HSC Mathematics Extension 1

Practice Paper 6

Question 1

(a) Solve $\dfrac{2x+5}{x} \leq 1$.

(b) Differentiate $x^2 \tan^{-1}x$.

(c) If $\log_a\left(\frac{b}{c}\right) = 1{\cdot}25$, what is the value of $\log_a\left(\frac{c}{b}\right)$?

(d) Find the exact value of $\displaystyle\int_0^3 \frac{1}{\sqrt{12-x^2}}\,dx$.

(e) Using the substitution $u = 3x+1$, or otherwise, evaluate

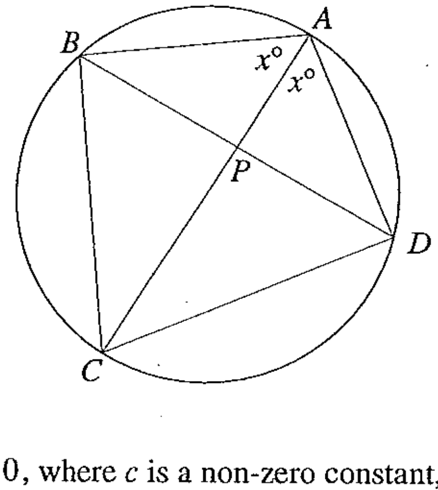
$$\int_0^2 \frac{3x-2}{3x+1}\,dx.$$

Question 2

(a) A, B, C and D are points on the circumference of a circle.

AC and BD intersect at P.

$\angle BAC = \angle DAC = x°$.



(i) State why $\angle ACB = \angle ADB$.

(ii) Prove that $\angle ABC = \angle APD$.

(iii) Deduce that $\angle ADC = \angle CPD$.

(b) The roots of the equation $4x^3 + 6x^2 + c = 0$, where c is a non-zero constant, are α, β, and $\alpha\beta$.

(i) Show that $\alpha\beta \neq 0$.

(ii) Show that $\alpha\beta + \alpha^2\beta + \alpha\beta^2 = 0$ and deduce the value of $\alpha + \beta$.

(iii) Show that $\alpha\beta = -\frac{1}{2}$.

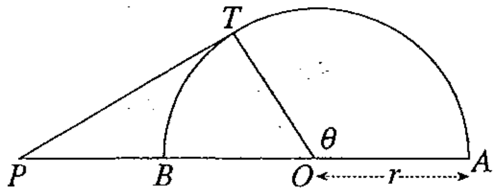
(iv) Solve the equation.

Question 3

(a) (i) On the same set of axes, sketch graphs of $y = \theta$ and $y = -\tan\theta$ for $-\pi \le \theta \le \pi$.

(ii) Use your graphs to state the number of solutions to the equation $\theta + \tan\theta = 0$ for $-\pi \le \theta \le \pi$.

(b)



The point T lies on the circumference of a semicircle, radius r and diameter AB, as shown. The point P lies on AB produced and PT is the tangent at T.

The arc AT subtends an angle of θ at the centre, O, and the area of ΔOPT is equal to that of the sector AOT.

(i) Show that $\theta + \tan\theta = 0$.

(ii) Taking 2 as an approximation to θ, use Newton's method once to find a better approximation to two decimal places.

(c) In a game of Tringles, three players, A, B and C, each have a regular six-sided dice. A throws first. B is then allowed up to two attempts to match A's number. If he does so, he wins. If B fails, C is allowed up to two attempts to match either A's number or B's second number. If C fails, A wins.

Which player has:

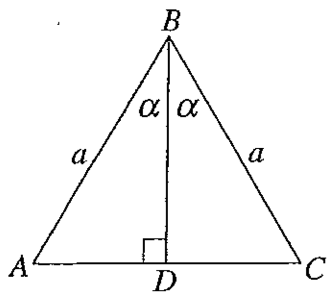
(i) the greatest chance of winning?

(ii) the least chance of winning?

Question 4

(a) The triangle ABC is isosceles, with $AB = BC = a$, and BD is perpendicular to AC.

Let $\angle ABD = \angle CBD = \alpha$.

