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HSC Mathematics Extension 1

Practice Paper 9

Question 1

(a) Differentiate $\tan^{-1}(\sin x)$.

(b) Evaluate $\log_8 e$, correct to two decimal places.

(c) Find the point dividing the interval from $(-3, 4)$ to $(5, -2)$ in the ratio $1:3$.

(d) Evaluate $\displaystyle\int_0^3 \frac{1}{\sqrt{16-x^2}}\,dx$, correct to three decimal places.

(e) How many different arrangements can be made using the seven letters of the word ARRANGE?

(f) Evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \left(\cos^2 x - \sin^2 x\right)\,dx$.

Question 2

(a) Let $f(x) = \dfrac{x^6+3}{x^2+1}$.

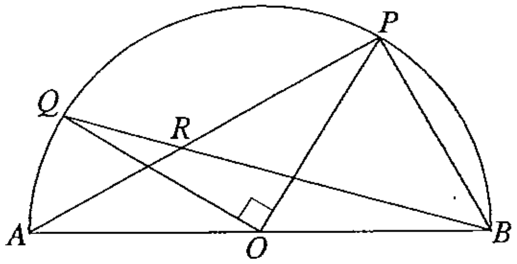
(i) By division, or otherwise, express $f(x)$ in the form

$$P(x) + \frac{R}{x^2+1},$$

where $P(x)$ is a polynomial and R is a constant.

(ii) Evaluate $\displaystyle\int_0^1 f(x)\,dx$.

(b)



O is the centre of a semicircle, diameter AB.
OP and OQ are perpendicular; AP and BQ intersect at R.

(i) Explain why $\angle PBQ = 45°$.

(ii) Prove that $PB = PR$.

(c) Use the substitution $u = \sqrt{x}$ to evaluate

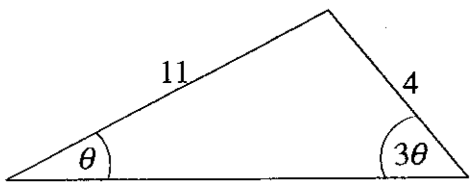
$$\int_1^{25} \frac{1}{x+\sqrt{x}}\, dx.$$

Question 3

(a) (i) Using standard expressions for $\sin 2\theta$ and $\cos 2\theta$, prove that

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

(ii)



Find θ to the nearest minute.

(b) (i) Expand $\left(\sqrt{x}-1\right)^4$.

(ii) The curve $y = \left(\sqrt{x}-1\right)^2$ meets the x axis at A, the y axis at B, and O is the origin.

The region enclosed by OA, OB and the arc AB makes a revolution about the x axis. Find the volume of the solid generated.

(iii) Express $\left(\sqrt{2}-1\right)^4$ in the form $a - b\sqrt{2}$, where a and b are rational.

(iv) How does part (iii) provide the approximation

$$\sqrt{2} \doteqdot \frac{17}{12}?$$

Question 4

(a) The acceleration of a particle, moving in a straight line, is, when x m from an origin, $-4x$ m s^{-2}.

(i) Show that the position function $x = A\cos(2t+\alpha)$, where t is the time in seconds and A and α constants, will provide such acceleration.

(ii) If its velocity is v m s^{-1}, prove that

$$v^2 = 4\left(A^2 - x^2\right).$$

(iii) Initially, the particle was at $x = 3$ with velocity 8 m s^{-1}.
Find the values of A (given that it is positive), $\cos\alpha$ and $\sin\alpha$.

(iv) Find the exact position of the particle when $t = \frac{\pi}{8}$.

(b) A sequence of numbers is defined by $u_1 = \frac{1}{3}$, and, if n is any positive integer,

$$u_{n+1} = \frac{1+3u_n}{3+u_n}.$$

(i) Find u_2.

(ii) Prove, by induction, that $u_n = \dfrac{2^n - 1}{2^n + 1}$.

Question 5

(a) Let $f(p) = 5p^5 - 5p^4 + 0{\cdot}2$, where $0 \le p \le 1$.

