NSW INDEPENDENT SCHOOLS

2004
Higher School Certificate
Trial Examination

Mathematics
Extension 2

General Instructions

- Reading time – 5minutes
- Working time – 3 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided with this paper.
- All necessary working should be shown in every question

Total marks - 120

Attempt Questions 1 – 8

All questions are of equal value

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME: ....

HSC STANDARD INTEGRALS

$\int x^n \, dx$	$= \frac{1}{n+1} x^{n+1}, \; n \neq 1; \; x \neq 0, \text{ if } n < 0$
$\int \frac{1}{x} \, dx$	$= \ln x, \; x > 0$
$\int e^{ax} \, dx$	$= \frac{1}{a} e^{ax}, \; a \neq 0$
$\int \cos ax \, dx$	$= \frac{1}{a} \sin ax, \; a \neq 0$
$\int \sin ax \, dx$	$= -\frac{1}{a} \cos ax, \; a \neq 0$
$\int \sec^2 ax \, dx$	$= \frac{1}{a} \tan ax, \; a \neq 0$
$\int \sec ax \tan ax \, dx$	$= \frac{1}{a} \sec ax, \; a \neq 0$
$\int \frac{1}{a^2 + x^2} dx$	$= \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$
$\int \frac{1}{\sqrt{a^2 - x^2}} dx$	$= \sin^{-1} \frac{x}{a}, \; a > 0, \; -a < x < a$
$\int \frac{1}{\sqrt{x^2 - a^2}} dx$	$= \ln\left(x + \sqrt{x^2 - a^2}\right), \; x > a > 0$
$\int \frac{1}{\sqrt{x^2 + a^2}} dx$	$= \ln\left(x + \sqrt{x^2 + a^2}\right)$

NOTE: $\ln x = \log_e x, \quad x > 0$

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Question 1 **Begin a new page** Marks

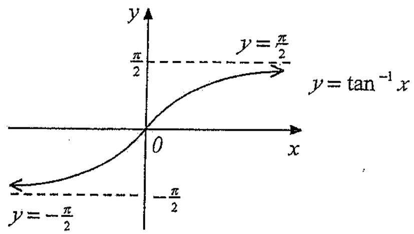
(a) Consider the function $f(x)=|1+x|+|1-x|$.

(i) Show that $f(x)$ is an even function. 1

(ii) Sketch the graph of $y=f(x)$ showing clearly the coordinates of any points of intersection with the axes and the coordinates of any sharp corners. 2

(iii) Use the graph to find the set of values of the real number k for which the equation $f(x)=k$ has exactly 2 real solutions. 1

(b) The diagram shows the graph of the function $y=\tan^{-1}x$.



(i) On separate diagrams sketch the graphs of the following curves. In each case show the equations of any asymptotes. 3

(1) $y=\dfrac{1}{\tan^{-1}x}$ (2) $y=(\tan^{-1}x)^2$ (3) $y^2=\tan^{-1}x$.

(ii) Now consider the function $g(x)=\log_e \tan^{-1}x$.

(1) Sketch the graph of $y=g(x)$ showing clearly the coordinates of any points of intersection with the axes and the equations of any asymptotes. 2

(2) Find the equation of the inverse function $y=g^{-1}(x)$. 1

(3) Find the domain of the inverse function $y=g^{-1}(x)$. 1

(c) Consider the function given by the equations $x=\theta+\frac{1}{2}\sin 2\theta$ and $y=\theta-\frac{1}{2}\sin 2\theta$ where θ is a real number.

(i) Show that $\dfrac{dy}{dx}=\tan^2\theta$. 2

(ii) Show that $\dfrac{d^2y}{dx^2}=\tan\theta\sec^4\theta$. 2

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Question 2 **Begin a new page** Marks

(a)(i) Find $\displaystyle\int \frac{e^{2x}-1}{e^x-1}\,dx$. 1

(ii) Find $\displaystyle\int \frac{\tan x}{\tan 2x}\,dx$. 2

(b) Use the substitution $x=u^2$ $(u>0)$ to find $\displaystyle\int \frac{1}{x\left(1+\sqrt{x}\right)}\,dx$. 3

(c) Use the substitution $t=\tan\frac{x}{2}$ to find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{5+3\cos x-4\sin x}\,dx$. 4

(d)(i) If $I_n=\int_{-1}^{0} x^n\sqrt{1+x}\,dx$ for $n=0,1,2,\ldots$ show that 3

$I_n=\dfrac{-2n}{2n+3}I_{n-1}$ for $n=1,2,3\ \ldots$.

(ii) Hence evaluate $\int_{-1}^{0} x^2\sqrt{1+x}\,dx$. 2

Student name / number ....

Marks

Question 3 **Begin a new page**

(a) $z=1+i$ is a root of the equation $z^2-aiz+b=0$ where a and b are real numbers.
(i) Find the values of a and b. **2**

(ii) Find the other root of the equation. **2**

(b)(i) Show that $z\bar{z}=|z|^2$ for any complex number z. **1**

(ii) A sequence of complex numbers z_n is given by the rule $z_1=w$ and $z_n=c\overline{z_{n-1}}$ for $n=2,3,4\ldots$, where w is a given complex number and c is a complex number with modulus 1. Show that $z_3=w$. **2**

(c)(i) Express the complex number $z_1=\sqrt{2}-i\sqrt{2}$ in modulus argument form. **2**

(ii) On an Argand diagram OPQ is an equilateral triangle where O is the origin, P is the point representing the complex number z_1 and Q is a point in the first quadrant representing the complex number z_2. Express z_2 in the form $z_2=a+ib$ where a and b are real numbers. **2**

(d)(i) On an Argand diagram shade in the region containing all the points representing complex numbers z such that both $|z|\le 2$ and $\frac{\pi}{4}\le \arg(z+2)\le\frac{\pi}{2}$. **2**

(ii) Find the possible values of $|z|$ and $\arg z$ for such complex numbers z. **2**

Student name / number ....

