Co-ordinate Geometry

1 Shade the region of the plane specified by $|x + 2y| < 4$

2 i) Show that $\tan 75° = 2 + \sqrt{3}$

ii) The lines $y = mx$ and $x = y\sqrt{3}$ meet at an angle of 75°. Find the value of m.

3 A and B are points with co-ordinates (3,0) and (6,0) respectively. A point P moves so that the length of PB is twice that of PA.

i) Find the locus of P

ii) Sketch this locus

4 i) The points A(1,-3), B(10,9) and P(4,1) are collinear. In what ratio does the point P divide the line joining A and B?

ii) In what ratio does B divide the line AP?

5 Find the acute angle between the lines $3x + y + 1 = 0$ and $2y - x + 4 = 0$

6 The curve $y = 4x^5$ is reflected in the y-axis. Find the equation of the resulting curve.

7 A line passing through (7,5) makes a right angle with the line $4x - y = 6$ at the point P. Find the co-ordinates of P.

8 Find the co-ordinates of the point which divides the line joining the points (-1,3) and (5,-7) externally in the ratio 4:3

9 The lines $4x + 3y + 8 = 0$ and $3x + 8y + 7 = 0$ meet at A. Find the equation of the line through A and B(1,9)

10 Sketch the graph $|x| + |2y| = 8$

11 The lines $Ax + By = 11$ and $Bx - Ay = 2$ meet at P(2,1). Find the values of A and B.

12 The lines $y - mx + 3m = 0$ and $my + x - 8 = 0$ intersect at R and meet the x-axis at P and Q respectively.

i) Explain why R lies on a circle on PQ as diameter

ii) Show by referring to a diagram, that when triangle PQR has greatest area, then R lies on the line $x = 5.5$

iii) Find the co-ordinates of R when the area of the triangle PQR is greatest

iv) Show that the greatest area is 6.25 u^2

13 The line $y = 4x + 6$ meets the circle $x^2 + y^2 = 8y$ at A and B. Find the co-ordinates of M, the mid-point of AB.

14 The parabolas $y = x^2$ and $y = (x + 4)^2$ meet at the point P

i) Find the co-ordinates of P

ii) Find the acute angle between the tangents to the curve at P.

15 i) Show that the perpendicular distance from the point (4,5) to the line $y = mx$ is given by $\left|\dfrac{4m - 5}{\sqrt{m^2 + 1}}\right|$

ii) The line $y = mx$ is a tangent to the circle $(x - 4)^2 + (y - 5)^2 = 4$. Explain why $\left|\dfrac{4m - 5}{\sqrt{m^2 + 1}}\right| = 2$ and hence show that m satisfies the equation $12m^2 - 40m + 21 = 0$.

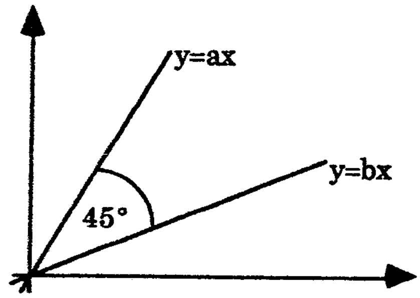
iii) The variable point P moves on the circle $(x - 4)^2 + (y - 5)^2 = 4$. Find the largest and smallest values of

a) angle XOP, where X is the positive x-axis and O is the origin.

b) the length OP.

16 The angle between the lines $y = ax$ and $y = bx$ is 45°.

Show that $b = \dfrac{a - 1}{a + 1}$



ii) $y'' - 4y' = 5y$

$k^2e^{kx} - 4ke^{kx} = 5e^{kx}$

i.e. $e^{kx}(k^2 - 4k - 5) = 0$

i.e. $k^2 - 4k - 5 = 0$, since $e^{kx} \neq 0$

$\therefore$ $k = -1, 5$

10 $I = \int_0^{1/4} \frac{3x}{(1+4x)^3}\,dx$

$u = 1 + 4x$
$du = 4dx$

When $x = 0$, $u = 1$
When $x = \frac{1}{4}$, $u = 2$

$= 3\int_1^2 \frac{\frac{u-1}{4} \cdot \frac{du}{4}}{u^3}$

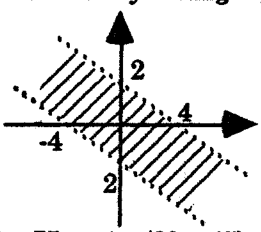
$= \frac{3}{16}\int_1^2 (u^{-2} - u^{-3})\,du$

