HSC Mathematics Extension 1

Practice Paper 8

Question 1

(a) Differentiate $x\sin^{-1}2x$.

(b) Given that $\log_b pq = 1{\cdot}544$ and $\log_b qr = 1{\cdot}113$, find the value of $\log_b\left(\frac{p}{r}\right)$.

(c) If $\tan\theta = m$ and $\tan\phi = 3$, find the value of m if $\theta - \phi = \frac{\pi}{4}$.

(d) Evaluate $\int_0^{\frac{\pi}{3}} \tan^2 x \; dx$.

(e) $P(x) = x^3 - 3x^2 - 3x + 10$

(i) Show that $x = 2$ is one root of $P(x) = 0$.

(ii) What is the product of the other two roots?

(f) Determine $\lim_{x \to 1} \frac{(x^2-1)+\sin(x-1)}{x-1}$.

Question 2

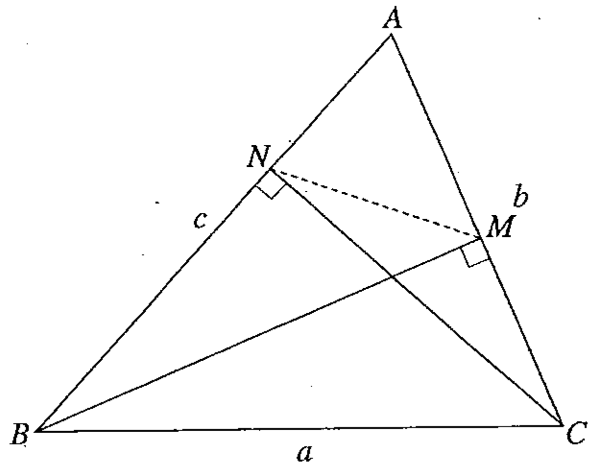
(a) $Q(x) = ax^2 + bx + c$

When $Q(x)$ is divided by either $(x-m)$ or $(x-n)$, the remainder is the same. Prove that, if $m \neq n$, then $(m+n)$ is equal to the sum of the roots of $Q(x) = 0$.

(b) The population of Oldtown, currently 7440, is decreasing at an annual rate of 5%. In Newtown, current population 1500, the population is increasing at an annual rate of 5%.

If these trends continue, in how many year's time will the populations be the same, and what will that population be?

(c)



Triangle ABC has sides of length a, b, c as shown. BM is perpendicular to AC and CN is perpendicular to AB.

(i) Show that $AM = c\cos A$ and $AN = b\cos A$.

(ii) Hence, using the cosine rule, prove that $MN = a\cos A$.

Question 3

(a) Prove that $15^n + 2^{3n} - 2$ is a multiple of seven for all positive integers n.

(b) (i) Express $\sqrt{3}\cos 2t - \sin 2t$ in the form $A\cos(2t + \alpha)$, with $A > 0$ and $0 < \alpha < \frac{\pi}{2}$.

(ii) Find, in exact form, the general solution to

$$\sqrt{3}\cos 2t - \sin 2t = 1.$$

(c) Two particles, A and B, start simultaneously from points on an x axis. At time t, their positions are given by

$$x(A) = \sqrt{3}\cos 2t \text{ and } x(B) = 1 + \sin 2t.$$

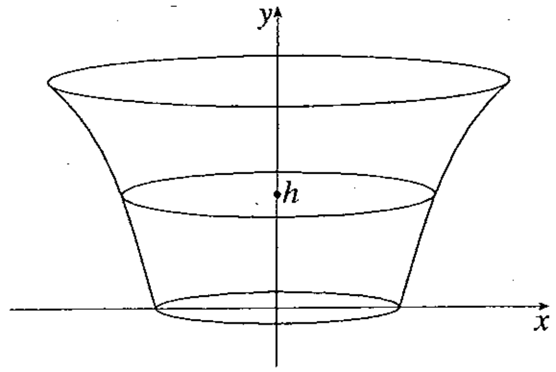
(i) State the amplitudes and periods of the motions.

(ii) Sketch, on one diagram, graphs of both position functions for $0 \le t \le \pi$.

(iii) Find the times in the interval $0 \le t \le \pi$ at which the particles occupy the same position and find their velocities at these times.

