Series and Sequences

1. The nth term of a series is given by $u_n = \frac{1}{n(n+1)}$
   i) Write down expressions for u_k and u_{k+1}
   ii) Assume that the formula for the sum of k terms of this series is given by $S_k = \frac{k}{k+1}$.
   Hence prove that $S_{k+1} = \frac{k+1}{k+2}$
   iii) Explain carefully why the sum of the first n terms of the series is given by $S_n = \frac{n}{n+1}$
   iv) Give a neat sketch of the values of S_n for n = 1, 2, 3, 4, 5, 6
   v) Explain why $S_\infty = 1$
   vi) The limiting sum of the geometric series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ is also 1. Compare and discuss the behaviour of both series as n increases.

2. How many terms of the arithmetic series 96 + 93 + 90 + ... must be taken to give a sum of zero?

3. Show whether 2197 is a member of the arithmetic progression -47, -43, -39, ...

4. i) For what values of x will the geometric progression $1 + \frac{1}{1-x} + \frac{1}{(1-x)^2} + \ldots$ have a limiting sum.
   ii) If x = 3, find this limiting sum.

5. The sum of the first n terms of a certain series is given by $S_n = \frac{n}{3}(n+1)(n+2)$
   i) Show that the nth term is given by $u_n = n(n+1)$
   ii) Find the sum of the second 50 terms.

6. I deposited $1000 in a bank and left it there for 10 years. Interest is compounded at 12% per annum. How much will be in the account if interest is calculated
   i) annually
   ii) twice a year
   iii) quarterly
   iv) monthly

7. I need to have $10 000 in 8 years time. How much must I deposit in a bank account which pays interest at 1% per month in order to reach this goal?

8. A girl deposits $200 at the start of each month in a bank which pays 9% per annum. If the interest is calculated monthly, how much will be in the account at the end of 5 years? Answer to the nearest dollar.

9. A man wishes to have $20 000 capital in 15 years' time. He proposes to make a finance company 15 equal payments commencing now (the last payment being made 1 year before the $20 000 is to be paid to him). Assuming interest is paid by the company at 12% per annum, find what should be the amount of each payment to nearest dollar.

10. A function f is defined as $f(P) = PR - M$. Write down, in terms of P, R and M an expression for
   i) $f(f(P))$
   ii) $f(f(f(P)))$

11. I borrow $200 000 at 18% per annum reducible interest. It is to be repaid over 25 years by equal monthly instalments and the interest is also calculated monthly on the balance outstanding.
   i) Find the value of each instalment.
   ii) Find how much will still be owed after 5 years.
   iii) Find the total amount of interest which will be paid over the life of the loan.
   iv) Express this interest as a flat rate correct to 3 significant figures.

12 A loan of \$50 000 is to be repaid over 20 years by equal monthly instalments. Interest is calculated monthly on the balance owing at a rate of 12% per annum.
- i) Find M the size of each instalment
- ii) Describe what would happen if the borrower paid less than this amount each month.
- iii) The borrower decides to pay twice this amount each month i.e. \$2M each month. How long would it then take to pay off the loan?

13 The first two terms of an arithmetic progression are -96 and -89.
- i) Find an expression for the n'th term
- ii) Find the sum of the first n terms.
- iii) Find the least number of terms which must be added to give a positive sum.

14 The first 3 terms of an arithmetic sequence are 98, 91, 84.
- i) Write down a formula for the nth term.
- ii) The last term of this sequence is - 28. How many terms are there in this sequence?
- iii) Find the sum of the series.

15 Two geometric progressions are given:

A: $1 + (\sqrt{3} - 1) + (\sqrt{3} - 1)^2 + \ldots$

B: $1 + (\sqrt{3} + 1) + (\sqrt{3} + 1)^2 + \ldots$

A new series is formed by taking the product of corresponding terms

i.e. $1 \times 1 + (\sqrt{3} - 1)(\sqrt{3} + 1) + (\sqrt{3} - 1)^2 (\sqrt{3} + 1)^2 + \ldots$
- i) Explain whether this new series is arithmetic, geometric or neither.
- ii) Find the nth term of this series
- iii) Which will be the first term of this series which is greater than 10^6 ?