$= \frac{3}{16}\left[-u^{-1} + \frac{1}{2u^2}\right]_1^2$

$= \frac{3}{128}$

Co-ordinate Geometry

1 The boundaries of the region will be $x + 2y = \pm 4$ and by testing regions we find :



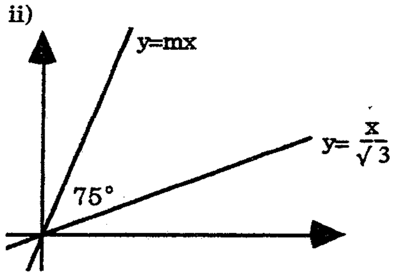
2 i) $\tan 75 = \tan(30 + 45)$

$= \frac{\frac{1}{\sqrt{3}} + 1}{1 - \frac{1}{\sqrt{3}}.1} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

$= \frac{1+\sqrt{3}}{\sqrt{3}-1} \cdot \frac{\sqrt{3}+1}{\sqrt{3}+1}$

$= 2 + \sqrt{3}$

ii)



$\tan\alpha = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$

$\tan 75 = \left|\frac{m - \frac{1}{\sqrt{3}}}{1 + \frac{m}{\sqrt{3}}}\right|$

$2 + \sqrt{3} = \left|\frac{m\sqrt{3} - 1}{\sqrt{3} + m}\right|$

We have two cases :

Case 1 : $2 + \sqrt{3} = \frac{m\sqrt{3} - 1}{\sqrt{3} + m}$

i.e. $2\sqrt{3} + 2m + 3 + m\sqrt{3} = m\sqrt{3} - 1$

i.e. $m = -2 - \sqrt{3}$

Case 2 : $2 + \sqrt{3} = \frac{1 - m\sqrt{3}}{\sqrt{3} + m}$

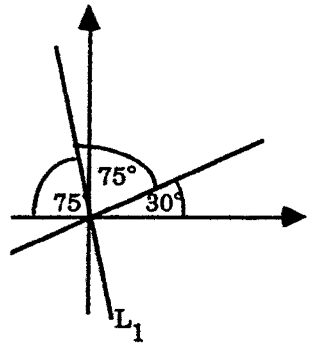
i.e. $2\sqrt{3} + 2m + 3 + m\sqrt{3} = 1 - m\sqrt{3}$

i.e. $2m\sqrt{3} + 2m = -2 - 2\sqrt{3}$

i.e. $m = \frac{-2 - 2\sqrt{3}}{2\sqrt{3} + 2}$

$= -1$

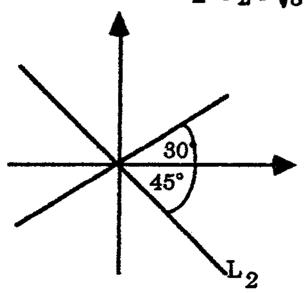
Alternative Method



Gradient of L_1 $= \tan 105$

$= -\tan 75$

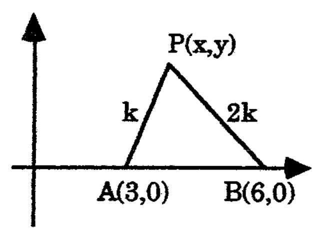
$= -2 - \sqrt{3}$, from i)



Gradient of L_2 $= \tan 135$

$= -1$

3 i)