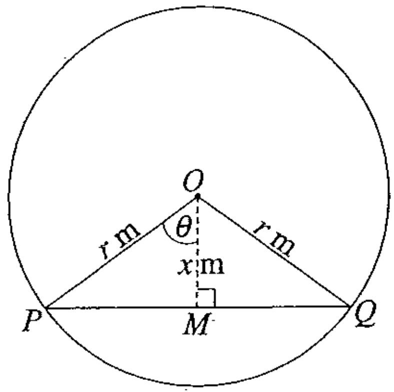
Sketch a graph of $y = f(p)$, showing any stationary points and the values of the function at the extremities of its domain.

(b) An event has probability of success in a single trial of p. It is required that the probability of exactly four successes in a sequence of five trials should be 0·2.

(i) Prove that $5p^5 - 5p^4 + 0{\cdot}2 = 0$.

(ii) Use your graph in part (a) to show that there are two possible values of p, and by taking suitable first approximations to them, use Newton's method to find approximations to two decimal places.

(c)



O is the centre of a fixed circle, radius r m. OM is perpendicular to PQ.
$OM = x$ m and $\angle POM = \theta$ radians.
The area of the minor segment formed by PQ is A m^2.

(i) Express A in terms of r and θ and prove that

$$\frac{dA}{d\theta} = 2r^2 \sin^2 \theta.$$

(ii) The length of OM is increasing at a constant rate of R m s^{-1}.
Prove that

$$\frac{d\theta}{dt} = -\frac{R}{r \sin \theta}.$$

(iii) Find the rate at which A is changing when $\theta = \frac{\pi}{6}$.

Question 6

(a) Let $f(x) = 3 + \sqrt{x-1}$.

(i) State the domain of $f(x)$.

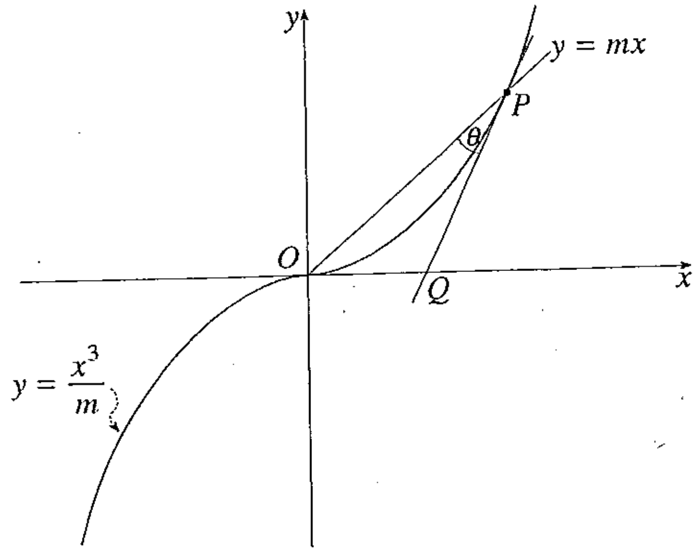
(ii) Prove that $f(x)$ is an increasing function and hence state its range.

(iii) Since $f(x)$ is monotonic, an inverse function exists. State the domain and range of $f^{-1}(x)$.

(iv) Find $f^{-1}(x)$ and sketch, on one number plane, graphs of $y = f^{-1}(x)$ and $y = f(x)$.

(v) Find where the graphs in (iv) intersect.

(b)



The diagram shows the graphs of $y = mx$ and $y = \frac{x^3}{m}$, where $m > 0$, intersecting at P.

The tangent to $y = \frac{x^3}{m}$ at P meets the x axis at Q and $\angle OPQ = \theta$.

(i) Find the coordinates of P.

(ii) Prove that $\tan\theta = \frac{2m}{1+3m^2}$.

(iii) Prove that, as m varies, $\tan\theta$ (and, hence, θ) has a maximum value and that, when this maximum occurs, $OQ = QP$.

Question 7

(a) A projectile is fired from a point on horizontal ground with initial speed V m s^{-1} at an angle of elevation of θ.

Its horizontal and vertical displacements from its firing point are x m and y m and its equations of motion are $\ddot{x} = 0$ and $\ddot{y} = -g$.

(i) What are the initial values of $\dot{x}$ and $\dot{y}$?

