HSC Mathematics Extension 1

Practice Paper 7

Question 1

(a) Differentiate $\dfrac{x}{\operatorname{cosec} x}$.

(b) Evaluate $\displaystyle\int_{-3}^{3} \frac{1}{9+x^2}\, dx$.

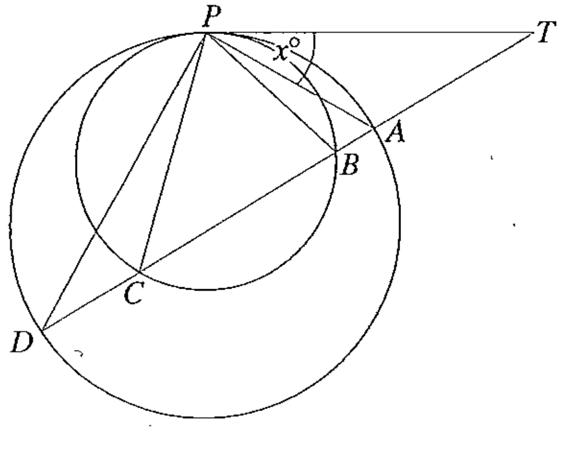
(c) On a number plane, the point P divides the interval from $(t, 2)$ to $(-1, t)$ in the ratio 2 : 1. If P lies on $y = x$, find the value of t.

(d) Given that $\log_b\left(\dfrac{p}{q}\right) = 3$ and $\log_b\left(\dfrac{q}{r}\right) = 1{\cdot}6$, find $\log_b\left(\dfrac{p}{r}\right)$.

(e) Using the substitution $u = e^x$, or otherwise, find $\displaystyle\int_{0}^{\ln 5} \frac{1}{1+e^{-x}}\, dx$.

Question 2

(a) PT is the common tangent to the two circles which touch internally at P. $\angle TPA = x°$.



(i) Explain why $\angle PDA = x°$.

(ii) Prove that $\angle APB = \angle CPD$.

(b) A particle is moving in simple harmonic motion. Its displacement, x cm, at time t seconds is given by $x = A\cos nt$, where A and n are positive constants.

(i) Find functions for its velocity and acceleration in terms of t.

(ii) When $t = \frac{\pi}{n}$, $x = -36$ and the acceleration is 9 cm s^{-2}.
Find the amplitude and period of the motion.

(iii) What is the particle's greatest speed?

(c) The roots of $x^3 - x^2 - 5x + 2 = 0$ are α, β and γ.

(i) Write down the values of 1. $\alpha\beta\gamma$
2. $\beta\gamma + \gamma\alpha + \alpha\beta$.

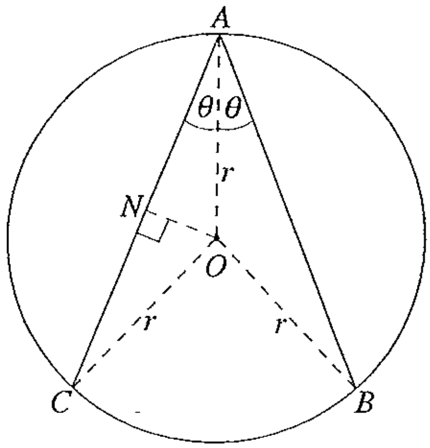
(ii) Show that $\beta + \gamma = 1 - \alpha$.

(iii) Using part (ii), and similar results, evaluate

$$\frac{\beta+\gamma}{\alpha} + \frac{\gamma+\alpha}{\beta} + \frac{\alpha+\beta}{\gamma}.$$

Question 3

(a) O is the centre of the circle, radius r.



$AB = AC$ and $\angle OAB = \angle OAC = \theta$.

(i) Explain why $\angle BOC = 4\theta$.

(ii) P is the perimeter of the region bounded by AB, AC and the minor arc BC.
Show that $P = 4r(\theta + \cos\theta)$.

(iii) If $P = 5r$, show that $\cos\theta + \theta - 1{\cdot}25 = 0$.

(iv) Take $\theta = 0{\cdot}3$ as an approximation and use Newton's method once to improve this to three decimal places.

(b) (i) Beginning with the expansion for $\cos(\alpha+\beta)$, prove

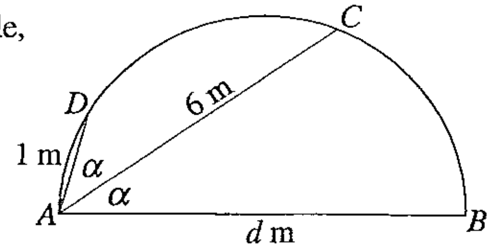
$$\cos 2\alpha = 2\cos^2\alpha - 1.$$

(ii) The figure shows a semicircle, diameter d metres.