Marks

Question 4 **Begin a new page**

(a) Consider the hyperbola $\frac{x^2}{4}-\frac{y^2}{12}=1$.

(i) Sketch the graph of the hyperbola showing clearly any intercepts made on the coordinate axes, the coordinates of the foci, the equations of the directrices and the equations of the asymptotes. **3**

(ii) $T(x_1,y_1)$ is a point on the hyperbola. Use differentiation to show that the tangent to the hyperbola at T has equation $\frac{xx_1}{4}-\frac{yy_1}{12}=1$. **3**

(iii) $P(4,6)$ and $Q(14,24)$ are two points on the hyperbola. M is the midpoint of PQ and $O(0,0)$ is the origin. The tangents to the hyperbola at P and Q intersect at the point R. Show that the points R, O and M are collinear. **3**

(b) $P(a\cos\theta, b\sin\theta)$ is a point in the first quadrant on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ where $a>b>0$. Q, R and S are other points on the ellipse.

(i) If $PQRS$ is a rectangle find the maximum area of $PQRS$ and the exact coordinates of the point P at which this maximum area occurs. **3**

(ii) If $PQRS$ is a square show that $PQRS$ has area $\frac{4a^2b^2}{a^2+b^2}$. **3**

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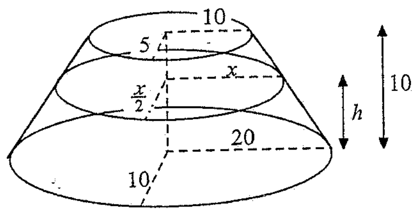
Marks

Question 5 **Begin a new page**

(a)(i) Show that the area contained by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab. **2**

(ii)



A solid of height 10 metres stands on horizontal ground. The base of the solid is an ellipse with semi-axes 20 metres and 10 metres. Horizontal cross sections taken parallel to the base and at height h metres above the base are ellipses with semi-axes x metres and $\frac{x}{2}$ metres so that the centres of these elliptical cross sections lie on a vertical straight line, and the extremities of their semi-axes lie on sloping straight lines as shown in the diagram. The top of the solid is an ellipse with semi-axes 10 metres and 5 metres.

Show that the volume $V\,\text{m}^3$ of the solid is given by $V=\frac{\pi}{2}\int_0^{10}(20-h)^2\ dh$, **5**

and hence find the volume of the solid correct to the nearest cubic metre.

(b)(i) Find the general solution of the equation $\cos 4\theta=\frac{1}{2}$. **2**

(ii) Use De Moivre's Theorem to show that $\cos 4\theta = 8\cos^4\theta-8\cos^2\theta+1$. **2**

(iii) Show that the equation $16x^4-16x^2+1=0$ has roots $x=\cos\frac{\pi}{12}$, $x=-\cos\frac{\pi}{12}$, $x=\cos\frac{5\pi}{12}$ and $x=-\cos\frac{5\pi}{12}$. **2**

(iv) By considering this equation as a quadratic equation in x^2, show that **2**

$\cos\frac{\pi}{12}=\frac{\sqrt{2+\sqrt{3}}}{2}$.

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Marks

Question 6 **Begin a new page**

(a) P and Q are points on the curve $y=x^4+4x^3$ where $x=\alpha$ and $x=\beta$ respectively. The line $y=mx+b$ is a tangent to the curve at both points P and Q.

(i) Explain why the equation $x^4+4x^3-mx-b=0$ has roots α, α, β and β. **1**

(ii) Use the relationships between the roots and the coefficients of this equation to find the values of m and b. **4**

(b) A particle P of mass m kg moves vertically in a medium in which the resistance to motion has magnitude $\frac{mv}{10}$ when the speed of the particle is $v\text{ ms}^{-1}$. The acceleration due to gravity is 10 ms^{-2}.

(i) If the particle falls vertically from rest, show that its acceleration $a\text{ ms}^{-2}$ is given by $a=10-\frac{v}{10}$. Hence show that the terminal velocity of the particle is 100 ms^{-1}. **2**

(ii) The particle is projected vertically upwards with speed 100 ms^{-1}.

(1) Show that its acceleration $a\text{ ms}^{-2}$ is given by $a=-10-\frac{v}{10}$. **1**

(2) Find expressions for the speed $v\text{ ms}^{-1}$ and the height x metres of the particle after time t seconds. **5**

(3) Find the time taken by the particle to reach its maximum height, and the maximum height attained. **2**

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Question 7 **Begin a new page** **Marks**

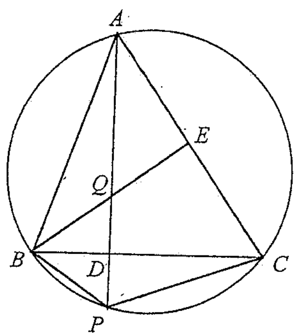
(a) The angles A, B and C are consecutive terms in an arithmetic sequence. Show that $\cos A \cos C - \cos^2 B = \sin A \sin C - \sin^2 B$. **3**

(b) Consider the function $y = \tan^{-1} x - x + \frac{1}{3}x^3$.

(i) Show that $\frac{dy}{dx} > 0$ for all values of $x > 0$. **2**

(ii) Show that $\tan^{-1} x > x - \frac{1}{3}x^3$ for all values of $x > 0$. **2**

(c)



ABC is an acute angled triangle inscribed in a circle. D is the point on BC such that AD is perpendicular to BC. AD produced meets the circle at P. Q is the point on AD such that $DQ = DP$. AQ produced meets CA at E.