$PB = 2PA$

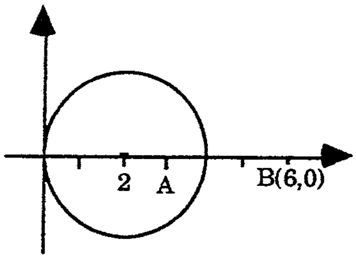
$\sqrt{(x-6)^2 + y^2} = 2\sqrt{(x-3)^2 + y^2}$

$x^2 - 12x + 36 + y^2 = 4(x^2 - 6x + 9) + 4y^2$

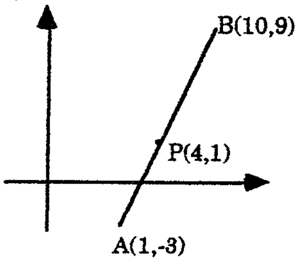
$x^2 - 12x + 36 + y^2 = 4x^2 - 24x + 36 + 4y^2$

$(x-2)^2 + y^2 = 4$

ii)

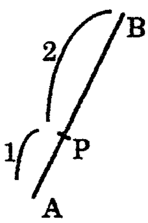


4 i)



$AP = \sqrt{9 + 16}$
$= 5$
$PB = \sqrt{36 + 64}$
$= 10$
$\therefore \frac{AP}{PB} = \frac{5}{10}$
$= \frac{1}{2}$
$\therefore$ Ratio is 1 : 2

ii)

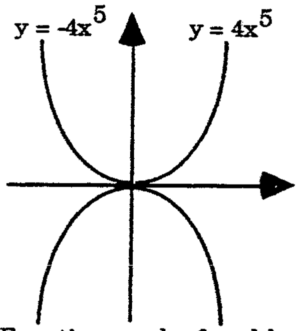


From the diagram, B divides AP externally in the ratio 3:2

5 $3x + y + 1 = 0$
$\therefore y = -3x - 1$
i.e. $m_1 = -3$
$2y - x + 4 = 0$
$\therefore y = \frac{1}{2}x - 2$
i.e. $m_2 = \frac{1}{2}$

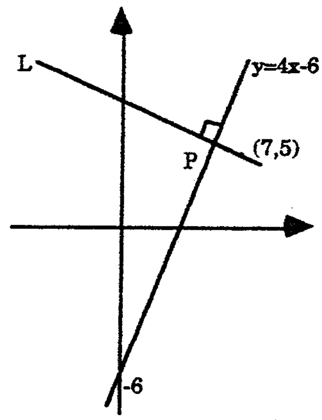
$$\tan\alpha = \left| \frac{-3 - \frac{1}{2}}{1 + -3 \cdot \frac{1}{2}} \right|$$
$= 7$
$\therefore \alpha = 81°52'$

6



Equation can be found by substituting - x instead of x :
$y = 4(-x)^5$
i.e. $y = -4x^5$

7



Gradient of L will be $-\frac{1}{4}$ since it is perpendicular to $y = 4x - 6$

Equation of L : $y - 5 = -\frac{1}{4}(x - 7)$
$x + 4y = 27$

To find P we solve $x + 4y = 27$
and $y = 4x - 6$
i.e. $x + 4(4x - 6) = 27$
$17x = 51$
$\therefore x = 3$
and hence $y = 6$
i.e. P is (3,6)

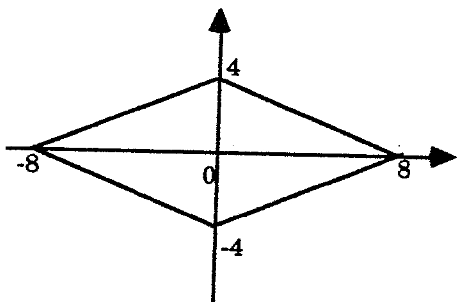
8 (-1,3) (5,-7)
$(x_1\ y_1)$ $(x_2\ y_2)$
4 : - 3
$m_1 : m_2$
$P\left(\frac{4(5) + (-3)(-1)}{4 + -3}, \frac{4(-7) + (-3)(3)}{4 + -3}\right)$
i.e. P(23, -37)