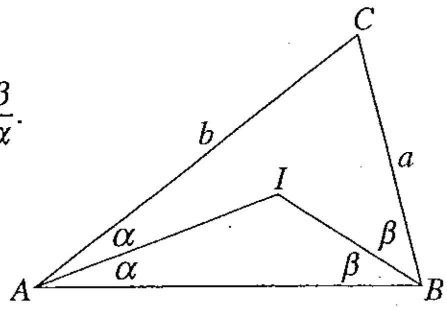


(i) Show that the area of ΔABD is $\dfrac{a^2 \sin\alpha \cos\alpha}{2}$.

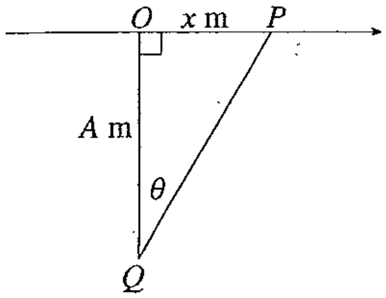
(ii) By considering the area of ΔABC, prove that $\sin 2\alpha = 2\sin\alpha\cos\alpha$.

(iii) IA and IB bisect $\angle$s CAB and CBA as shown.

Prove that $\dfrac{IB}{IA} = \dfrac{a\cos\beta}{b\cos\alpha}$.



(b) P is a point oscillating in simple harmonic motion on an x axis, the centre of the motion being the origin, O. The amplitude of the motion is A m, the period 2π seconds, and, when $t = 0$, the point is at O, moving in the positive direction.



(i) Express x as a sine function of t.

(ii) OQ is perpendicular to the axis, $OQ = A$ m and $\angle OQP = \theta$. Show that $x = A\tan\theta$ and deduce that

$$\frac{dx}{d\theta} = A\left(1 + \sin^2 t\right).$$

(iii) Find $\dfrac{d\theta}{dt}$ as a function of t.

(iv) Find the times at which θ is increasing at a rate of $\dfrac{2}{7}$ radians per second.

Question 5

(a) A particle moves on an x axis so that, when x m from an origin, its acceleration is $-2x^3$ m s^{-2}. The particle is initially at rest at $x = 4$.

(i) In which direction will the particle first move, and why?

(ii) Show that its velocity, v m s^{-1}, is given by $v^2 = 256 - x^4$.

(iii) Where will the particle next come to rest and what will be the subsequent motion?

(iv) The particle takes T seconds to travel from $x = -3$ to $x = 3$.

Show that $T = \int_{-3}^{3} \frac{1}{\sqrt{256 - x^4}}\, dx$.

(v) Use Simpson's rule with three function values to find a two-place decimal approximation to T.

(b) (i) Write, in sigma notation, the expansion of $(1+a)^n$.

(ii) Prove that $\sum_{r=1}^{n} \binom{n}{r} a^r = \sum_{r=1}^{n} a(1+a)^{r-1}$.

Question 6

(a) (i) Find the stationary points on the graph of $y = \frac{12x - x^3}{8}$.

(ii) A function is defined by $f(x) = \frac{12x - x^3}{8}$, for $-2 \le x \le 2$.

Sketch a graph of $y = f(x)$ and explain why an inverse function, $f^{-1}(x)$, exists.

(iii) Sketch $y = f^{-1}(x)$ on the same set of axes.

(iv) If θ is the acute angle formed by the tangents to the two curves at the origin, find the value of $\tan\theta$.

(v) Find the value of $\int_{0}^{2} f^{-1}(x)\, dx$.

(b) Jo borrows \$50 000 at 9% per year reducible interest, calculated monthly. The loan is to be repaid in 240 equal monthly payments.

(i) Show that the amount of each payment will be \$449·86.

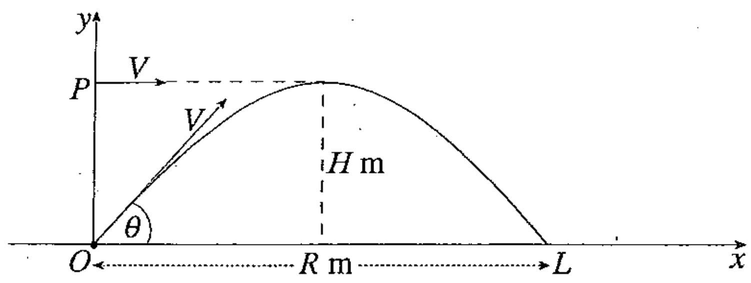
(ii) After Jo has made twelve such payments, the interest rate rises to 12% per year. Show that, if Jo is ever going to repay the loan, she must increase the amount of her payments.