(i) Copy the diagram showing the above information. **1**

(ii) Show that $\Delta BDP \equiv \Delta BDQ$. **2**

(iii) Show that $BDEA$ is a cyclic quadrilateral. **4**

(iv) Show that BE is perpendicular to CA. **1**

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Question 8 **Begin a new page** **Marks**

(a) The lengths of the sides of a triangle are the first three terms in an arithmetic sequence with first term 1 and common difference d. Find the set of possible values of d. **3**

(b) The letters of the word EQUATIONS are arranged at random in a row from left to right.

(i) Find the probability that no two of the vowels (A, E, I, O, U) are next to each other. **2**

(ii) Find the probability that the vowels occur in alphabetical order (A E I O U) reading from left to right. **2**

(c) The real number x is such that $x^2 = x + 1$. The Fibonacci sequence of numbers T_n $(n = 1, 2, 3....)$ is given by $T_1 = 1$, $T_2 = 1$ and $T_n = T_{n-1} + T_{n-2}$ $(n = 3, 4, 5...)$.

(i) Use Induction to show that $x^n = T_{n-1} + T_n x$ for all positive integers $n \geq 2$. **4**

(ii) Use Induction to show that $1 + x + x^2 + ... + x^n = T_{n+1} + (T_{n+2} - 1)x$ for all positive integers $n \geq 2$. **4**

1a. Outcomes Assessed: (i) P5 (ii) P5 (iii) P5

Marking Guidelines

Criteria	Marks
i • Shows that $f(-x)=f(x)$	1
ii • Horizontal interval between $(-1,2)$ and $(1,2)$ with y-intecept 2	1
• Sloping rays from $(-1,2)$ and $(1,2)$ with gradients $-2, 2$ respectively.	1
iii ¥ Values for k obtained from graph.	1

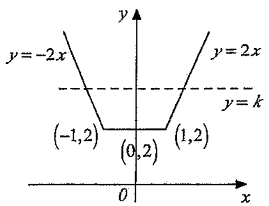
Answer

i.

$$f(-x)=|1+(-x)|+|1-(-x)|$$
$$=|1-x|+|1+x|$$
$$=f(x)$$

$\therefore f(x)$ is even.

ii.



iii.

Require k such that line $y=k$ intersects graph exactly twice.
Hence $\{k: k>2\}$.

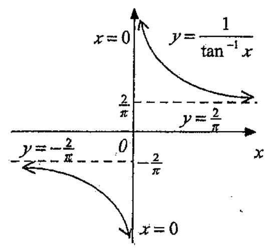
1b. Outcomes Assessed: (i) E6 (ii) HE4 , E6

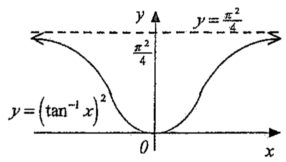
Marking Guidelines

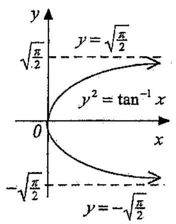
Criteria	Marks
i • Shape of curve with equations of asymptotes	1, 1, 1
ii • Shape of curve with equations of asymptotes	1
• Intercept made by curve on x axis	1
• Equation of inverse function	1
• Domain of inverse function	1

Answer

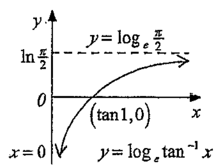
i.







ii.



$$y=\log_e \tan^{-1} x$$
$$e^y=\tan^{-1} x$$
$$\tan e^y = x$$

$\therefore g^{-1}(x)=\tan e^x$
with domain $\{x: x<\log_e \frac{\pi}{2}\}$.

/

1c. Outcomes Assessed: (i) H5 , HE5 (ii) H5 , HE5

Marking Guidelines

Criteria	Marks
i • Expressions for $\frac{dy}{d\theta}, \frac{dx}{d\theta}$ in terms of $\cos 2\theta$	1
• Using appropriate trig. identities to simplify $\frac{dy}{dx}$ into required form.	1
ii • Using the chain rule to express $\frac{d^2y}{dx^2}$ in form $\frac{d}{d\theta}\left(\frac{dy}{dx}\right).\frac{d\theta}{dx}$	1
• Obtaining $\frac{d^2y}{dx^2}$ as function of θ in required form.	1

Answer

i.

$$y=\theta-\tfrac{1}{2}\sin 2\theta \qquad x=\theta+\tfrac{1}{2}\sin 2\theta$$
$$\frac{dy}{d\theta}=1-\cos 2\theta \qquad \frac{dx}{d\theta}=1+\cos 2\theta$$
$$=2\sin^2\theta \qquad =2\cos^2\theta$$
$$\therefore \frac{dy}{dx}=\frac{2\sin^2\theta}{2\cos^2\theta}=\tan^2\theta$$

ii.

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{d\theta}\left(\frac{dy}{dx}\right).\frac{d\theta}{dx}$$
$$\therefore \frac{d^2y}{dx^2}=2\tan\theta\sec^2\theta.\frac{1}{2\cos^2\theta}$$
$$=\tan\theta\sec^4\theta$$

2a Outcomes Assessed: (i) H3 , H5 (ii) H5

Marking Guidelines

Criteria	Marks
i • Rearrangement of integrand and resulting primitive	1
ii • Using expression for $\tan 2x$ to simplify integrand	1
• Primitive function	1

Answer

i.

$$\int \frac{e^{2x}-1}{e^x-1}\,dx=\int \frac{(e^x-1)(e^x+1)}{e^x-1}\,dx$$
$$=\int (e^x+1)\,dx$$
$$=e^x+x+c$$

ii.