$AC = 6$ metres and

$AD = 1$ metre.

$\angle BAC = \angle CAD = \alpha$.



Write expressions for $\cos\alpha$ and $\cos 2\alpha$ and, using part (i), find the value of d.

Question 4

(a)

NW			NE
2	5	1	8
12	15	11	14
3	6	4	7
9	16	10	13
SW			SE

The integers from one to sixteen are divided into four quadrants as shown.

Three of these numbers are taken at random.

Find as decimals, correct to two places (where necessary), the probability that

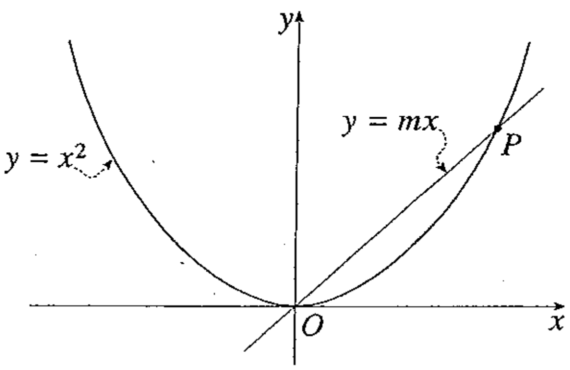
(i) all three are in the same quadrant

(ii) all three are in the north

(iii) either all three are in the north, or all three are in the east

(iv) exactly two are in the west.

(b) The diagram shows the curve $y = x^2$ and the line $y = mx$, where $m > 0$, intersecting at the origin, O, and the point P.



(i) Find the coordinates of P.

(ii) Show that the area of the region enclosed by the interval OP and the arc OP is $\frac{m^3}{6}$ square units.

(iii) The angle, θ, formed by OP and the positive x axis, is increasing at a rate of 0·5 radians per second.

Find the rate at which the area in part (ii) is increasing when $\theta = \frac{\pi}{3}$.

Question 5

(a) Jack and Jill compete for a prize. Jack will toss an ordinary six-faced dice six times, in an endeavour to toss exactly two 6's. Jill will toss a coin fourteen times, endeavouring to toss exactly seven 'heads'.

Who has the greater chance of success?

(b) A particle is released from rest at an origin on an x axis. Its acceleration is determined by its position so that, at position x, it is $2\cos x$.

(i) In which direction will it first move?

(ii) Show that its velocity is given by $v^2 = 4\sin x$.

(iii) Where will the particle next come to rest?

(iv) Describe the subsequent motion.

(v) What is the particle's greatest speed?

(vi) If the particle takes time T to move from $x = \frac{\pi}{6}$ to $x = \frac{5\pi}{6}$, show that

$$2T = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sqrt{\operatorname{cosec} x}\ dx.$$

(vii) Use Simpson's rule with three function values to find a two-place decimal approximation to T.

Question 6

Consider the function $f(x) = \dfrac{1}{1+e^{-x}}$.

(a) Show that $y = f(x)$ has no stationary points.

(b) Describe the behaviour of $f(x)$ as x increases without limit in

(i) the positive direction;

(ii) the negative direction.

(c) Sketch a graph of $y = f(x)$ using the same scale on both axes and showing its intersection with the y axis.

(d) Explain why an inverse function, $f^{-1}(x)$, exists, and sketch its graph on the same axes as $y = f(x)$.

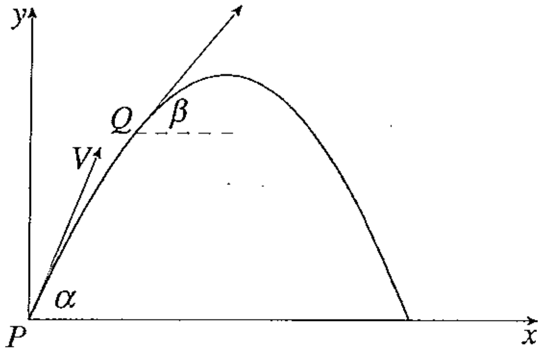
(e) Find $f^{-1}(x)$ and verify that $f^{-1}\left(\frac{1}{2}\right) = 0$ and $f^{-1}\left(\frac{5}{6}\right) = \ln 5$.