Question 4

(a)



The diagram shows a bowl whose curved sides have been made by rotating part of the curve $x^2 - y^2 = 36$ about the y axis, the scales on both axes being cm.

The bowl contains water to a depth of h cm.

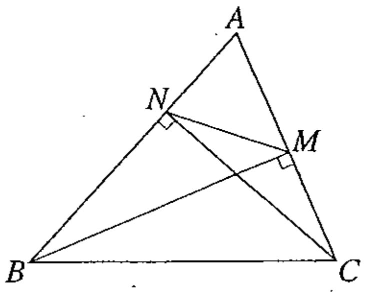
(i) Prove that the volume, V cm^3, of water is given by

$$V = \pi\left(36h + \frac{h^3}{3}\right).$$

(ii) The water is entering the bowl at a rate of 1 L/minute.
Find the rate of increase in the depth when

1. the depth is 2 cm
2. the surface area of the water is 200 cm^2.

(b)



BM is perpendicular to AC and CN is perpendicular to AB.

(i) State why $BNMC$ is a cyclic quadrilateral.

(ii) Prove that ΔABC and ΔAMN are similar.

(iii) Hence prove that if $BC = a$, $MN = a\cos A$.

Question 5

(a) $f(n) = \sum_{r=1}^{n}\left(2r + 2^{r-1}\right)$ Find $f(n)$.

(b) The bowl in Question 4(a) has a height of 6 cm.

(i) Find its exact capacity.

(ii) When the bowl is half full of water, the depth is h cm. Prove that

$$h^3 + 108h - 432 = 0.$$

(iii) Show that h lies between 3 and 4, and, taking $h = 3$ as a first approximation, use Newton's method to find a better one.

(c) A wholesale fruiterer buys mangos from two farms. He markets them in boxes of sixteen, putting eight from each farm in each box.

(i) If three mangos are taken at random from a box, show that the probability of all three coming from the same farm is 0·2.

(ii) If three mangos are taken at random from each of seven boxes, find, correct to three decimal places, the probability that all three come from the same farm

1. on only one occasion
2. on at least two occasions.

Question 6

(a) Consider the function $f(x) = \dfrac{x^3 + 8x}{8}$.

(i) Show that $f(x)$ is an increasing function.

(ii) Find the equation of the tangent to $y = f(x)$ at the origin.

(iii) Using equal scales on the axes, sketch a graph of $y = f(x)$ over the domain $-2 \le x \le 2$ and draw its tangent at the origin.

(iv) Explain why an inverse function exists and draw a graph of $y = f^{-1}(x)$ on your diagram.

(v) Solve $f^{-1}(x) = 8$.

(vi) Evaluate $\int_0^3 f^{-1}(x)\, dx$.

(b) The acceleration, a m s^{-2}, of a particle P, moving in a straight line is given by

$$a = 10x - 2x^3,$$

where x is its displacement from an origin in metres.

The particle is released from rest at A, where $x = 1$.

(i) In which direction will it first move, and why?

(ii) Show that its velocity, v m s^{-1}, is given by

$$v^2 = -9 + 10x^2 - x^4.$$

(iii) Find the position of B, the next point at which P comes to rest, and the subsequent direction of the motion.

(iv) Find P's maximum velocity in moving from A to B.

(v) Julia decided to investigate P's speed when at C, where $x = -2$. She wrote:

'When $x = -2$

$$v^2 = -9 + 40 - 16$$
$$= 15.$$

Hence P's speed will be $\sqrt{15}$ m s^{-1}.'

Do you agree with Julia? (Justify your answer.)

Question 7

(a) Find the exact values of the following integrals, using the given substitutions if required.

(i) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{8\sin x\cos x}{1+4\sin^2 x}\,dx,\qquad u = 4\sin^2 x$

(ii) $\displaystyle\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{8\cos x}{1+4\sin^2 x}\,dx,\qquad v = 2\sin x$

(b) A rocket is fired from a point P on horizontal ground with initial speed $V \text{ m s}^{-1}$ at an angle of elevation of θ. You may assume that, after t seconds its horizontal (x m) and vertical (y m) displacements from P are given by

$$x = Vt\cos\theta, \qquad y = Vt\sin\theta - \frac{1}{2}gt^2.$$

(i) Show that the range, R m, is given by

$$R = \frac{2V^2\sin\theta\cos\theta}{g}.$$

(ii) The angle of projection is fixed, but the speed of projection may be varied. Also, there is a restriction on the flight, this being that it must not rise more than H m above the ground.