16 The nth term of a series is given by $u_n = 2n + (\frac{1}{2})^n$. Find the sum of the first 10 terms of this series.

17 Express 0.515151.... as a geometric progression and hence express it as a rational number.

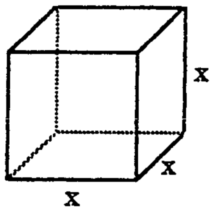
18 What number must be added to each of $\frac{1}{10}, \frac{1}{4}, \frac{9}{20}$ in order to form a geometric progression?

19 The sum of the first two terms of an infinite geometric series is 5 and each term is 3 times the sum of all terms that follow it. Find the series.

20
- i) Find the limiting sum of the geometric progression $\frac{1}{16} + \frac{1}{64} + \ldots$
- ii) When a beam of light hits a certain pane of glass, half of it is reflected, 25% of it is absorbed by the glass and the remainder passes through the glass. Two panes of this glass are placed close together. Show that a total of $\frac{1}{12}$ of the original amount of light passes completely through the two panes of glass.

Also, $\frac{dx}{dt} = \frac{dx}{dL} \cdot \frac{dL}{dt}$
$= \frac{5}{4} \cdot \frac{3}{2}$
$= 1\frac{7}{8} ms^{-1}$

7



Let the side of the cube be x metres.
We know that $\frac{dV}{dt} = 0.1$
We want $\frac{dS}{dt}$ when $x = 4$
We know that $V = x^3$ and $S = 6x^2$
Hence, $\frac{dS}{dt} = \frac{dS}{dV} \cdot \frac{dV}{dt}$
But we don't have a quick method of finding $\frac{dS}{dV}$
So, we expand $\frac{dS}{dV}$:

$$\frac{dS}{dt} = \frac{dS}{dx} \cdot \frac{dx}{dV} \cdot \frac{dV}{dt}$$
$$= 12x \cdot \frac{1}{3x^2} \cdot 0.1$$
$$= 12(4) \cdot \frac{1}{3(4)^2} \cdot 0.1, \text{ when } x = 4$$
$$= 0.1\ m^2 s^{-1}$$

Series and Sequences

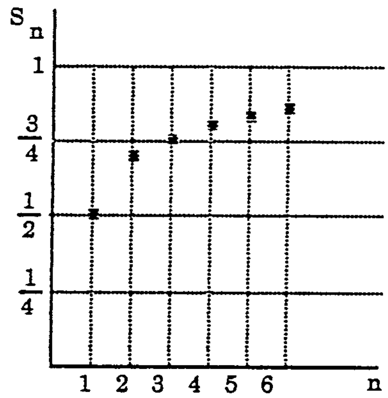
1 i) $u_k = \frac{1}{k(k+1)}$

$u_{k+1} = \frac{1}{(k+1)(k+2)}$

ii) We are told that
$S_k = u_1 + u_2 + ... + u_k$
$= \frac{k}{k+1}$
We need to prove that
$S_{k+1} = (u_1 + u_2 + ... + u_k) + u_{k+1}$
$= \frac{k+1}{k+2}$
i.e. we need to prove that
$\frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k+1}{k+2}$
Now, LHS $= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$
$= \frac{k(k+2)+1}{(k+1)(k+2)}$
$= \frac{(k+1)^2}{(k+1)(k+2)}$
$= \frac{k+1}{k+2}$
= RHS

iii) From $u_n = \frac{1}{n(n+1)}$, we know that $u_1 = \frac{1}{2}$
And obviously $S_1 = u_1$
Now, from $S_n = \frac{n}{n+1}$, we know that
$S_1 = \frac{1}{1+1} = \frac{1}{2}$
So, the formula $S_n = \frac{n}{n+1}$ is true for n = 1.
Also, we know that if the statement S_k is true, then the statement S_{k+1} is true.
So, since the statement is true for n = 1, it must be true for n = 2, and so on for all positive integral values of n.
Hence, $S_n = \frac{n}{n+1}$.
Note: This is a mathematical induction question.

iv)



v) As n -> ∞, n and n+1 become relatively closer together. Hence, $\frac{n}{n+1} \to 1$
Or we could argue:
$S_n = \frac{n}{n+1}$
$= \frac{n/n}{1 + 1/n}$
$= \frac{1}{1+1/n}$, dividing top and bottom by n
As $n \to \infty$, $\frac{1}{n} \to 0$
$\therefore S_n \to \frac{1}{1+0} = 1$
i.e. $S_\infty = 1$

vi) The series $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ...$ is a geometric progression. The first series is not.
Hence, for the **second series**,
$u_n = \frac{1}{2}\left(\frac{1}{2}\right)^{n-1}$
i.e. $u_n = \left(\frac{1}{2}\right)^n$