(ii) By writing its vertical acceleration as $\frac{d}{dy}\left(\frac{1}{2}\dot{y}^2\right)$, prove that

$$\dot{y}^2 = V^2\sin^2\theta - 2gy.$$

(iii) If, when y m above the ground, its speed is S m s^{-1}, prove that

$$S^2 = V^2 - 2gy.$$

(b) The mass, M, of a radioactive element decreases at a rate proportional to the mass,

i.e. $$\frac{dM}{dt} = -kM, \text{ where } k \text{ is a constant.}$$

(i) Show that the function $M = M_0 e^{-kt}$, where M_0 is the initial mass, provides such a rate.

(ii) The 'half-life' period of an element is the time taken for any given mass to be reduced by half. If the half-life period is T, prove that

$$k = \frac{\ln 2}{T}.$$

(iii) A substance contains two radioactive elements A and B, with half-life periods T_A and T_B, where $T_A > T_B$. Initially the mass of B present was twice that of A.

Prove that the substance will contain the same mass of both elements after a time of

$$\frac{T_A T_B}{T_A - T_B}.$$

Practice Paper 9

Question 1

(a) $f'(x) = \dfrac{\cos x}{1+\sin^2 x}$

(b) $\log_8 e = \dfrac{\ln e}{\ln 8}$
$= \dfrac{1}{\ln 8}$
$= 0{\cdot}48$

(c) $x = \dfrac{-9+5}{4} = -1, \quad y = \dfrac{12-2}{4} = 2\frac{1}{2}$

(d) $\displaystyle\int_0^3 \frac{1}{\sqrt{16-x^2}}\,dx = \left[\sin^{-1}\frac{x}{4}\right]_0^3$
$= \sin^{-1} 0{\cdot}75$
$= 0{\cdot}848$

(e) $\dfrac{7!}{2!\,2!} = 1260$

(f) $\displaystyle\int_0^{\frac{\pi}{6}} \cos 2x\,dx = \frac{1}{2}\Big[\sin 2x\Big]_0^{\frac{\pi}{6}}$
$= \dfrac{1}{2}\left(\dfrac{\sqrt{3}}{2} - 0\right)$
$= \dfrac{\sqrt{3}}{4}$

Question 2

(a) (i) $f(x) = \dfrac{(x^6+1)+2}{x^2+1}$
$= \dfrac{(x^2+1)(x^4-x^2+1)}{x^2+1} + \dfrac{2}{x^2+1}$
$= x^4 - x^2 + 1 + \dfrac{2}{x^2+1}$

(ii) $\displaystyle\int_0^1 f(x)\,dx = \left[\frac{x^5}{5} - \frac{x^3}{3} + x + 2\tan^{-1}x\right]_0^1$
$= \dfrac{1}{5} - \dfrac{1}{3} + 1 + 2\tan^{-1}1$
$= \dfrac{13}{15} + \dfrac{\pi}{2}$

(b) (i) The angle at the centre is twice the angle at the circumference.

(ii) $\angle APB = 90°$ ($\angle$ in semicircle)
$\angle PBR = 45°$ (proved)
$\therefore \angle PRB = 45°$ ($\angle$ sum of Δ)
$\therefore PB = PR$ (opp. equal angles)

(c) $u = x^{\frac{1}{2}}$
$du = \dfrac{1}{2\sqrt{x}}\,dx = \dfrac{1}{2u}\,dx$
$dx = 2u\,du$
$\displaystyle\int_1^{25} \frac{1}{x+\sqrt{x}}\,dx = \int_1^5 \frac{2u}{u^2+u}\,du$
$\displaystyle= \int_1^5 \frac{2}{u+1}\,du$
$= 2\Big[\ln(u+1)\Big]_1^5$
$= 2(\ln 6 - \ln 2)$
$= 2\ln 3$

Question 3

(a) (i) $\sin(2\theta+\theta)$
$= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta$
$= 2\sin\theta\cos^2\theta + (1-2\sin^2\theta)\sin\theta$
$= 2\sin\theta(1-\sin^2\theta) + \sin\theta - 2\sin^3\theta$
$= 3\sin\theta - 4\sin^3\theta$