(iii) If the new interest rate is maintained, and Jo does not increase her payments, what will be the amount owing after a further twenty-four payments?

Question 7

(a) Sketch, on the same number plane, graphs of $y = \sin\theta$ and $y = \sin 2\theta$, for $0 \le \theta \le \frac{\pi}{2}$.

(b)



A particle is projected from a point O on horizontal ground with speed V m s^{-1} at an angle of elevation of θ, landing at L, as shown. You may assume that the displacements of this particle after t seconds are given by

$$x = Vt\cos\theta \quad \text{and} \quad y = Vt\sin\theta - \frac{g}{2}t^2.$$

(i) Show that the range, R m, and the greatest height reached, H m, are given by $R = \dfrac{V^2 \sin 2\theta}{g}$ and $H = \dfrac{V^2 \sin^2\theta}{2g}$.

(ii) A second particle is projected at the same time as the first, with speed V m s^{-1}, horizontally from the point P, H m above O.

Prove that its displacements are given by

$$x = Vt \quad \text{and} \quad y = H - \frac{1}{2}gt^2.$$

(iii) Prove that, when the second particle lands, the first is at the top of its flight.

(iv) Let the range of the second particle be S m and find the value of θ for which $R = S$.

(v) Describe the manner in which $|R - S|$ varies as θ increases from 0 to $\frac{\pi}{2}$.

[You may use part (a).]

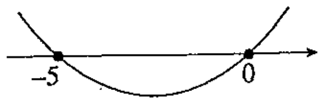
Practice Paper 6

Question 1

(a) $x(2x+5) \le x^2$

$x^2+5x \le 0$

$x(x+5) \le 0$



$\therefore \quad -5 \le x < 0$, since $x \ne 0$.

(b) $f'(x) = x^2\left(\dfrac{1}{1+x^2}\right)+2x\tan^{-1}x$

$= \dfrac{x^2}{1+x^2}+2x\tan^{-1}x$

(c) $\log\left(\dfrac{c}{b}\right) = \log\left(\dfrac{b}{c}\right)^{-1}$

$= -\log\left(\dfrac{b}{c}\right)$

$= -1\cdot25$

(d) $\displaystyle\int_0^3 \frac{1}{\sqrt{12-x^2}}\,dx = \left[\sin^{-1}\frac{x}{2\sqrt{3}}\right]_0^3$

$= \sin^{-1}\dfrac{3}{2\sqrt{3}} - \sin^{-1}0$

$= \sin^{-1}\dfrac{\sqrt{3}}{2}$

$= \dfrac{\pi}{3}$

(e) $u = 3x+1, \quad x=0, \quad u=1$

$\dfrac{du}{dx} = 3, \quad x=2, \quad u=7$

$\displaystyle\int_0^2 \frac{3x-2}{3x+1}\,dx = \frac{1}{3}\int_1^7 \frac{u-3}{u}\,du$

$= \dfrac{1}{3}\displaystyle\int_1^7 \left(1-\frac{3}{u}\right)du$

$= \dfrac{1}{3}\Big[u-3\ln u\Big]_1^7$

$= \dfrac{1}{3}(7-3\ln 7-1)$

$= 2-\ln 7$

Question 2

(a) (i) Angles in the same segment of a circle are equal.

(ii) Let $\angle ACB = \angle ADB = y°$

Then $\angle ABC = 180° - (x+y)°$

($\angle$ sum of ΔABC)

and $\angle APD = 180° - (x+y)°$

($\angle$ sum of ΔAPD)

$\therefore \quad \angle ABC = \angle APD.$

(iii) $\angle ADC = 180° - \angle ABC$

(opp. $\angle$s of cyclic quad.)

$\angle CPD = 180° - \angle APD$

(straight line)

$\therefore \angle ADC = \angle CPD.$

(b) (i) $\alpha\beta$ is a root of the equation, but zero is not.