9 Consider the straight line
$4x + 3y + 8 + k(3x + 8y + 7) = 0$ ------------(1)
If (a,b) is the point of intersection of $4x + 3y + 8 = 0$ and $3x + 8y + 7 = 0$ then $4a + 3b + 8 = 0$ and $3a + 8b + 7 = 0$.
Therefore, on substitution we see that (a,b) will certainly solve (1). Hence, (1) represents a straight line through the point of intersection of the original two lines. If it passes through B(1,9) then we can substitute B's co-ordinates :

i.e. $4(1) + 3(9) + 8 + k(3(1) + 8(9) + 7) = 0$
$4 + 27 + 8 + k(3 + 72 + 7) = 0$
$k = \frac{-39}{82}$

$\therefore$ Equation of line is :
$4x + 3y + 8 - \frac{39}{82}(3x + 8y + 7) = 0$
i.e. $211x - 66y + 383 = 0$

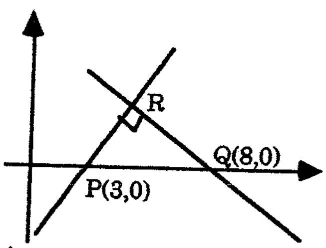
10



11 Since both lines pass through (2,1) we can substitute (2,1) into each equation :
$2A + B = 11$
$2B - A = 2$
Solving simultaneously:
$\therefore$ $A = 4$
$\therefore$ $B = 3$

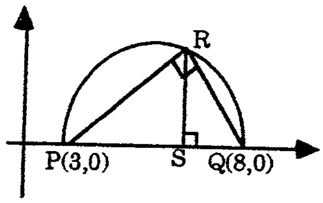
12 The equations are $y = mx - 3m$
and $y = -\frac{1}{m}x + \frac{8}{m}$
The lines are perpendicular
since $(m).(-\frac{1}{m}) = -1$

i)



As m varies, the locus of R is a circle on PQ as diameter because of the converse of the angle in a semi circle is a right angle theorem.

ii)



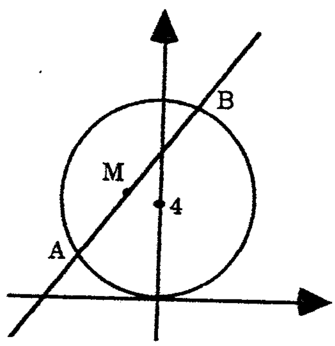
Area of triangle $= \frac{1}{2} . PQ . RS$
$= \frac{5}{2} . RS$

Hence, area will be greatest when RS is greatest.
By symmetry this occurs when S is at the midpoint of PQ i.e. $x = \frac{3+8}{2}$
$= 5.5$

iii) Equation of circle :
$(x - 5.5)^5 + y^2 = 2.5^2$
When $x = 5.5$, $y = \pm 2.5$
$\therefore$ R is (5,5, ± 2.5)

iv) $\text{Max(area)} = \frac{1}{2} . 5 . 2.5$
$= 6.25 \text{ u}^2$

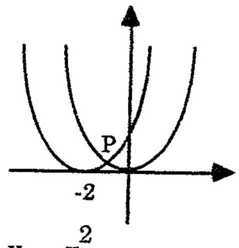
13



$y = 4x + 6$
$x^2 + y^2 = 8y$
Solving simultaneously :
$x^2 + (4x + 6)^2 = 8(4x + 6)$
$17x^2 + 16x - 12 = 0$
The roots of this equation, x_1 and x_2, are the x-co-ordinates of A and B.
We know that $x_1 + x_2 = \frac{-b}{a}$
$= \frac{-16}{17}$

The x-co-ordinate of M is $\frac{x_1 + x_2}{2} = -\frac{8}{17}$, using the mid-point formula.
Substituting $x = -\frac{8}{17}$ into $y = 4x + 6$ gives
$y = -\frac{32}{17} + 6 = \frac{70}{17}$
$\therefore$ M is $(-\frac{8}{17}, \frac{70}{17})$