(ii) $\dfrac{11}{\sin 3\theta} = \dfrac{4}{\sin\theta}$
$4(3\sin\theta - 4\sin^3\theta) = 11\sin\theta$
$12 - 16\sin^2\theta = 11 \quad (\sin\theta \neq 0)$
$\sin^2\theta = \dfrac{1}{16}$
$\sin\theta = \dfrac{1}{4} \quad (\sin\theta > 0)$
$\theta = 14°\,29'$

(b) (i) $(\sqrt{x}-1)^4$
$= (\sqrt{x})^4 - 4(\sqrt{x})^3 + 6(\sqrt{x})^2 - 4\sqrt{x} + 1$
$= x^2 - 4x\sqrt{x} + 6x - 4\sqrt{x} + 1$

(ii) $\displaystyle V = \pi\int_0^1 (\sqrt{x}-1)^4\,dx$
$\displaystyle= \pi\int_0^1 \left(x^2 - 4x^{\frac{3}{2}} + 6x - 4x^{\frac{1}{2}} + 1\right)dx$
$= \pi\left[\dfrac{x^3}{3} - \dfrac{8}{5}x^{\frac{5}{2}} + 3x^2 - \dfrac{8}{3}x^{\frac{3}{2}} + x\right]_0^1$
$= \pi\left(\dfrac{1}{3} - \dfrac{8}{5} + 3 - \dfrac{8}{3} + 1\right)$
$= \dfrac{\pi}{15}$ units^3

(iii) $(\sqrt{2}-1)^4 = 4 - 8\sqrt{2} + 12 - 4\sqrt{2} + 1$
$= 17 - 12\sqrt{2}$

(iv) Since $0 < \sqrt{2}-1 < 1$, $(\sqrt{2}-1)^4 \doteqdot 0$
$17 - 12\sqrt{2} \doteqdot 0$
$\sqrt{2} \doteqdot \dfrac{17}{12}$

Question 4

(a) (i) $x = A\cos(2t+\alpha)$
$v = \dfrac{dx}{dt}$
$= -2A\sin(2t+\alpha)$
$a = \dfrac{dv}{dt}$
$= -4[A\cos(2t+\alpha)]$
$= -4x$

(ii) $v^2 = 4A^2\sin^2(2t+\alpha)$
$= 4A^2[1-\cos^2(2t+\alpha)]$
$= 4A^2\left(1 - \dfrac{x^2}{A^2}\right)$
$= 4(A^2 - x^2)$

(iii) When $t = 0$, $x = 3$ and $v = 8$.
$64 = 4(A^2 - 9)$
$A^2 = 25$
$A = 5 \quad (A > 0)$
$3 = 5\cos\alpha$ and $8 = -10\sin\alpha$
$\cos\alpha = \dfrac{3}{5} \qquad \sin\alpha = -\dfrac{4}{5}$

(iv) $x = 5\cos\left(\frac{\pi}{4}+\alpha\right)$

$= 5\left(\cos\frac{\pi}{4}\cos\alpha - \sin\frac{\pi}{4}\sin\alpha\right)$

$= 5\left(\frac{1}{\sqrt{2}}\cdot\frac{3}{5}+\frac{1}{\sqrt{2}}\cdot\frac{4}{5}\right)$

$= 5\left(\frac{7}{5\sqrt{2}}\right)$

$= \frac{7}{\sqrt{2}}.$

(b) (i) $u_2 = \dfrac{1+3\left(\frac{1}{3}\right)}{3+\frac{1}{3}} = \dfrac{6}{10} = \dfrac{3}{5}$

(ii) $\dfrac{2^1-1}{2^1+1} = \dfrac{1}{3}$,

and the result is true for the integer 1.

Let k be an integer for which the result is true.

i.e. $u_k = \dfrac{2^k-1}{2^k+1}$

Then. $u_{k+1} = \dfrac{1+3\left(\frac{2^k-1}{2^k+1}\right)}{3+\frac{2^k-1}{2^k+1}}$

$= \dfrac{2^k+1+3\cdot 2^k-3}{3\cdot 2^k+3+2^k-1}$

$= \dfrac{4\cdot 2^k-2}{4\cdot 2^k+2}$

$= \dfrac{2\left(2\cdot 2^k-1\right)}{2\left(2\cdot 2^k+1\right)}$

$= \dfrac{2^{k+1}-1}{2^{k+1}+1}$,

and the result is also true for the integer $(k+1)$.