(f) With the aid of Question 1(e), evaluate

$$\int_{\frac{1}{2}}^{\frac{5}{6}} f^{-1}(x)\, dx.$$

Question 7

(a) A particle is projected from a point P on horizontal ground, with speed V m s^{-1} at an angle of elevation to the horizontal of α.



Its equations of motion are $\ddot{x} = 0$, $\ddot{y} = -g$.

(i) Write down expressions for its horizontal (x) and vertical (y) displacements from P after t seconds.

(ii) Determine the time of flight of the particle.

(iii) The particle reaches a point Q, as shown, where the direction of the flight makes an angle β with the horizontal.

Show that the time taken to travel from P to Q is

$$\frac{V\sin(\alpha-\beta)}{g\cos\beta} \text{ seconds.}$$

(iv) Consider the case where $\beta = \frac{\alpha}{2}$.

If the time taken to travel from P to Q is then one-third of the total time of flight, find the value of α.

(b) (i) For positive integers n and r, with $r < n$, show that

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}.$$

(ii) Prove that $\sum_{r=0}^{n} \binom{n+r}{r} = \binom{2n+1}{n}$.

Practice Paper 7

Question 1

(a) $f(x) = x\sin x$

$f'(x) = x\cos x + \sin x$

(b) $\int_{-3}^{3} \frac{1}{9+x^2}\,dx = \frac{1}{3}\left[\tan^{-1}\frac{x}{3}\right]_{-3}^{3}$

$= \frac{1}{3}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]$

$= \frac{\pi}{6}$

(c) $x = \frac{t-2}{3}, \quad y = \frac{2+2t}{3}$

$\frac{2+2t}{3} = \frac{t-2}{3}$

$t = -4$

(d) $\log\left(\frac{p}{r}\right) = \log\left(\frac{p}{q}\cdot\frac{q}{r}\right)$

$= \log\left(\frac{p}{q}\right) + \log\left(\frac{q}{r}\right)$

$= 4\cdot 6$

(e) $u = e^x, \quad x = 0, \quad u = 1$

$\frac{du}{dx} = e^x, \quad x = \ln 5, \quad u = 5$

$\int_0^{\ln 5} \frac{1}{1+e^{-x}}\,dx = \int_1^5 \frac{1}{1+u^{-1}}\,\frac{du}{u}$

$= \int_1^5 \frac{1}{u+1}\,du$

$= \left[\ln(u+1)\right]_1^5$

$= \ln 6 - \ln 2$

$= \ln 3$

Question 2

(a) (i) Angle between tangent to a circle and a chord drawn to the point of contact is equal to the angle in the alternate segment.

(ii) Let $\angle PTB = y°$.

Then $\angle PCB = y°$ (alt. seg.)

$\angle APB = (y-x)°$, by subtraction

$\angle CPD = (y-x)°$ (ext. $\angle$, ΔCPD)

$\therefore \angle APB = \angle CPD$.

(b) (i) $x = A\cos nt$

$v = -nA\sin nt$

$a = -n^2 A\cos nt$

(ii) $-36 = A\cos\pi$

$A = 36$

$9 = -n^2(36)\cos\pi$

$n^2 = \frac{1}{4}$

$n = \frac{1}{2}$

Amplitude = 36; period = 4π.

(iii) Greatest speed $= \left(\frac{1}{2}\right)(36) = 18$ cm s^{-1}

(c) (i) (1) $\alpha\beta\gamma = -2$

(2) $\beta\gamma + \gamma\alpha + \alpha\beta = -5$

(ii) $\alpha + \beta + \gamma = 1$

$\beta + \gamma = 1 - \alpha$

(iii) $\frac{1-\alpha}{\alpha} + \frac{1-\beta}{\beta} + \frac{1-\gamma}{\gamma} = \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} - 3$

$= \frac{\beta\gamma + \gamma\alpha + \alpha\beta}{\alpha\beta\gamma} - 3$

$= -\frac{1}{2}$

Question 3

(a) (i) Angle at the centre is twice the angle at the circumference.