Prove that

1. in order to maximise the range, $V^2 = \dfrac{2gH}{\sin^2\theta}$

2. the maximum range is $4H\cot\theta$.

(c) (i) Show that, in the binomial expansion of $\left(x - \frac{1}{x}\right)^{2n}$, the term independent of x is $(-1)^n\, {}^{2n}C_n$.

(ii) Show that $(1+x)^{2n}\left(1 - \frac{1}{x}\right)^{2n} \equiv \left(x - \frac{1}{x}\right)^{2n}$.

(iii) Deduce that

$$\left({}^{2n}C_0\right)^2 - \left({}^{2n}C_1\right)^2 + \left({}^{2n}C_2\right)^2 \cdots + \left({}^{2n}C_{2n}\right)^2 = (-1)^n\, {}^{2n}C_n.$$

Practice Paper 8

Question 1

(a) $f'(x) = \sin^{-1} 2x + x\left(\frac{2}{\sqrt{1-4x^2}}\right)$

$= \sin^{-1} 2x + \frac{2x}{\sqrt{1-4x^2}}$

(b) $\log_b\left(\frac{p}{r}\right) = \log_b\left(\frac{pq}{qr}\right)$

$= \log_b pq - \log_b qr$

$= 0{\cdot}431$

(c) $\tan(\theta - \phi) = 1$

$\frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} = 1$

$\frac{m-3}{1+3m} = 1$

$m - 3 = 1 + 3m$

$m = -2$

(d) $\int_0^{\frac{\pi}{3}} (\sec^2 x - 1)\ dx = \left[\tan x - x\right]_0^{\frac{\pi}{3}}$

$= \left(\sqrt{3} - \frac{\pi}{3}\right) - 0$

$= \sqrt{3} - \frac{\pi}{3}$

(e) (i) $P(2) = 8 - 12 - 6 + 10$

$= 0$

(ii) $2\alpha\beta = -10$

$\alpha\beta = -5$

(f) $\lim_{x\to1}\left(\frac{x^2-1}{x-1} + \frac{\sin(x-1)}{x-1}\right)$

$= \lim_{x\to1}\left(x + 1 + \frac{\sin(x-1)}{x-1}\right)$

$= 2 + 1$

$= 3$

Question 2

(a)

$am^2 + bm + c = an^2 + bn + c$

$am^2 - an^2 + bm - bn = 0$

$a(m-n)(m+n) + b(m-n) = 0$

$a(m+n) + b = 0 \qquad (m - n \neq 0)$

$m + n = -\frac{b}{a}$

$= \alpha + \beta$

(b) After n years, $7440(0{\cdot}95)^n = 1500(1{\cdot}05)^n$

$\frac{7440}{1500} = \left(\frac{1{\cdot}05}{0{\cdot}95}\right)^n$

$\ln\left(\frac{7440}{1500}\right) = n\ln\left(\frac{1{\cdot}05}{0{\cdot}95}\right)$

$n = \frac{\ln\left(\frac{7440}{1500}\right)}{\ln\left(\frac{1{\cdot}05}{0{\cdot}95}\right)}$

$= 16$

Population $= 7440(0{\cdot}95)^{16} = 3275$

(c) (i) In ΔAMB, $\cos A = \frac{AM}{c}$

$AM = c\cos A$

In ΔANC, $\cos A = \frac{AN}{b}$

$AN = b\cos A$

(ii) $MN^2 = AN^2 + AM^2 - 2AN \cdot AM \cdot \cos A$

$= b^2\cos^2 A + c^2\cos^2 A - 2bc\cos^3 A$

$= \cos^2 A\left(b^2 + c^2 - 2bc\cos A\right)$

$= \cos^2 A\left(a^2\right)$

$MN = a\cos A$

Question 3

(a) $15^1 + 2^3 - 2 = 21$

$= 7(3)$

and the result is true for the integer 1.

Let k be an integer for which the result is true.

i.e. $15^k + 2^{3k} - 2 = 7r \qquad (r \text{ an integer})$

Then $15^{k+1} + 2^{3k+3} - 2$

$= 15.15^k + 8.2^{3k} - 2$

$= 15\left(7r - 2^{3k} + 2\right) + 8.2^{3k} - 2$

$= 7(15r) - 7.2^{3k} + 28$

$= 7\left(15r - 2^{3k} + 4\right)$

and the result is also true for the integer $(k+1)$.