Also, $S_n = \dfrac{\frac{1}{2}[1 - (\frac{1}{2})^n]}{1 - \frac{1}{2}}$

i.e. $S_n = 1 - (\frac{1}{2})^n$

Thus we have:

	n	1	2	3	4	5	6
First series	$u_n = \frac{1}{n(n+1)}$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{1}{20}$	$\frac{1}{30}$	$\frac{1}{42}$
Second series	$u_n = \frac{1}{2^n}$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$
First series	$S_n = \frac{n}{n+1}$	$\frac{1}{2}$	$\frac{2}{3}$	$\frac{3}{4}$	$\frac{4}{5}$	$\frac{5}{6}$	$\frac{6}{7}$
Second series	$S_n = 1 - (\frac{1}{2})^n$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{7}{8}$	$\frac{15}{16}$	$\frac{31}{32}$	$\frac{63}{64}$

The first term of both series is $\frac{1}{2}$. However, for $n > 1$, the terms of the first series are smaller than the terms of the second. This implies that for any value of $n > 1$, **the sum of the terms of the first series will be less than the sum of the terms of the second.** As $n \to \infty$, therefore, even though the sums of both series tend to 1, the first series will tend to 1 at a slower rate than the second.

2 $a = 96$, $d = -3$, $S_n = 0$, $n = ?$

$S_n = \frac{n}{2}[2a + (n-1)d]$

$0 = \frac{n}{2}[192 + (n-1).(-3)]$

$0 = n(195 - 3n)$

$\therefore n = 0, 65$

Hence, 65 terms must be taken to give a sum of zero.

3 $-47, -43, -39$

$a = -47$, $d = 4$, $T_n = 2197$, $n = ?$

$T_n = a + (n-1)d$

$\therefore \quad 2197 = -47 + (n-1)4$

$2244 = 4n - 4$

$4n = 2248$

$n = 562$

Since n is a positive integer, 2197 is a member of the series. It is the 562nd term.

4 i) $r = \frac{u_2}{u_1} = \frac{1}{1-x}$

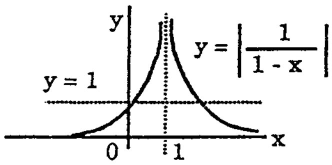
Series will have a limiting sum if $|r| < 1$

i.e. if $\left|\frac{1}{1-x}\right| < 1$

We need that part of the curve **below** the line y = 1. Solving simultaneously: $|1 - x| = 1$

$\therefore x = 0, 2$

$\therefore$ Series will have a limiting sum if $x < 0$ or $x > 2$



ii) If $x = 3$, series becomes $1 - \frac{1}{2} + \frac{1}{4} - \ldots$

$\therefore S_\infty = \frac{1}{1 - (\frac{-1}{2})} = \frac{2}{3}$

5 $S_n = \frac{n}{3}(n+1)(n+2)$

$S_{n-1} = \frac{n-1}{3}(n)(n+1)$

Using $T_n = S_n - S_{n-1}$

$= \frac{n}{3}(n+1)(n+2) - \frac{n-1}{3}(n)(n+1)$

$= \frac{n(n+1)(n+2) - (n-1)(n)(n+1)}{3}$

$= n(n+1)$, as required

$S_{50} = \frac{50}{3}(50+1)(50+2)$

$= 44\,200$

$S_{100} = \frac{100}{3}(100+1)(100+2)$

$= 343\,400$

The sum of the second fifty terms is $S_{100} - S_{50} = 299\,200$

6 i) $P = 1000$ $r = 12\%$ p.a.