$$\int \frac{\tan x}{\tan 2x}\,dx=\int \tan x.\frac{1-\tan^2 x}{2\tan x}\,dx$$
$$=\tfrac{1}{2}\int (2-\sec^2 x)\,dx$$
$$=x-\tfrac{1}{2}\tan x+c$$

2b Outcomes Assessed: HE6 , E8

Marking Guidelines

Criteria	Marks
• Substitution to obtain integral with respect to u	1
• Rearrangement of integrand into partial fractions	1
• Primitive function	1

Answer

$$x=u^2,\ u>0$$
$$dx=2u\,du$$
$$\therefore \int \frac{1}{x(1+\sqrt{x})}\,dx=\int \frac{1}{u^2(1+u)}2u\,du$$

$$2\int \frac{1}{u(1+u)}\,du=2\int\left(\frac{1}{u}-\frac{1}{1+u}\right)du$$
$$=2\ln\left|\frac{u}{1+u}\right|+c$$
$$=2\ln\frac{\sqrt{x}}{1+\sqrt{x}}+c$$

2c Outcomes Assessed: HE6

Marking Guidelines

Criteria	Marks
• Expressing dx in terms of dt	1
• Using t-formulae to write $\sin x$, $\cos x$ in terms of t	1
• Simplifying integrand and obtaining primitive function	1
• Using appropriate limits to evaluate the definite integral	1

Answer

$t=\tan\frac{x}{2}$

$dt=\frac{1}{2}\sec^2\frac{x}{2}\,dx$

$dt=\frac{1}{2}(1+t^2)\,dx$

$dx=\frac{2}{1+t^2}\,dt$

$x=0\Rightarrow t=0$

$x=\frac{\pi}{2}\Rightarrow t=1$

$$5+3\cos x-4\sin x=\frac{5(1+t^2)+3(1-t^2)-8t}{1+t^2}=\frac{2(t^2-4t+4)}{1+t^2}=\frac{2(t-2)^2}{1+t^2}$$

$$I=\int_0^{\frac{\pi}{2}}\frac{1}{5+3\cos x-4\sin x}\,dx=\int_0^1\frac{1+t^2}{2(t-2)^2}\cdot\frac{2}{1+t^2}\,dt=\int_0^1\frac{1}{(t-2)^2}\,dt$$

$$\therefore I=\left[\frac{-1}{t-2}\right]_0^1=1-\frac{1}{2}=\frac{1}{2}$$

2d Outcomes Assessed: (i) E8 (ii) H5 , E8

Marking Guidelines

Criteria	Marks
i • Integration by parts	1
• Rearrangement of resulting integrand in terms of I_{n-1}, I_n	1
• Simplification to obtain required recurrence formula	1
ii • Evaluation of I_0	1
• Using recurrence formula to evaluate I_2	1

Answer

i.

$$I_n=\int_{-1}^0 x^n\sqrt{1+x}\,dx$$
$$=\frac{2}{3}\left[x^n(1+x)^{\frac{3}{2}}\right]_{-1}^0-\frac{2}{3}\int_{-1}^0 nx^{n-1}(1+x)\sqrt{1+x}\,dx$$
$$=0-\frac{2n}{3}\left\{\int_{-1}^0 x^{n-1}\sqrt{1+x}\,dx+\int_{-1}^0 x^n\sqrt{1+x}\,dx\right\}$$

$$\therefore\quad I_n=-\frac{2n}{3}I_{n-1}-\frac{2n}{3}I_n$$
$$\left(1+\frac{2n}{3}\right)I_n=-\frac{2n}{3}I_{n-1}$$
$$(3+2n)I_n=-2n\,I_{n-1}$$
$$I_n=\frac{-2n}{2n+3}I_{n-1}$$

ii.

$$I_0=\int_{-1}^0\sqrt{1+x}\,dx=\frac{2}{3}\left[(1+x)^{\frac{3}{2}}\right]_{-1}^0=\frac{2}{3}(1-0)=\frac{2}{3}$$

$$\therefore I_2=-\frac{4}{7}I_1=\left(-\frac{4}{7}\right)\left(-\frac{2}{5}\right)I_0=\frac{4}{7}\frac{2}{5}\frac{2}{3}=\frac{16}{105}$$

3a Outcomes Assessed: (i) E3 (ii) E3

Marking Guidelines

Criteria	Marks
i • Substituting $1+i$ and rearranging into form $c+id$	1
• Equating real and imaginary parts to evaluate a and b.	1
ii • Using sum or product of roots to find an expression involving second root.	1
• Evaluating second root in form $c+id$.	1

Answer

i. $z=1+i\Rightarrow z^2-aiz+b=0$
$$(1+i)^2-ai(1+i)+b=0$$
$$(a+b)+i(2-a)=0$$

Equating real and imaginary parts, $a=2$, $b=-2$.

ii. Let the second root be β. Then
$$(1+i)+\beta=2i$$
$$\beta=-1+i$$

3b Outcomes Assessed: (i) E3 (ii) E3

Marking Guidelines

Criteria	Marks
i • Using $z=a+ib$ and expressions for $\bar{z}$, $\lvert z\rvert^2$ in terms of a, b to prove result	1
• Using recurrence formula to write expression for z_3 in terms of c and w.	1
• Using $c\bar{c}=\lvert c\rvert^2$ to obtain result.	1

Answer

i. Let $z=a+ib$, a, b real.
$$z\bar{z}=(a+ib)(a-ib)=a^2-i^2b^2=a^2+b^2=\lvert z\rvert^2$$

ii. $z_2=c\bar{w}$
$$z_3=c\left(\overline{c\bar{w}}\right)=c\bar{c}\,w=\lvert c\rvert^2 w=w$$

3c Outcomes Assessed: (i) E3 (ii) E3

Marking Guidelines

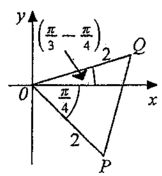
Criteria	Marks
i • Identifying modulus as 2	1
• Identifying argument as $-\frac{\pi}{4}$	1
ii • Finding surd expressions for cos and sin of $\arg z_2$	1
• Using these expressions to write z_2 in form $a+ib$	1

Answer

i. $z_1=\sqrt{2}-i\sqrt{2}$
$$=2\left(\tfrac{1}{\sqrt{2}}-\tfrac{1}{\sqrt{2}}i\right)$$
$$=2\left\{\cos\left(-\tfrac{\pi}{4}\right)+i\sin\left(-\tfrac{\pi}{4}\right)\right\}$$

ii. ΔOPQ is equilateral.