Since it is true for the integer 1, it is therefore true for 2 and therefore true for 3, etc. (i.e. it is true for all positive integers).

(b) (i) $\sqrt{3}\cos 2t - \sin 2t = 2\left(\frac{\sqrt{3}}{2}\cos 2t - \frac{1}{2}\sin 2t\right)$

$= 2\left(\cos 2t\cos\frac{\pi}{6} - \sin 2t\sin\frac{\pi}{6}\right)$

$= 2\cos\left(2t + \frac{\pi}{6}\right)$

(ii) $2\cos\left(2t + \frac{\pi}{6}\right) = 1$

$\cos\left(2t + \frac{\pi}{6}\right) = \frac{1}{2}$

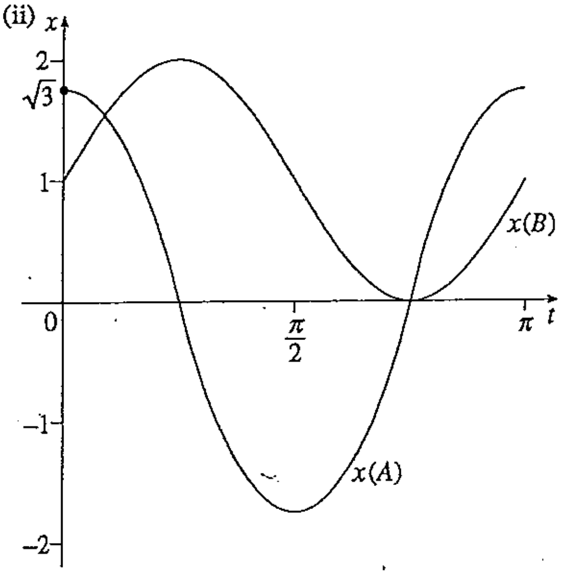
$2t + \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{3}$

$2t = 2n\pi + \frac{\pi}{6}$ or $2t = 2n\pi - \frac{\pi}{2}$

$t = n\pi + \frac{\pi}{12}$ or $t = n\pi - \frac{\pi}{4}$

(c) (i) For A: Amp. $= \sqrt{3}$ Period $= \pi$
For B: Amp. $= 1$ Period $= \pi$

(ii)



(iii)
$$\sqrt{3}\cos 2t = 1+\sin 2t$$
$$\sqrt{3}\cos 2t - \sin 2t = 1$$
From (b)(ii), $t = 0+\frac{\pi}{12}$ or $t = \pi - \frac{\pi}{4}$
$= \frac{\pi}{12}$ or $= \frac{3\pi}{4}$
$$v_A = -2\sqrt{3}\sin 2t;\ v_B = 2\cos 2t.$$
When $t = \frac{\pi}{12}$, $v_A = -2\sqrt{3}\left(\frac{1}{2}\right)$ $v_B = 2\left(\frac{\sqrt{3}}{2}\right)$
$= -\sqrt{3}$ $= \sqrt{3}$
When $t = \frac{3\pi}{4}$ $v_A = -2\sqrt{3}(-1)$ $v_B = 2(0)$
$= 2\sqrt{3}$ $= 0$

Question 4

(a) (i)
$$V = \pi\int_0^h x^2\,dy$$
$$= \pi\int_0^h \left(36+y^2\right)\,dy$$
$$= \pi\left[36y+\frac{y^3}{3}\right]_0^h$$
$$= \pi\left(36h+\frac{h^3}{3}\right)$$

(ii)
$$\frac{dV}{dt} = 1000$$
$$\frac{dh}{dt} = \frac{dh}{dV}\cdot\frac{dV}{dt}$$
$$= \frac{1000}{\pi\left(36+h^2\right)}$$

1. $$\frac{dh}{dt} = \frac{1000}{\pi(36+4)}$$
$$= \frac{25}{\pi}\text{ cm/m}$$

2. Surface area $= \pi x^2$
$$= \pi\left(36+h^2\right)$$
$$\therefore \quad \frac{dh}{dt} = \frac{1000}{200}$$
$$= 5\text{ cm/m}$$

(b) (i) BC subtends equal angles at M and N.