$R = 1 + \frac{12}{100}$

$= 1.12$

$n = 10$ years

$A = PR^n$

$= 1000 \times (1.12)^{10}$

$= \$3105.85$

ii) $P = 1000$

$r = 6\%$ per half year

$R = 1 + \frac{6}{100}$

$= 1.06$

$n = 20$ half years

$A = PR^n$

$A = 1000 \times (1.06)^{20}$

$= \$3207.14$

iii) $P = 1000$

$r = 3\%$ per quarter

$R = 1 + \frac{3}{100}$

$= 1.03$

$n = 40$ quarters

$A = PR^n$

$A = 1000 \times (1.03)^{40}$

$= \$3262.04$

iv) $P = 1000$

$r = 1\%$ per month

$R = 1 + \frac{1}{100}$

$= 1.01$

$n = 120$ months

$A = PR^n$

$A = 1000 \times (1.01)^{120}$

$= \$3300.39$

7 Let the amount of the deposit be \$P

$n = 8$ years

$= 96$ months

$r = 1\%$ per month

$R = 1 + \frac{1}{100} = 1.01$

$A = PR^n$

$10\,000 = P \times 1.01^{96}$

$P = \frac{10\,000}{1.01^{96}}$

$= \$3847.23$

8 The first \$200 will be in the account for 5 x 12 = 60months. It will earn interest at $\frac{9}{12}$ % per month.

Hence, $R = 1 + \frac{r}{100}$

$= 1 + \frac{9/12}{100}$

$= 1.0075$

It will grow to $200(1.0075)^{60}$ during this time. The second deposit will earn interest for 59 months, and so on. Diagrammatically:

\$200 —— 60 months ——▶ $200(1.0075)^{60}$

\$200 —— 59 months ——▶ $200(1.0075)^{59}$

....

\$200 —— 1 ——▶ $200(1.0075)^{1}$

Total

Total $= 200(1.0075) + 200(1.0075)^2 + \ldots + 200(1.0075)^{60}$

$= 200(1.0075 + 1.0075^2 + \ldots + 1.0075^{60})$

$= 200.\frac{1.0075(1.0075^{60} - 1)}{1.0075 - 1}$

= \$15 198, to the nearest dollar.

9 We need to find the value of P where \$P is the amount of each deposit. We are told that the total is \$20 000

$r = 12\%$, $R = 1 + \frac{12}{100} = 1.12$, $n = 15$

\$P —— 15 ——▶ $P(1.12)^{15}$

\$P —— 14 ——▶ $P(1.12)^{14}$

....

\$P —— 1 ——▶ $P(1.12)^{1}$

Total

Total $= P(1.12)^1 + P(1.12)^2 + \ldots + P(1.12)^{15}$

$20\,000 = P(1.12 + 1.12^2 + \ldots + 1.12^{15})$

$20\,000 = P.\frac{1.12(1.12^{15} - 1)}{1.12 - 1}$

$\therefore P = \frac{20\,000(0.12)}{1.12(1.12^{15} - 1)}$

$= \$479$

10 i) $f(P) = PR - M$

$f(f(P)) = Rf(P) - M$

$= R(PR - M) - M$

$= PR^2 - M(1 + R)$

ii) $f(f(f(P))) = Rf(f(P)) - M$

$= R(PR^2 - M(1 + R)) - M$

$= PR^3 - M(1 + R + R^2)$

Note: If we continue this, we will get the formula for the amount remaining to be repaid after making n repayments of a standard loan.

11 i) $M = \frac{PR^N(R - 1)}{R^N - 1}$

$P = 200\,000$

$r = 1.5\%$ p.m.

$R = 1.015$

$N = 300$ m

$M = \frac{200\,000 \times 1.015^{300}(1.015 - 1)}{1.015^{300} - 1}$

$= \$3034.86$

ii) $A_n = PR^n - \frac{M(R^n - 1)}{R - 1}$

$A_{60} = 200\,000 \times (1.015)^{60} - \frac{3034.86(1.015^{60} - 1)}{1.015 - 1}$

$= \$196\,645.96$

iii) Total repayments $= 300 \times 3034.86$

$= \$910\,458$

Principal $= \$200\,000$

$\therefore$ Total interest $= \$710\,458$

iv) $I = \frac{Pnr}{100}$

$710\,458 = \frac{200\,000 \times 25 \times r}{100}$

$r = 14.2\%$ p.a.