$\therefore OQ=2$ and $\angle POQ=\frac{\pi}{3}$



$\lvert z_2\rvert=2$. Let $\arg z_2=\theta$.

Then $\theta=\frac{\pi}{3}-\frac{\pi}{4}$.

$$\therefore\cos\theta=\tfrac{1}{2}\cdot\tfrac{1}{\sqrt{2}}+\tfrac{\sqrt{3}}{2}\cdot\tfrac{1}{\sqrt{2}}=\tfrac{\sqrt{3}+1}{2\sqrt{2}}$$
$$\sin\theta=\tfrac{\sqrt{3}}{2}\cdot\tfrac{1}{\sqrt{2}}-\tfrac{1}{2}\cdot\tfrac{1}{\sqrt{2}}=\tfrac{\sqrt{3}-1}{2\sqrt{2}}$$

$$\therefore z_2=\tfrac{\sqrt{3}+1}{\sqrt{2}}+\tfrac{\sqrt{3}-1}{\sqrt{2}}\,i$$

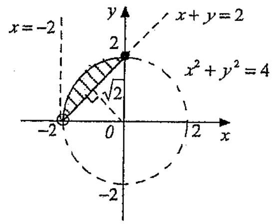
3d Outcomes Assessed: (i) E3 (ii) E3

Marking Guidelines

Criteria	Marks
i • Shading region inside the circle of radius 2 centred at the origin.	1
• Shading segment cut off by chord joining $(-2,0)$ (excluded), and $(0,2)$ (included).	1
ii • Stating possible values of $\lvert z \rvert$	1
• Stating possible values of $\arg z$.	1

Answer

i.



ii. Distance from the origin to the line $x+y-2=0$ is $\left|\frac{0+0-2}{\sqrt{1^2+1^2}}\right|=\sqrt{2}$. $\therefore \sqrt{2}\le|z|\le 2$.

Minimum value of $\arg z$ occurs when z is represented by the point $(0,2)$. Since $\arg 0$ is not defined, the point $(-2,0)$, representing $z=-2$, is excluded from the region so that the limiting maximum value for $\arg z$ is not attained.

$\therefore \frac{\pi}{2}\le \arg z<\pi$.

4a Outcomes Assessed: (i) E3 (ii) E4 (iii) E3 , E4

Marking Guidelines

Criteria	Marks
i • Showing asymptotes and intercepts on x axis	1
• Showing coordinates of foci	1
• Showing equations of directrices	1
ii • Finding gradient of tangent by differentiation	1
• Using an appropriate method to find equation of tangent in unsimplified form	1
• Using fact that T lies on hyperbola to simplify equation of tangent into required form	1
iii • Writing down coordinates of M	1
• Solving simultaneous equations to find coordinates of R	1
• Deducing that R, O, M are collinear by comparison of gradients.	1

Answer

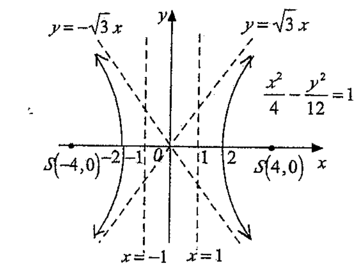
i. $a^2=4,\ b^2=12$

$12=4(e^2-1)\Rightarrow e^2=4\ \therefore e=2$

$ae=2\times 2\Rightarrow$ foci are $(\pm 4,0)$

$\frac{a}{e}=\frac{2}{2}\Rightarrow$ directrices are $x=\pm 1$

$\frac{b}{a}=\frac{\sqrt{12}}{\sqrt{4}}\Rightarrow$ asymptotes are $y=\pm\sqrt{3}\,x$



ii. $\frac{x^2}{4}-\frac{y^2}{12}=1$

$\frac{x}{2}-\frac{y}{6}\frac{dy}{dx}=0$

$\frac{dy}{dx}=\frac{3x}{y}$

Hence tangent at T has gradient $\frac{3x_1}{y_1}$ and equation

$3x_1x-y_1y=k$ for some constant k.

T lies on both tangent and hyperbola.

$\therefore k=3x_1^2-y_1^2=12\left(\frac{x_1^2}{4}-\frac{y_2^2}{12}\right)=12$

Hence equation of tangent is

$3x_1x-y_1y=12,\quad or\quad \frac{xx_1}{4}-\frac{yy_1}{12}=1$

iii. $P(4,6),\ Q(14,24)\quad \therefore M(9,15)$

Tangent at P: $\frac{4x}{4}-\frac{6y}{12}=1$

Tangent at Q: $\frac{14x}{4}-\frac{24y}{12}=1$

At R $\quad 2x-y=2\ (1)$

$7x-4y=2\ (2)$

$4\times(1)-(2)\Rightarrow x=6\quad \therefore R(6,10)$

Gradient OM is $\frac{15}{9}=\frac{5}{3}$. Gradient OR is $\frac{10}{6}=\frac{5}{3}$

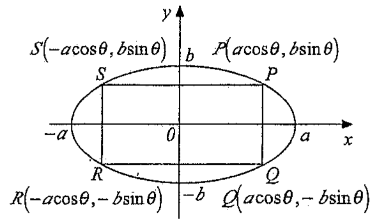
$\therefore R, O, M$ are collinear.