Since it is true for the integer 1, it is therefore true for 2 and therefore true for 3, etc.

Question 5

(a) $f(p) = 5p^5-5p^4+0{\cdot}2$

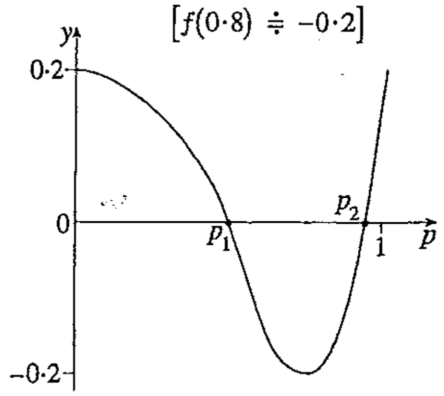
$f(0) = 0{\cdot}2; \qquad f(1) = 0{\cdot}2$

$f'(p) = 25p^4-20p^3$

$= 5p^3(5p-4)$

Stationary when $p = 0$ and $p = 0{\cdot}8$

$\left[f(0{\cdot}8) \doteqdot -0{\cdot}2\right]$



(b) (i) ${}^5C_1\, p^4(1-p) = 0{\cdot}2$

$5p^4-5p^5 = 0{\cdot}2$

$5p^5-5p^4+0{\cdot}2 = 0$

(ii) Take $p_1 \doteqdot 0{\cdot}5$:

$0{\cdot}5-\dfrac{5(0{\cdot}5)^5-5(0{\cdot}5)^4+0{\cdot}2}{25(0{\cdot}5)^4-20(0{\cdot}5)^3} \doteqdot 0{\cdot}55$

Take $p_2 \doteqdot 1$: $1-\dfrac{5(1)-5(1)+0{\cdot}2}{25(1)-20(1)} = 0{\cdot}96$

(c) (i) $A = \dfrac{r^2}{2}(2\theta-\sin 2\theta)$

$\dfrac{dA}{d\theta} = \dfrac{r^2}{2}(2-2\cos 2\theta)$

$= r^2(1-\cos 2\theta)$

$= 2r^2\sin^2\theta$

(ii) $\dfrac{dx}{dt} = R$

$x = r\cos\theta$

$\therefore \quad \dfrac{dx}{d\theta} = -r\sin\theta$

Now $\dfrac{d\theta}{dt} = \dfrac{d\theta}{dx}\cdot\dfrac{dx}{dt}$

$= -\dfrac{R}{r\sin\theta}$

(iii) $\dfrac{dA}{dt} = \dfrac{dA}{d\theta}\cdot\dfrac{d\theta}{dt}$

$= 2r^2\sin^2\theta\left(-\dfrac{R}{r\sin\theta}\right)$

$= -2rR\sin\theta$

$= -2rR\left(\frac{1}{2}\right)$

$= -rR \text{ m}^2\text{ s}^{-1}$

Question 6

(a) (i) $x \geq 1$

(ii) $f'(x) = \frac{1}{2}(x-1)^{-\frac{1}{2}}$

$= \dfrac{1}{2\sqrt{x-1}} > 0$ for all $x > 1$.

Since $f(1) = 3$,

$f(x) \geq 3$

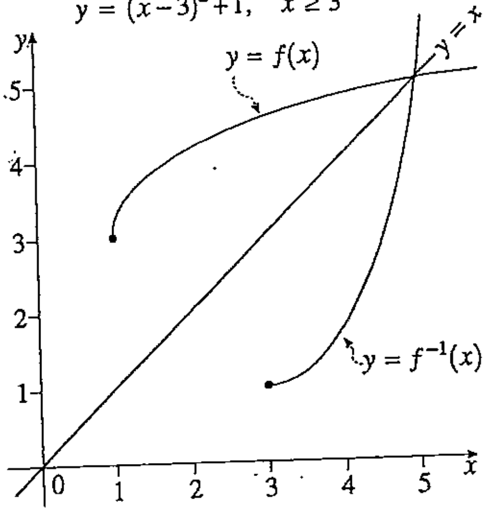
(iii) Domain: $x \geq 3$ Range: $f^{-1}(x) \geq 1$