12 i) $M = \frac{PR^N(R - 1)}{R^N - 1}$

$= \frac{50\,000(1.01)^{240}(0.01)}{1.01^{240} - 1}$

$= \$550.54$

ii) The amount owing would increase and the loan would never be paid off.

iii) $M = \frac{PR^N(R - 1)}{R^N - 1}$

$MR^N - M = PRR^N - PR^N$

i.e. $R^N(M - PR + P) = M$

$R^N = \frac{M}{M - PR + P}$

We want $M = 2 \times 550.54$

$P = 50\,000$

$R = 1.01$

$\therefore 1.01^N = \frac{2 \times 550.54}{2 \times 550.54 - 50\,000 \times 1.01 + 50\,000}$

$= 1.832$

$\therefore N = 60.8$ months

13 i) $a = -96$, $d = 7$

$T_n = -96 + (n - 1)7$

$= 7n - 103$

ii) $S_n = \frac{n}{2}[-192 + (n - 1)7]$

$= \frac{n}{2}[7n - 199]$

iii) $0 < \frac{n}{2}[7n - 199]$

$\therefore\ n > \frac{199}{7}$

$> 28\frac{3}{7}$

$\therefore\ n = 29$

14 i) $a = 98, d = -7$

$T_n = 98 + (n - 1)(-7)$

$= 105 - 7n$

ii) $-28 = 105 - 7n$

$7n = 133$

$n = 19$

iii) $S_{19} = \frac{19}{2}[196 + (19 - 1)(-7)]$

$= 665$

15 i) $1 \times 1 + (\sqrt{3} - 1)(\sqrt{3} + 1) + (\sqrt{3} - 1)^2 (\sqrt{3} + 1)^2 + ...$

$1 + (3 - 1) + (3 - 1)^2 + ...$

i.e. $1 + 2 + 2^2 + ...$

Therefore new series is geometric with $a = 1$ and $r = 2$

ii) $T_n = 2^{n-1}$

iii) $2^{n-1} > 10^6$

By calculator, using trial and error, $n = 21$.

Using logarithms: $\log_{10}2^{n-1} > 6$

$(n-1)\log_{10}2 > 6$

$n > \frac{6}{\log_{10}2} + 1$

> 20.9

$\therefore\ n = 21$, since n must be an integer.

Hence, the first term greater than 10^6 will be $T_{21} = 1\ 048\ 576$

16 $u_1 = 2 + \frac{1}{2}$

$u_2 = 4 + \frac{1}{4}$

$u_3 = 6 + \frac{1}{8}$

....

$u_{10} = 20 + \frac{1}{1024}$

$\therefore\ S_{10} = (2 + 4 + 6 + ... + 20) + (\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + ... + \frac{1}{1024})$

$= \frac{10}{2}[4 + 9 \times 2] + \frac{\frac{1}{2}(1 - (\frac{1}{2})^{10})}{1 - \frac{1}{2}}$

$= 110 + \frac{1023}{1024}$

$= 110\frac{1023}{1024}$

17 $S_\infty = \frac{51}{100} + \frac{51}{10000} + ...$

$S_\infty = \frac{a}{1 - r},\ |r| < 1$

$= \frac{\frac{51}{100}}{1 - \frac{1}{100}}$

$= \frac{51}{99}$

18 Let the number be x

Then $(\frac{1}{10} + x), (\frac{1}{4} + x), (\frac{9}{20} + x)$ is in geometric progression.

$\therefore\ (\frac{1}{4} + x)^2 = (\frac{1}{10} + x)(\frac{9}{20} + x)$

$\frac{1}{16} + \frac{x}{2} + x^2 = \frac{9}{200} + \frac{x}{10} + \frac{9x}{20} + x^2$

i.e. $\frac{1}{16} + \frac{x}{2} = \frac{9}{200} + \frac{11x}{20}$

$\therefore\ \frac{x}{20} = \frac{7}{400}$

$x = \frac{7}{20}$

19 $a + ar = 5$ .... (i)

$ar^{n-1} = 3(ar^n + ar^{n+1} + ...)$

i.e. $r^{n-1} = 3(r^n + r^{n+1} + ...)$, since $a \neq 0$

$\therefore\ r^{n-1} = 3 . \frac{r^n}{1 - r}$

i.e. $r^{n-1} - r^n = 3r^n$

$r^{n-1} = 4r^n$

i.e. $1 = 4r$

i.e. $r = \frac{1}{4}$

Therefore, from (i), $a + \frac{a}{4} = 5$

$\therefore\ a = 4$

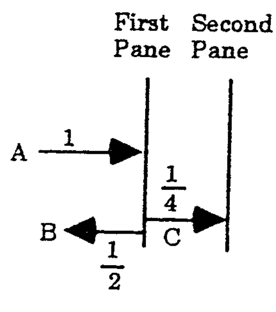
$\therefore$ the series is $4 + 1 + \frac{1}{4} + ...$

20 i) $a = \frac{1}{16}, r = \frac{1}{4}$

$S_\infty = \frac{\frac{1}{16}}{1 - \frac{1}{4}}$

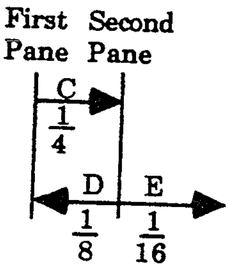
$= \frac{1}{12}$

ii) Let us call the original ray A and let it be 1 unit in size.