4b Outcomes Assessed: (i) E3 , E4 (ii) E3 , E4

Marking Guidelines

Criteria	Marks
i • Expressing area of $PQRS$ in terms of θ	1
• Finding maximum area	1
• Finding coordinates of P for maximum area	1
ii • Finding value of $\tan\theta$ when $PQRS$ is a square	1
• Using t formula to evaluate $\sin 2\theta$	1
• Finding area of $PQRS$ and simplifying to required form	1

Answers



i. Area $PQRS$ is $2a\cos\theta.2b\sin\theta=2ab\sin 2\theta$.

Hence maximum area is $2ab$ when $\theta=\frac{\pi}{4}$ and $P\left(\frac{a}{\sqrt{2}},\frac{b}{\sqrt{2}}\right)$.

ii. If $PQRS$ is a square, $2a\cos\theta=2b\sin\theta\Rightarrow\tan\theta=\frac{a}{b}$

$\therefore \sin 2\theta=\frac{2\left(\frac{a}{b}\right)}{1+\left(\frac{a}{b}\right)^2}=\frac{2ab}{a^2+b^2}$

Hence area of $PQRS$ is $2ab\sin 2\theta=\frac{4a^2b^2}{a^2+b^2}$.

5a Outcomes Assessed: (i) H8 (ii) H8 , E7

Marking Guidelines

Criteria	Marks
i • Writing area of ellipse as definite integral	1
• Evaluating this definite integral	1
ii • Using similarity to relate x and h	1
• Writing area of slice in terms of h	1
• Expressing V as limiting sum of slice volumes as h varies	1
• Finding primitive function of integral for V	1
• Substituting to calculate volume to nearest cubic metre	1

Answer

i $y^2=\frac{b^2}{a^2}(a^2-x^2)$

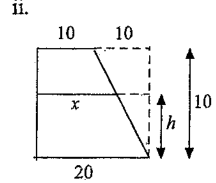
Area A of ellipse is given by

$A=4\frac{b}{a}\int_0^a\sqrt{a^2-x^2}\ dx$

$=4\frac{b}{a}.\frac{1}{4}\pi a^2$

$=\pi ab$

ii.



Using similar triangles, $\frac{20-x}{10}=\frac{h}{10}$

$\therefore x=20-h$

Area of slice is $A=\pi x\frac{x}{2}=\frac{\pi}{2}(20-h)^2$

Hence volume of solid is given by

$V=\lim_{\delta h\to 0}\sum_{h=0}^{h=10}\frac{\pi}{2}(20-h)^2\,\delta h$

$=\frac{\pi}{2}\int_0^{10}(20-h)^2\,dh$

$V=-\frac{\pi}{6}\left[(20-h)^3\right]_0^{10}=-\frac{\pi}{6}(10^3-20^3)$

Volume is 3665m^3 (to nearest m^3).

5b Outcomes Assessed: (i) H5 (ii) E3 (iii) P4 , H5 (iii) P4 , H5

Marking Guidelines

Criteria	Marks
i • Finding at least 2 solutions for 4θ	1
• Finding general solution for θ	1
ii • Using De Moivre's Theorem by taking $\text{Re}(\cos\theta + i\sin\theta)^4$	1
• Simplifying terms in binomial expansion to obtain required result	1
iii • Substituting $x = \cos\theta$ to reduce equation to $\cos 4\theta = \frac{1}{2}$	1
• Using general solution to obtain 4 distinct solutions for x in required form.	1
iv • Solving the equation as quadratic in x^2	1
• Deducing value of $\cos\frac{\pi}{12}$	1

Answer

i. $\cos 4\theta = \frac{1}{2}$

$4\theta = \pm\frac{\pi}{3} + 2n\pi$

$\theta = \frac{\pi}{12}(6n \pm 1), \quad n = 0, \pm 1, \pm 2, \ldots$

ii. $\cos 4\theta = \text{Re}(\cos\theta + i\sin\theta)^4$

$= \cos^4\theta + 6\cos^2\theta(i\sin\theta)^2 + (i\sin\theta)^4$

$= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2$

$= 8\cos^4\theta - 8\cos^2\theta + 1$

iii. Let $x = \cos\theta$. Then equation becomes

$16\cos^4\theta - 16\cos^2\theta + 1 = 0$

$2(8\cos^4\theta - 8\cos^2\theta + 1) = 1$

$\cos 4\theta = \frac{1}{2}$

$\therefore x = \cos\{\frac{\pi}{12}(6n \pm 1)\}, \quad n = 0, \pm 1, \pm 2, \ldots$

giving 4 distinct solutions

$x = \cos\frac{\pi}{12}, \ \cos\frac{5\pi}{12}, \ \cos\frac{7\pi}{12}, \ \cos\frac{11\pi}{12}$

$x = \cos\frac{\pi}{12}, \ \cos\frac{5\pi}{12}, \ \cos(\pi - \frac{5\pi}{12}), \ \cos(\pi - \frac{\pi}{12})$

$\therefore x = \pm\cos\frac{\pi}{12}, \ \pm\cos\frac{5\pi}{12}$

iv. $16x^4 - 16x^2 + 1 = 0$

$\Delta = 16^2 - 4.16 = 64 \times 3$

$\therefore x^2 = \frac{16 \pm 8\sqrt{3}}{32} = \frac{2 \pm \sqrt{3}}{4}$

$\therefore x = \pm\frac{\sqrt{2 \pm \sqrt{3}}}{2}$

But $\cos\frac{\pi}{12} > \cos\frac{5\pi}{12} > 0$.