(iv) $x = 3+\sqrt{y-1}$

$\sqrt{y-1} = x-3$

$y-1 = (x-3)^2, \qquad x \geq 3$

$y = (x-3)^2+1, \quad x \geq 3$



(v) $(x-3)^2+1 = x$

$x^2-7x+10 = 0$

$(x-5)(x-2) = 0$

$x = 5 \qquad (x \geq 3)$

$y = 5$

(b) (i) $\left.\begin{aligned} y &= mx \\ y &= \frac{x^3}{m} \end{aligned}\right\}$

$$x^3 = m^2 x$$
$$x(x^2 - m^2) = 0$$
$$x = m \quad (m > 0)$$

i.e. $P(m, m^2)$

(ii)
$$y = \frac{x^3}{m}$$
$$\frac{dy}{dx} = \frac{3x^2}{m}$$
$$= 3m \text{ at } P$$

$$\tan\theta = \left|\frac{3m - m}{1 + 3m^2}\right|$$
$$= \frac{2m}{1 + 3m^2} \quad (m > 0)$$

(iii) Let $f(m) = \dfrac{2m}{1+3m^2}$

$$f'(m) = \frac{(1+3m^2)2 - 2m(6m)}{(1+3m^2)^2}$$
$$= \frac{2(1-3m^2)}{(1+3m^2)^2}$$

When $f'(m) = 0$,

$$1 - 3m^2 = 0$$
$$m = \frac{1}{\sqrt{3}} \quad (m > 0)$$

$f'(0) > 0$ and $f'(1) < 0$

$\therefore$ max. at $m = \dfrac{1}{\sqrt{3}}$

$\left(\text{Hence } \angle POQ = \frac{\pi}{6}\right)$

$$f\left(\frac{1}{\sqrt{3}}\right) = \frac{\frac{2}{\sqrt{3}}}{1+1}$$
$$= \frac{1}{\sqrt{3}}$$

$\therefore$ $\theta = \frac{\pi}{6}$ and $\angle POQ = \theta$

$\therefore$ $OQ = QP$

Question 7

(a) (i) Initial $\dot{x} = V\cos\theta$
Initial $\dot{y} = V\sin\theta$

(ii)
$$\frac{d}{dy}\left(\frac{1}{2}\dot{y}^2\right) = -g$$
$$\frac{1}{2}\dot{y}^2 = -gy + c$$

When $y = 0$, $\dot{y} = V\sin\theta$

$$\frac{1}{2}V^2\sin^2\theta = c$$

$\therefore$
$$\dot{y}^2 = V^2\sin^2\theta - 2gy$$

(iii)
$$S^2 = \dot{x}^2 + \dot{y}^2$$
$$= V^2\cos^2\theta + V^2\sin^2\theta - 2gy$$
$$= V^2(\cos^2\theta + \sin^2\theta) - 2gy$$
$$= V^2 - 2gy$$

(b) (i)
$$\frac{dM}{dt} = -k\left(M_0\,e^{-kt}\right)$$
$$= -kM$$

(ii)
$$\frac{M_0}{2} = M_0\,e^{-kT}$$
$$e^{-kT} = \frac{1}{2}$$
$$e^{kT} = 2$$
$$kT = \ln 2$$
$$k = \frac{\ln 2}{T}$$

(iii) $M_A = M_0\,e^{-k_A t}$; $M_B = 2M_0\,e^{-k_B t}$

$$M_0\,e^{-k_A t} = 2M_0\,e^{-k_B t}$$
$$e^{-k_A t} = 2e^{-k_B t}$$
$$e^{k_B t - k_A t} = 2$$
$$t(k_B - k_A) = \ln 2$$
$$t = \frac{\ln 2}{\frac{\ln 2}{T_B} - \frac{\ln 2}{T_A}}$$
$$= \frac{T_A T_B}{T_A - T_B}$$