14 i)



$y = x^2$

$y = (x+4)^2$

$\therefore\ x^2 = (x+4)^2$

$x^2 = x^2 + 8x + 16$

$\therefore\ x = -2$

$\therefore\ y = 4$

P is (-2,4)

ii) $y = x^2$

$y' = 2x$

$= -4$, at $x = -2$

$y = (x+4)^2$

$y' = 2(x+4) = 4$, at $x = -2$

$\tan\alpha = \left| \frac{-4-4}{1+(-4)(4)} \right|$

$= \frac{8}{15}$

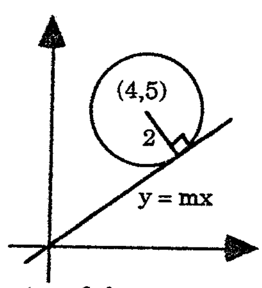
$\therefore\ \alpha = 28°4'$

15 i) $mx - y = 0$ (4,5)

$d = \left| \frac{ax_1 + by_1 + 2}{\sqrt{a^2+b^2}} \right|$

$= \left| \frac{4m-5}{\sqrt{m^2+1}} \right|$, as required

ii)



The centre of the circle is (4,5) and its radius is 2. The radius and the tangent are perpendicular. Hence, if y = mx is a tangent, the perpendicular distance from the centre to y = mx must equal the radius.

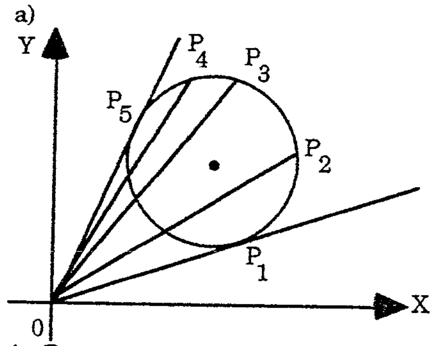
$\therefore\ \left| \frac{4m-5}{\sqrt{m^2+1}} \right| = 2$

Squaring both sides:

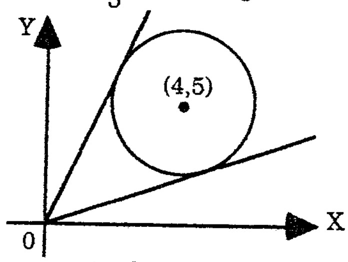
$\frac{16m^2 - 40m + 25}{m^2+1} = 4$

i.e. $12m^2 - 40m + 21 = 0$

iii) a)



As P moves around the circle, the largest and smallest values of angle XOP will occur when OP is a tangent to the circle - at the positions P_1 and P_5 in the diagram



At these points, the slope m of the tangent satisfies $12m^2 - 40m + 21 = 0$

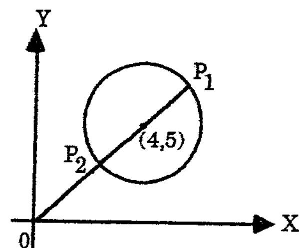
i.e. $m = \frac{40 \pm \sqrt{592}}{24}$

i.e. $m = 0.6529, 2.6805$

i.e. $\tan\alpha = 0.6529, 2.6805$

$\therefore\ \alpha = 33°8', 69°32'$

b)



From the diagram the largest and smallest values of OP occur when OP passes through the centre (4,5)

Distance from (0,0) to (4,5) is $\sqrt{41}$ units. Radius is 2 units

$\therefore\ \max(OP) = \sqrt{41} + 2$

$\min(OP) = \sqrt{41} - 2$

16 $\tan\alpha = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$

$m_1 = a, \quad m_2 = b$

From the diagram $a > b > 0$

$\therefore\ \tan 45 = \frac{a-b}{1+ab}$

$1 = \frac{a-b}{1+ab}$

$1 + ab = a - b$

$b(a+1) = a - 1$

$b = \frac{a-1}{a+1}$