(ii) $\alpha\beta + \alpha(\alpha\beta) + \beta(\alpha\beta) = 0$

$\alpha\beta + \alpha^2\beta + \alpha\beta^2 = 0$

$\alpha\beta(1+\alpha+\beta) = 0$

But $\alpha\beta \ne 0$,

$\therefore \quad 1+\alpha+\beta = 0$

$\alpha+\beta = -1$

(iii) $\alpha+\beta+\alpha\beta = -\dfrac{6}{4}$

$\alpha\beta = -\dfrac{1}{2}$

(iv) α and β are the roots of $x^2-(-1)x-\dfrac{1}{2} = 0$

that is, $2x^2+2x-1 = 0$

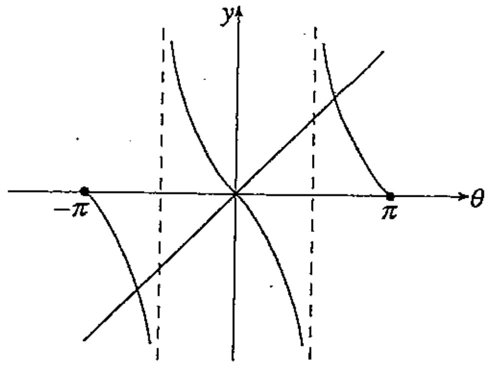
$x = \dfrac{-2\pm\sqrt{4+8}}{4}$

$= \dfrac{-1\pm\sqrt{3}}{2}$

Hence the roots are $-\dfrac{1}{2}$ and $\dfrac{-1\pm\sqrt{3}}{2}$.

Question 3

(a) (i)



(ii) Three

(b) (i) $OT = r, \quad \angle OTP = 90°$

$\tan(\pi-\theta) = \dfrac{PT}{r}$

$-\tan\theta = \dfrac{PT}{r}$

$PT = -r\tan\theta$

Area $\Delta OPT = -\dfrac{r^2\tan\theta}{2}$

$\therefore \quad -\dfrac{r^2\tan\theta}{2} = \dfrac{r^2\theta}{2}$

$\theta+\tan\theta = 0$

(ii) $f'(\theta) = 1+\sec^2\theta$

Approximation $= 2-\dfrac{2+\tan 2}{1+\sec^2 2}$

$= 2\cdot03$

(c) $P(B) = \dfrac{1}{6}+\dfrac{5}{6}\times\dfrac{1}{6} = \dfrac{11}{36}$

For C to win, B must fail, and then C succeed.

$P(C) = \dfrac{25}{36}\left(\dfrac{2}{6}+\dfrac{4}{6}\times\dfrac{2}{6}\right) = \dfrac{125}{324}$

$P(B) = \dfrac{99}{324}$

$P(A) = 1-\left(\dfrac{125}{324}+\dfrac{99}{324}\right) = \dfrac{100}{324}$

(i) C has the greatest chance

(ii) B has the least chance

Question 4

(a) (i) $AD = a\sin\alpha$

$BD = a\cos\alpha$

Area $\Delta ABD = \dfrac{a^2\sin\alpha\cos\alpha}{2}$

(ii) Area $\Delta ABC = \dfrac{a^2\sin 2\alpha}{2}$

Area $\Delta ABC = 2\times$ area ΔABD

$\dfrac{a^2\sin 2\alpha}{2} = a^2\sin\alpha\cos\alpha$

$\sin 2\alpha = 2\sin\alpha\cos\alpha$

(iii) $\dfrac{IB}{IA} = \dfrac{\sin\alpha}{\sin\beta}$ (sine rule, ΔIAB)

Also, $\dfrac{a}{b} = \dfrac{\sin 2\alpha}{\sin 2\beta}$ (sine rule, ΔABC)

$$= \frac{2\sin\alpha\cos\alpha}{2\sin\beta\cos\beta}$$
$$= \frac{\sin\alpha}{\sin\beta}\cdot\frac{\cos\alpha}{\cos\beta}$$

that is, $\dfrac{a}{b} = \dfrac{IB}{IA}\cdot\dfrac{\cos\alpha}{\cos\beta}$

$$\frac{IB}{IA} = \frac{a\cos\beta}{b\cos\alpha}$$

(b) (i) $x = A\sin t$

(ii)
$$\frac{x}{A} = \tan\theta$$
$$x = A\tan\theta$$
$$\frac{dx}{d\theta} = A\sec^2\theta$$
$$= A\left(1+\tan^2\theta\right)$$
$$= A\left(1+\frac{x^2}{A^2}\right)$$
$$= A\left(1+\sin^2 t\right)$$