(ii) $\angle AMN = \angle ABC$ (ext. $\angle$ cyclic quad.)
$\angle ANM = \angle ACB$ (" " ")
$\angle A$ is common
$\therefore$ Δs ABC and AMN are equiangular.

(iii)
$$\frac{MN}{BC} = \frac{AM}{AB} \quad (\text{similar } \Delta\text{s})$$
$$\frac{MN}{a} = \cos A$$
$$MN = a\cos A$$

Question 5

(a)
$$f(n) = (2+4+\cdots+2n)+\left(1+2+\cdots+2^{n-1}\right)$$
$$= \frac{n}{2}(2+2n)+\frac{1\left(2^n-1\right)}{2-1}$$
$$= n(n+1)+2^n-1$$

(b) (i) $V = \pi\left(216+\frac{6^3}{3}\right) = 288\pi$

(ii)
$$\pi\left(36h+\frac{h^3}{3}\right) = 144\pi$$
$$108h+h^3 = 432$$
$$h^3+108h-432 = 0$$

(iii) $3^3+108(3)-432 = -81 < 0$
$4^3+108(4)-432 = 64 > 0$
Hence a root lies between 3 and 4.
$f'(h) = 3h^2+108$
$f'(3) = 135$
Approximation $= 3-\frac{-81}{135}$
$= 3{\cdot}6$

(c) (i) $P_7 = \frac{{}^8C_3+{}^8C_3}{{}^{16}C_3} = 0{\cdot}2$

(ii) 1. $P = {}^7C_1(0{\cdot}2)(0{\cdot}8)^6$
$= 0{\cdot}367$

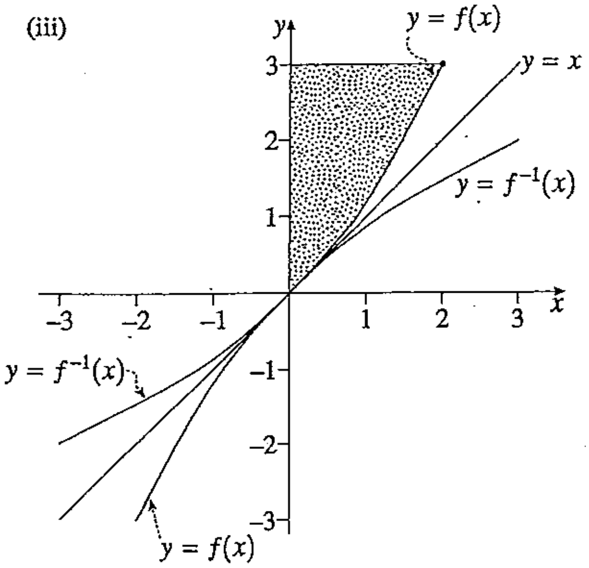
2. P(no occasions) $= (0{\cdot}8)^7$
P(at least two) $= 1-\left(0{\cdot}367+0{\cdot}8^7\right)$
$= 0{\cdot}423$

Question 6

(a) (i) $f'(x) = \frac{3x^2+8}{8} > 0$ for all x

(ii) $f'(0) = \frac{8}{8} = 1$
Tangent: $y = x$

(iii)



(iv) $f(x)$ is monotonic

(v) $x = f(8) = 72$

(vi) The integral measures the area of the shaded region $= 6 - \int_0^2 \frac{x^3+8x}{8}\,dx$

$$= 6 - \frac{1}{8}\left[\frac{x^4}{4} + 4x^2\right]_0^2$$

$$= 6 - \frac{1}{8}(4+16)$$

$$= 3 \cdot 5$$

(b) (i) Positive, since it is at rest at $x = 1$, where acceleration (and hence force) is in the positive direction.

(ii) $$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 10x - 2x^3$$

$$\frac{1}{2}v^2 = \int\left(10x - 2x^3\right)dx$$

$$\frac{1}{2}v^2 = 5x^2 - \frac{1}{2}x^4 + c$$

$v = 0$ at $x = 1$,

$$\therefore \quad 0 = 5 - \frac{1}{2} + c$$

$$\frac{1}{2}v^2 = 5x^2 - \frac{1}{2}x^4 - \frac{9}{2}$$

$$v^2 = -9 + 10x^2 - x^4$$

(iii) $v = 0$: $$x^4 - 10x^2 + 9 = 0$$

$$\left(x^2-1\right)\left(x^2-9\right) = 0$$

$$(x-1)(x+1)(x-3)(x+3) = 0$$

From (i), B is at $x = 3$, where acc. < 0 and P will move in the negative direction.