$\therefore \cos\frac{\pi}{12} = \frac{\sqrt{2+\sqrt{3}}}{2}$.

6a Outcomes Assessed: (i) E4 (ii) E4

Marking Guidelines

Criteria	Marks
i • Explanation of existence of repeated roots	1
ii • Using the sum of roots to evaluate $\alpha + \beta$	1
• Using sum of products taken two at a time to evaluate $\alpha\beta$	1
• Evaluating m	1
• Evaluating b	1

Answer

i. Since the line $y = mx + b$ is tangent to the curve $y = x^4 + 4x^3$ at P where $x = \alpha$, and at Q where $x = \beta$, solving these equations simultaneously gives the equation $x^4 + 4x^3 - mx - b = 0$ with repeated roots $\alpha, \alpha, \beta, \beta$.

ii. Using the sum of roots is -4 and sum of products taken two at a time is 0 :

$2\alpha + 2\beta = -4$

$\alpha^2 + \beta^2 + 4\alpha\beta = 0 \Rightarrow (\alpha+\beta)^2 + 2\alpha\beta = 0$

$\therefore \alpha + \beta = -2$ and $\alpha\beta = -2$

Using the sum of products of roots taken three at a time is m, and the product of roots is $-b$:

$m = 2\beta\alpha^2 + 2\alpha\beta^2 = 2\alpha\beta(\alpha+\beta) = 8$

$b = -\alpha^2\beta^2 = -4$

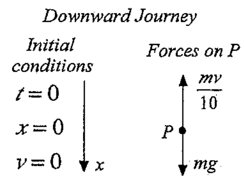
6b Outcomes Assessed: (i) E5 (ii) E5

Marking Guidelines

Criteria	Marks
i • Showing forces on P and applying Newton's 2nd Law to obtain equation of motion	1
• Deducing terminal velocity	1
ii • Showing forces on P and applying Newton s 2nd Law to derive equation of motion	1
• Integration to find t as a function of v, using initial conditions to evaluate constant	2
• Rearranging to find v as a function of t	1
• Integration to find x as a function of t, using initial conditions to evaluate constant	2
• Substituting $v = 0$ to find t and x at maximum height	2

Answer

i.



By Newton's 2nd Law,

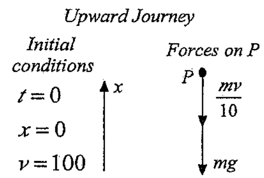
$ma = mg - \frac{mv}{10}$

$\therefore a = 10 - \frac{v}{10}$

$a \to 0$ as $v \to 100$

Hence terminal velocity is $100\,\text{ms}^{-1}$.

ii.



By Newton's 2nd Law,

$ma = -mg - \frac{mv}{10}$

$\therefore a = -10 - \frac{v}{10}$

$-10\frac{dv}{dt} = 100 + v$

$-\frac{1}{10}\frac{dt}{dv} = \frac{1}{100+v}$

$-\frac{1}{10}t = \ln(100+v)A$

$t = 0, \ v = 100 \Rightarrow A = \frac{1}{200}$

$\therefore e^{-\frac{1}{10}t} = \frac{100+v}{200}$

$v = 100\left(2e^{-\frac{1}{10}t} - 1\right)$

$\frac{dx}{dt} = 100\left(2e^{-\frac{1}{10}t} - 1\right)$

$x = 100\left(-20e^{-\frac{1}{10}t} - t + c\right)$

$t = 0, \ x = 0 \Rightarrow c = 20$

$\therefore x = 100\left(20 - 20e^{-\frac{1}{10}t} - t\right)$

At maximum height,

$v = 0 \Rightarrow e^{-\frac{1}{10}t} = \frac{1}{2}$

$\therefore t = 10\ln 2, \quad x = 1000(1 - \ln 2)$

Hence particle attains maximum height of $1000(1-\ln 2)$ metres after $10\ln 2$ seconds.

7a Outcomes Assessed: H5

Marking Guidelines

Criteria	Marks
• Using defining relation for an AP to show $A + C = 2B$	1
• Expanding $\cos(A+C)$	1
• Using identity for $\cos 2B$ and rearranging to obtain required result	1

Answer

$C - B = B - A$

$A + C = 2B$

$\cos(A+C) = \cos 2B$

$\cos A\cos C - \sin A\sin C = \cos^2 B - \sin^2 B$

$\cos A\cos C - \cos^2 B = \sin A\sin C - \sin^2 B$

7b Outcomes Assessed: (i) H5 , PE3 (ii) H5 , PE3

Marking Guidelines

Criteria	Marks
i • Finding the derivative	1
• Rearranging to show this derivative is positive for $x>0$	1
ii • Using $y=0$ when $x=0$ and y is an increasing function to deduce required result	2

Answer

i. $y=\tan^{-1}x - x + \frac{1}{3}x^3$

$$\frac{dy}{dx}=\frac{1}{1+x^2}-1+x^2$$
$$=\frac{1+(x^2-1)(1+x^2)}{1+x^2}$$
$$=\frac{x^4}{1+x^2}$$
$$\therefore \frac{dy}{dx}>0 \text{ for all } x>0.$$

ii. $y=0$ when $x=0$, and y is an increasing function of x for $x>0$ (since $\frac{dy}{dx}>0$ for $x>0$)

$\therefore y>0$ for all $x>0$

$\therefore \tan^{-1}x > x-\frac{1}{3}x^3$ for all $x>0$.

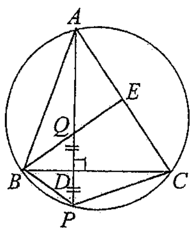
7c Outcomes Assessed: (i) H5 (ii) H5 (iii) PE3 (iv) PE3

Marking Guidelines

Criteria	Marks
i • Showing given information on copied diagram	1
ii • Establishing SAS congruence	2
iii • Writing a sequence of deductions to show DE subtends equal angles at A and B	2
• Supporting each of these deductions with a reason	2
iv • Using properties of cyclic quadrilateral $BDEA$ to deduce $BE\perp CA$	1

Answer

i.



ii.