(iii)
$$\frac{d\theta}{dt} = \frac{d\theta}{dx}\cdot\frac{dx}{dt}$$
$$= \frac{A\cos t}{A\left(1+\sin^2 t\right)}$$
$$= \frac{\cos t}{1+\sin^2 t}$$

(iv)
$$\frac{\cos t}{1+\sin^2 t} = \frac{2}{7}$$
$$7\cos t = 2+2\left(1-\cos^2 t\right)$$
$$2\cos^2 t+7\cos t-4 = 0$$
$$(2\cos t-1)(\cos t+4) = 0$$
$$\cos t = \frac{1}{2} \qquad (\cos t \neq -4)$$
$$t = 2n\pi \pm \frac{\pi}{3}$$

Question 5

(a) (i) Negative, since it is initially at rest where acceleration is negative.

(ii)
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -2x^3$$
$$\frac{1}{2}v^2 = -\frac{1}{2}x^4+c$$

At $x=4$, $v=0$, $\quad 0 = -128+c$

$$v^2 = 256-x^4$$

(iii) For $v=0$, $\quad x^4 = 256$

$\therefore \quad x=4$ or $x=-4$

Hence next at rest at $x=-4$.

At $x=-4$, acceleration is positive, and the particle will move in the positive direction.

(iv) In moving from $x=-3$ to $x=3$, $v>0$.

Hence $\dfrac{dx}{dt} = \sqrt{256-x^4}$

$$\frac{dt}{dx} = \frac{1}{\sqrt{256-x^4}}$$
$$T = \int_{-3}^{3}\frac{1}{\sqrt{256-x^4}}\,dx$$

(v) $T \doteqdot \dfrac{3}{3}\left(\dfrac{1}{\sqrt{175}}+4\times\dfrac{1}{16}+\dfrac{1}{\sqrt{175}}\right) = 0{\cdot}40$

(b) (i) $(1+a)^n = \displaystyle\sum_{r=0}^{n}\binom{n}{r}a^r$

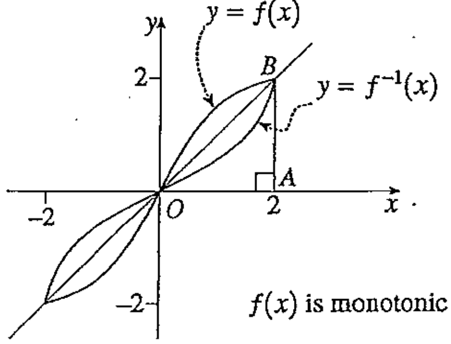
(ii)
$$\sum_{r=1}^{n}\binom{n}{r}a^r = \sum_{r=0}^{n}\binom{n}{r}a^r-\binom{n}{0}a^0$$
$$= (1+a)^n-1$$
$$\sum_{r=1}^{n}a(1+a)^{r-1} = a+a(1+a)+\cdots+a(1+a)^{n-1}$$
$$= \frac{a\left[(1+a)^n-1\right]}{(1+a)-1}$$
$$= (1+a)^n-1$$

Question 6

(a) (i) $\dfrac{dy}{dx} = \dfrac{3\left(4-x^2\right)}{8}$

Stationary at $(2, 2)$ and $(-2, -2)$.

(ii) and (iii)



(iv) Gradient of $f(x)$ at $(0, 0) = \dfrac{3}{2}$

Gradient of $f^{-1}(x)$ at $(0, 0) = \dfrac{2}{3}$

$$\tan\theta = \frac{\frac{3}{2}-\frac{2}{3}}{1+\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)} = \frac{5}{12}$$

(v)
$$\int_0^2 \frac{12x-x^3}{8}\,dx = \frac{1}{8}\left[6x^2-\frac{x^4}{4}\right]_0^2$$
$$= \frac{1}{8}(24-4)$$
$$= 2{\cdot}5$$

Area of $\Delta OAB = 2$

Hence $\displaystyle\int_0^2 f^{-1}(x)\,dx = 1{\cdot}5$.

