v = \int (6 - 6t)\, dt$$
$$v = 6t - 3t^2 + c$$
When $t = 0$, $v = 0$, hence $c = 0$
$$v = 6t - 3t^2$$

(iii)
$$x = \int \left(6t - 3t^2\right) dt$$
$$x = 3t^2 - t^3 + c$$
When $t = 0$, $x = 2$, hence $c = 2$
$$x = 2 + 3t^2 - t^3$$

(iv)
$$6t - 3t^2 = 0$$
$$3t(2 - t) = 0$$
$$t = 2 \qquad (t \neq 0)$$
$$x = 2 + 12 - 8 = 6$$
When $t = 2$, $6 - 6t < 0$ and B will move in the negative direction.

(c) (i)



A oscillates between -2 and 2, and hence will travel 8 m.
B moves from $x = 2$ to $x = 6$ and returns to $x = 2$, that is, 8 m.

(ii) A will have made one oscillation, which takes $\frac{2\pi}{\sqrt{6}}$ seconds.
For B, when $x = 2$,
$$2 = 2 + 3t^2 - t^3$$
$$0 = t^2(3 - t)$$
$$t = 3 \qquad (t \neq 0)$$
Now $\frac{2\pi}{\sqrt{6}} = 2{\cdot}5 \ldots$, hence A is first.

(iii) A will have made $1\frac{1}{4}$ oscillations, which will take $\frac{5}{4} \cdot \frac{2\pi}{\sqrt{6}} \doteqdot 3{\cdot}2$ seconds.
For B, when $t = 3{\cdot}2$,
$$x = 2 + 3(3{\cdot}2)^2 - (3{\cdot}2)^3$$
$$= -0{\cdot}048,$$
which is not far from zero.

Question 7

(a) (i)
$$\text{Probability} = {}^5C_3(0{\cdot}4)^3(0{\cdot}6)^2$$
$$= 0{\cdot}2304$$

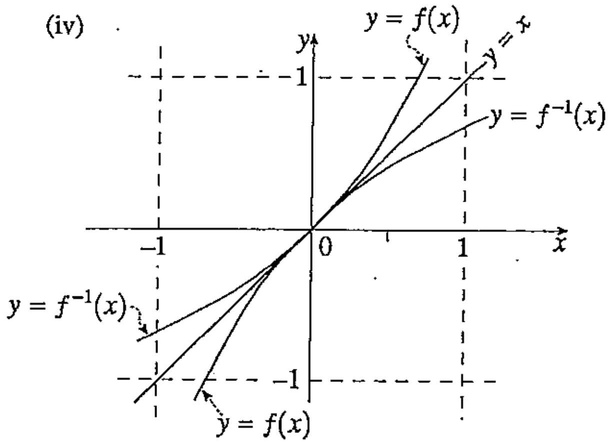
(ii) For this to occur, each must win two of the first four games, and Jerry the fifth.
$$\text{Probability} = {}^4C_2(0{\cdot}4)^2(0{\cdot}6)^2 \times 0{\cdot}4$$
$$= 0{\cdot}138\,24.$$

(b) (i)
$$1 - x^2 > 0$$
$$x^2 < 1$$
$$-1 < x < 1$$

(ii)
$$f'(x) = \frac{\left(1-x^2\right)^{\frac{1}{2}} - x\left(\frac{1}{2}\right)\left(1-x^2\right)^{-\frac{1}{2}}(-2x)}{\left(1-x^2\right)}$$
$$= \frac{\left(1-x^2\right) + x^2}{\left(1-x^2\right)^{\frac{1}{2}}\left(1-x^2\right)}$$
$$= \frac{1}{\left(1-x^2\right)^{\frac{3}{2}}}$$

(iii) $f'(x) > 0$ for all x.

(iv)



(v) $f'(0) = 1$. Tangent is $y = x$.

(vi) $f(x)$ is monotonic.

(vii)
$$x = \frac{y}{\sqrt{1-y^2}}$$
$$x^2 = \frac{y^2}{1-y^2}$$
$$x^2 - x^2y^2 = y^2$$
$$x^2 = y^2\left(1 + x^2\right)$$
$$y^2 = \frac{x^2}{1+x^2}$$
$$y = \frac{x}{\sqrt{1+x^2}}$$
(This option makes $y > 0$ when $x > 0$ and $y < 0$ when $x < 0$, as is required.)