(ii) $\frac{AN}{r} = \cos\theta$

$AN = r\cos\theta$

$P = 4AN + r(4\theta)$

$= 4r(\theta + \cos\theta)$

(iii) $4r(\theta + \cos\theta) = 5r$

$\theta + \cos\theta = 1\cdot 25$

$\cos\theta + \theta - 1\cdot 25 = 0$

(iv) $f'(\theta) = -\sin\theta + 1$

Approximation $= 0\cdot 3 - \frac{\cos 0\cdot 3 - 0\cdot 95}{1 - \sin 0\cdot 3}$

$= 0\cdot 292$

(b) (i) $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$

$\cos(\alpha+\alpha) = \cos^2\alpha - \sin^2\alpha$

$\cos 2\alpha = \cos^2\alpha - (1 - \cos^2\alpha)$

$= 2\cos^2\alpha - 1$

(ii) Join BC and BD. $\cos\alpha = \frac{6}{d}$, $\cos 2\alpha = \frac{1}{d}$

$\frac{1}{d} = 2\left(\frac{36}{d^2}\right) - 1$

$d = 72 - d^2$

$d^2 + d - 72 = 0$

$(d-8)(d+9) = 0$

$d = 8 \quad (d \neq -9)$

Question 4

(a) Number of outcomes $= {}^{16}C_3 = 560$

(i) Prob. $= \frac{4 \times {}^4C_3}{560} = 0\cdot 03$

(ii) Prob. $= \frac{{}^8C_3}{560} = \frac{56}{560} = 0\cdot 1$

(iii) Prob. $= \frac{{}^8C_3 + {}^8C_3 - {}^4C_3}{560} = 0\cdot 19$

(iv) Prob. $= \frac{{}^8C_2 \times {}^8C_1}{560} = 0\cdot 4$

(b) (i) $x^2 = mx$

$x(x-m) = 0$

$x = m \quad (x \neq 0)$

$y = m$

(ii) $A = \int_0^m \left(mx - x^2\right) dx = \left[\frac{mx^2}{2} - \frac{x^3}{3}\right]_0^m$

$= \frac{m^3}{2} - \frac{m^3}{3}$

$= \frac{m^3}{6}$ units^2

(iii) $\frac{d\theta}{dt} = 0{\cdot}5$

Since $\tan\theta = m$, $A = \frac{\tan^3\theta}{6}$

$$\frac{dA}{dt} = \frac{dA}{d\theta}\cdot\frac{d\theta}{dt}$$

$$= \frac{3\tan^2\theta\sec^2\theta}{6}\times 0{\cdot}5$$

$$= \frac{\tan^2\theta\sec^2\theta}{4}$$

$$= \frac{3\times 4}{4}$$

$= 3$ units^2/s

Question 5

(a) $\text{P(Jack)} = {}^6C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^4 = 0{\cdot}2009\ldots$

$\text{P(Jill)} = {}^{14}C_7\left(\frac{1}{2}\right)^7\left(\frac{1}{2}\right)^7 = 0{\cdot}209\ldots$

Hence Jill has the greater chance.

(b) (i) The positive direction.

(ii) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 2\cos x$

$\frac{1}{2}v^2 = 2\sin x + c$

At $x = 0$, $v = 0$, hence $c = 0$

$v^2 = 4\sin x$

(iii) The first positive root of $\sin x = 0$ is $x = \pi$.

(iv) At $x = \pi$, acceleration < 0, hence the particle will return to $x = 0$ and subsequently oscillate between $x = 0$ and $x = \pi$.

(v) Maximum speed where acceleration is zero.

That is, $x = \frac{\pi}{2}$

$v^2 = 4$

$\therefore$ max. speed $= 2$.

(vi) Since the particle is moving in the positive direction, $v > 0$.

$\therefore \quad \frac{dx}{dt} = 2\sqrt{\sin x}$

$\therefore \quad \frac{dt}{dx} = \frac{1}{2}\sqrt{\operatorname{cosec} x}$

$$2T = \int_{\frac{\pi}{6}}^{\frac{5\pi}{6}} \sqrt{\operatorname{cosec} x}\ dx$$

(vii) $2T \doteqdot \frac{\pi}{9}\left(\sqrt{2} + 4\times 1 + \sqrt{2}\right)$

$\therefore \quad T \doteqdot 1{\cdot}19$

Question 6

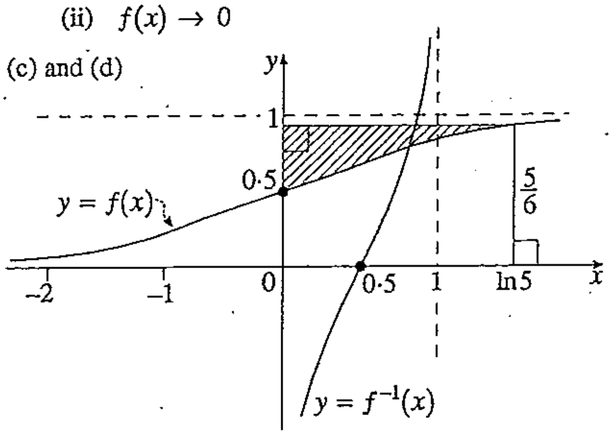
(a) $f'(x) = -\left(1 + e^{-x}\right)^{-2}\left(-e^{-x}\right)$

$= \frac{e^{-x}}{\left(1+e^{-x}\right)^2}$, which is positive for all x.