(b) (i) $50\,000(1{\cdot}0075)^{240} - \dfrac{P\left(1{\cdot}0075^{240}-1\right)}{1{\cdot}0075-1} = 0$

$$\therefore\ P = \frac{50\,000(1{\cdot}0075)^{240}(0{\cdot}0075)}{1{\cdot}0075^{240}-1}$$
$$= \$449{\cdot}86$$

(ii) The amount owing after 12 payments

$$= 50\,000(1{\cdot}0075)^{12} - \frac{449{\cdot}86\left(1{\cdot}0075^{12}-1\right)}{1{\cdot}0075-1}$$
$$= \$49\,063{\cdot}68.$$

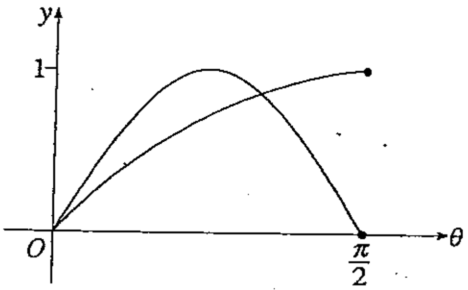
The next interest charge will be over \$490, which exceeds her payment.

(iii) The amount owing

$$= 49\,063{\cdot}68(1{\cdot}01)^{24} - \frac{449{\cdot}86\left(1{\cdot}01^{24}-1\right)}{1{\cdot}01-1}$$
$$= \$50\,163{\cdot}57.$$

Question 7

(a)



(b) (i) When $y = 0$, $t = \dfrac{2V\sin\theta}{g}$

$$R = V\cos\theta\left(\frac{2V\sin\theta}{g}\right)$$

That is, $R = \dfrac{V^2\sin 2\theta}{g}$

$$\frac{dy}{dt} = V\sin\theta - gt$$

When $\dfrac{dy}{dt} = 0$,

$$t = \frac{V\sin\theta}{g}$$

$$H = \frac{V^2\sin^2\theta}{g} - \frac{g}{2}\cdot\frac{V^2\sin^2\theta}{g^2}$$

That is, $H = \dfrac{V^2\sin^2\theta}{2g}$.

(ii) For this particle, put $\theta = 0$ and $y = H$ when $t = 0$.
Hence $x = Vt$, $y = H - \frac{1}{2}gt^2$.

(iii) When $H - \frac{1}{2}gt^2 = 0$,

$$t^2 = \frac{2H}{g} = \frac{V^2\sin^2\theta}{g^2}$$

$$t = \frac{V\sin\theta}{g},$$

which is the time taken by the first particle to reach the top of its flight, from part (i).

(iv) When $t = \dfrac{V\sin\theta}{g}$, $x = \dfrac{V^2\sin\theta}{g}$

That is, $S = \dfrac{V^2\sin\theta}{g}$.

If $R = S$, $\dfrac{V^2\sin 2\theta}{g} = \dfrac{V^2\sin\theta}{g}$

$$2\sin\theta\cos\theta = \sin\theta$$

$$2\cos\theta = 1 \qquad (\sin\theta \neq 0)$$

$$\cos\theta = \frac{1}{2}$$

$$\theta = \frac{\pi}{3}$$

(v) The difference between R and S is illustrated by the distance between the graphs in part (a), measured parallel to the y axis.
This begins at zero, increases to a local maximum (see below), decreases to zero at $\frac{\pi}{3}$ and then increases to $\dfrac{V^2}{g}$ when $\theta = \frac{\pi}{2}$.
For $(\sin 2\theta - \sin\theta)$ to be maximum,

$$2\cos 2\theta - \cos\theta = 0$$

$$4\cos^2\theta - \cos\theta - 2 = 0$$

$\therefore \quad \cos\theta = \dfrac{1+\sqrt{33}}{8} \quad \left(\dfrac{1-\sqrt{33}}{8} < 0\right)$

and $\quad \theta \doteqdot 0{\cdot}57$.