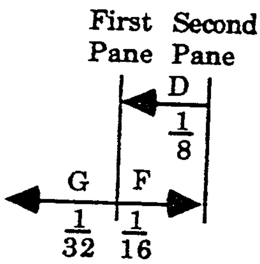
When the ray A hits the first plane, half of it is reflected - ray B - and $\frac{1}{4}$ of it passes through - ray C.

This means that ray B = $\frac{1}{2} \cdot 1 = \frac{1}{2}$ of the original ray and ray C = $\frac{1}{4} \cdot 1 = \frac{1}{4}$ of the original ray.



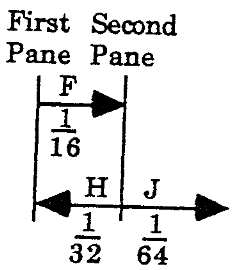
When the ray C hits the second pane, half of it is reflected - ray D - and $\frac{1}{4}$ of it passes through - ray E

This means that ray D = $\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}$ of the original ray and ray E = $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$ of the original ray.



When ray D hits the first pane, half of it will be reflected - ray F - and $\frac{1}{4}$ of it passes through - ray G.

This means that ray F = $\frac{1}{2} \cdot \frac{1}{8} = \frac{1}{16}$ of the original ray and ray E = $\frac{1}{4} \cdot \frac{1}{8} = \frac{1}{32}$ of the original ray.



When ray F hits the second pane, half of it is reflected - ray H - and $\frac{1}{4}$ of it passes through - ray J.

This means that ray H = $\frac{1}{2} \cdot \frac{1}{16} = \frac{1}{32}$ of the original ray and ray E = $\frac{1}{4} \cdot \frac{1}{16} = \frac{1}{64}$ of the original ray. The only rays to pass through the second pane so far are E and J, giving a total of $\frac{1}{16} + \frac{1}{64}$.

Continuing this process, we get a geometric series $\frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots$ giving a total of

$$S_\infty = \frac{\frac{1}{16}}{1 - \frac{1}{4}} = \frac{1}{12}\text{, as required.}$$

Motion and Simple Harmonic Motion

1 i) We know that the period of the cosine function is 2π.

$\therefore x = \cos nt$
$= \cos(nt + 2\pi)$
$= \cos n(t + \frac{2\pi}{n})$

Hence, at time t and at time $t + \frac{2\pi}{n}$ we will have the particle in the same position and moving in the same direction.

$\therefore$ in general, the period of $\cos nt$ is $\frac{2\pi}{n}$

$\therefore$ Period $= \frac{2\pi}{n} = \frac{2\pi}{\pi} = 2$ seconds

ii) Equilibrium position occurs when $x'' = 0$

$x = 4\cos\pi t$
$x' = -4\pi\sin\pi t$
$x'' = -4\pi^2\cos\pi t$
$= 0$

$\therefore \cos\pi t = 0$

$\pi t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots$

$t = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

$\therefore x = 4\cos\pi(\frac{1}{2}), 4\cos\pi(\frac{3}{2}), 4\cos\pi(\frac{5}{2}), \ldots$
$= 0, 0, 0, \ldots$

i.e. centre of motion = equilibrium position

$\Rightarrow x = 0$

iii) When $t = \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \ldots$

$x' = -4\pi, 4\pi, -4\pi, \ldots$

Therefore speed $= 4\pi$ ms^{-1}

iv) Extreme positions occur when $x' = 0$

i.e. $0 = -4\pi\sin\pi t$
i.e. $\sin\pi t = 0$
$\pi t = 0, \pi, 2\pi, \ldots$
$t = 0, 1, 2, \ldots$
$\therefore x'' = -4\pi^2, 4\pi^2, -4\pi^2, \ldots$

i.e. the acceleration is $\pm 4\pi^2$ ms^{-2} at the ends of the motion.