(iv) Acc. = 0: $2x\left(5 - x^2\right) = 0$

$$x = \sqrt{5} \text{ (since } 1 < x < 3)$$

$$v^2 = -9 + 50 - 25$$

$$= 16$$

$\therefore$ $v = 4$ (since P is moving in the positive direction)

(v) No. P will return to A, where the original situation occurs and P will oscillate between A and B. It will not reach C.

Question 7

(a) (i) $$u = 4\sin^2 x$$

$$du = 8\sin x\cos x\,dx$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{8\sin x\cos x}{1+4\sin^2 x}\,dx = \int_1^3 \frac{1}{1+u}\,du$$

$$= \Big[\ln(1+u)\Big]_1^3$$

$$= \ln 4 - \ln 2$$

$$= \ln 2$$

(ii) $$v = 2\sin x$$

$$dv = 2\cos x\,dx$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{8\cos x}{1+4\sin^2 x}\,dx = 4\int_1^{\sqrt{3}} \frac{1}{1+v^2}\,dv$$

$$= 4\left[\tan^{-1} v\right]_1^{\sqrt{3}}$$

$$= 4\left(\frac{\pi}{3} - \frac{\pi}{4}\right)$$

$$= \frac{\pi}{3}$$

(b) (i) When $y = 0$, $Vt\sin\theta - \frac{1}{2}gt^2 = 0$

$$V\sin\theta - \frac{1}{2}gt = 0 \qquad (t \neq 0)$$

$$t = \frac{2V\sin\theta}{g}$$

$$R = V\cos\theta\left(\frac{2V\sin\theta}{g}\right)$$

$$= \frac{2V^2\sin\theta\cos\theta}{g}$$

(ii) 1. $$\frac{dy}{dt} = V\sin\theta - gt$$

When $\frac{dy}{dt} = 0$, $t = \frac{V\sin\theta}{g}$

and $$y = \frac{V^2\sin^2\theta}{g} - \frac{V^2\sin^2\theta}{2g}$$

$$= \frac{V^2\sin^2\theta}{2g}$$

$$V^2 = \frac{2gy}{\sin^2\theta}$$

Since θ is fixed, R will be maximum when V^2 is maximum, and V^2 will be maximum when $y = H$.

$$\therefore \quad V^2 = \frac{2gH}{\sin^2\theta}.$$

2. $$R = \frac{4gH\sin\theta\cos\theta}{g\sin^2\theta}$$

$$= 4H\cot\theta$$

(c) (i)

$$T_{r+1} = {}^{2n}C_r\, x^{2n-r}\left(-\frac{1}{x}\right)^r$$
$$= (-1)^r\, {}^{2n}C_r\, x^{2n-2r}$$
$$2n-2r = 0$$
$$r = n$$

and $T_{n+1} = (-1)^n\, {}^{2n}C_n$

(ii) $$(1+x)^{2n}\left(1-\frac{1}{x}\right)^{2n} \equiv \left(1-\frac{1}{x}+x-1\right)^{2n}$$
$$\equiv \left(x-\frac{1}{x}\right)^{2n}$$

(iii) $$(1+x)^{2n}\left(1-\frac{1}{x}\right)^{2n}$$
$$= \left({}^{2n}C_0 + {}^{2n}C_1\, x + {}^{2n}C_2\, x^2 + \cdots + {}^{2n}C_{2n}\, x^{2n}\right)$$
$$\times\left({}^{2n}C_0 - {}^{2n}C_1 \frac{1}{x} + {}^{2n}C_2 \frac{1}{x^2} + \cdots + {}^{2n}C_{2n} \frac{1}{x^{2n}}\right)$$

Equating terms independent of x in parts (i) and (iii),

$$\left({}^{2n}C_0\right)^2 - \left({}^{2n}C_1\right)^2 + \cdots + \left({}^{2n}C_{2n}\right)^2$$
$$= (-1)^n\, {}^{2n}C_n.$$