(b) (i) $f(x) \to 1$

(ii) $f(x) \to 0$

(c) and (d)



(d) $f(x)$ is monotonic, from part (a).

(e) $x = \frac{1}{1+e^{-y}}$

$1 + e^{-y} = \frac{1}{x}$

$e^{-y} = \frac{1-x}{x}$

$e^{y} = \frac{x}{1-x}$

$f^{-1}(x) = \ln\left(\frac{x}{1-x}\right)$

$f^{-1}\left(\frac{1}{2}\right) = \ln 1, \qquad f^{-1}\left(\frac{5}{6}\right) = \ln\left(\frac{5/6}{1/6}\right)$

$= 0 \qquad\qquad = \ln 5$

(f) Required integral is given by the area of the region shaded above, which is

$$\frac{5}{6}\ln 5 - \int_0^{\ln 5} f(x)\ dx = \frac{5}{6}\ln 5 - \ln 3$$

from question 1(e).

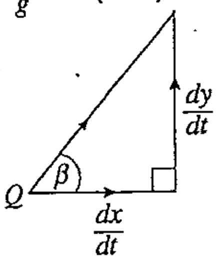
Question 7

(a) (i) $x = Vt\cos\alpha$, $y = Vt\sin\alpha - \frac{1}{2}gt^2$

(ii) When $y = 0$, $t = \frac{2V\sin\alpha}{g}$ $(t \neq 0)$

(iii) $\tan\beta = \frac{dy/dt}{dx/dt}$

$\frac{\sin\beta}{\cos\beta} = \frac{V\sin\alpha - gt}{V\cos\alpha}$



$V\sin\beta\cos\alpha = V\sin\alpha\cos\beta - gt\cos\beta$

$gt\cos\beta = V(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$= V\sin(\alpha - \beta)$

$$t = \frac{V\sin(\alpha-\beta)}{g\cos\beta}$$

(iv) $t = \dfrac{V\sin\frac{\alpha}{2}}{g\cos\frac{\alpha}{2}} = \dfrac{V}{g}\tan\dfrac{\alpha}{2}$

$$\frac{V}{g}\tan\frac{\alpha}{2} = \frac{1}{3}\cdot\frac{2V\sin\alpha}{g}$$

$$3\tan\frac{\alpha}{2} = 2\sin\alpha$$

$$3\tan\frac{\alpha}{2} = \frac{4\tan\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}$$

$$3 = \frac{4}{1+\tan^2\frac{\alpha}{2}} \qquad \left(\tan\frac{\alpha}{2} \neq 0\right)$$

$$3\tan^2\frac{\alpha}{2} = 1$$

$$\tan\frac{\alpha}{2} = \frac{1}{\sqrt{3}}$$

$$\frac{\alpha}{2} = \frac{\pi}{6}$$

$$\alpha = \frac{\pi}{3}$$

(b) (i)

$$\begin{aligned}\binom{n}{r}+\binom{n}{r+1} &= \frac{n!}{r!\,(n-r)!}+\frac{n!}{(r+1)!\,(n-r-1)!}\\ &= \frac{n!}{r!\,(n-r)!}+\frac{n!}{(r+1)!\,(n-r-1)!}\\ &= \frac{n!\left[(r+1)+(n-r)\right]}{(r+1)!\,(n-r)!}\\ &= \frac{n!\,(n+1)}{(r+1)!\,(n-r)!}\\ &= \frac{(n+1)!}{(r+1)!\,(n-r)!}\\ &= \binom{n+1}{r+1}\end{aligned}$$

(ii)

$$\begin{aligned}&\binom{n}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\cdots+\binom{2n}{n}\\ &= \binom{n+1}{0}+\binom{n+1}{1}+\binom{n+2}{2}+\cdots+\binom{2n}{n}\\ &= \binom{n+2}{1}+\binom{n+2}{2}+\cdots+\binom{2n}{n}\\ &= \binom{n+3}{2}+\binom{n+3}{3}+\cdots+\binom{2n}{n}\\ &\;\vdots\\ &= \binom{2n}{n-1}+\binom{2n}{n}\\ &= \binom{2n+1}{n}\end{aligned}$$