In ΔBDP, ΔBDQ

$DP=DQ$ (*given*)

$\angle BDP=\angle BDQ$ (*both* $90°$ *since* $AD\perp BC$)

BD is common

$\therefore \Delta BDP\equiv\Delta BDQ$ (SAS)

iii. $\angle PBD=\angle QBD$ (*corresponding* $\angle$*'s in congruent* Δ*'s are equal*)

$\therefore \angle PBC=\angle EBD$ (*B, D, C collinear and B, Q, E collinear*)

$\angle PBC=\angle PAC$ ($\angle$*'s subtended by arc PC at circumference are equal*)

$\therefore \angle EBD=\angle PAC$

$\therefore \angle EBD=\angle DAE$ (*P, D, A collinear and C, E, A collinear*)

$\therefore$ *BDEA* is a cyclic quadrilateral (*DE subtends equal angles at A, B on same side of DE*)

iv. $\angle BDA=\angle BEA$ (*chord AB subtends equal angles at D, E on circumference of circle BDEA*)

$\therefore \angle BEA=90°$ (*given* $\angle BDA=90°$)

$\therefore BE\perp CA$

8a Outcomes Assessed: H5

Marking Guidelines

Criteria	Marks
• Considering the fact that all sides are positive to deduce that $d>-\frac{1}{2}$	1
• Using the fact that the sum of any two sides exceeds the third to obtain 3 inequalities	1
• Considering all the restrictions on d to deduce the possible values for d	1

Answer

The sides are $1,\ 1+d,\ 1+2d$. Since all the sides are positive, $d>-\frac{1}{2}$.

Also the sum of any two sides exceeds the third side.

$1+(1+d)>1+2d \Rightarrow d<1$

$1+(1+2d)>1+d \Rightarrow d>-1$

$(1+d)+(1+2d)>1 \Rightarrow d>-\frac{1}{3}$

Hence since all the above inequalities for d must be true, $-\frac{1}{3}<d<1$

8b Outcomes Assessed: (i) PE3 (ii) PE3

Marking Guidelines

Criteria	Marks
i • Counting possible arrangements given the restriction	1
• Calculating the required probability	1
ii • Calculating the required probability with adequate explanation	2

Answer

i. Since there are 5 vowels (V) and 4 consonants (C), if no two vowels are together, the only possible pattern is V C V C V C V C V.

The number of such arrangements is $5!\times 4!$, while the total number of arrangements is $9!$

Hence $P(\textit{no 2 vowels together})=\frac{5!\,4!}{9!}=\frac{1}{126}$

ii. There are $5!$ arrangements of the vowels, only one of which will be alphabetical order left to right.

Hence the proportion of the possible arrangements with the vowels in alphabetical order is $\frac{1}{5!}$.

Hence $P(\textit{vowels in alphabetical order L to R})=\frac{1}{120}$

8c Outcomes Assessed: (i) HE2 , E9 (ii) HE2 , E9

Marking Guidelines

Criteria	Marks
i • Verifying statement true for $n=2$	1
• Showing if statement true for $n=k$ then true for $n=k+1$ $(k\ge 2)$	2
• Explaining how all such statements are true for $n\ge 2$ by Mathematical Induction.	1
ii • Verifying statement true for $n=2$	2
• Showing if statement true for $n=k$ then true for $n=k+1$ $(k\ge 2)$	2

Answer

i. Let $S(n)$ be the statement $x^n = T_{n-1} + T_n x$, $n = 2, 3, 4, \ldots$

Consider $S(2)$: $LHS = x^2 = x + 1$, $RHS = T_1 + T_2 x = 1 + x$ Hence $S(2)$ is true.

If $S(k)$ is true : $x^k = T_{k-1} + T_k x$ **

Consider $S(k+1)$, $k \geq 2$: $x^{k+1} = x.x^k$

$$\begin{aligned}
&= x\left(T_{k-1} + T_k x\right) \quad \text{if } S(k) \text{ is true, using **} \\
&= xT_{k-1} + x^2 T_k \\
&= xT_{k-1} + (x+1)T_k \\
&= T_k + \left(T_k + T_{k-1}\right)x \\
&= T_k + T_{k+1}x
\end{aligned}$$

Hence if $S(k)$ for some $k \geq 2$, then $S(k+1)$ is true. But $S(2)$ is true, hence $S(3)$ is true, and then $S(4)$ is true and so on. Hence by Mathematical Induction, $S(n)$ is true for all positive integers $n \geq 2$.

ii. The Fibonacci sequence begins 1, 1, 2, 3, 5, ...

Let $U(n)$ be the statement $1 + x + x^2 + \ldots + x^n = T_{n+1} + (T_{n+2} - 1)x$, $n = 2, 3, 4, \ldots$

Consider $U(2)$: $LHS = 1 + x + x^2 = 1 + x + 1 + x = 2 + 2x$

$RHS = T_3 + \left(T_4 - 1\right)x = 2 + (3-1)x = 2 + 2x$ Hence $U(2)$ is true.

If $U(k)$ is true : $1 + x + x^2 + \ldots + x^k = T_{k+1} + (T_{k+2} - 1)x$ ***

Consider $U(k+1)$, $k \geq 2$: $LHS = \left(1 + x + x^2 + \ldots + x^k\right) + x^{k+1}$

$$\begin{aligned}
&= T_{k+1} + (T_{k+2} - 1)x + x^{k+1} \quad \text{if } U(k) \text{ is true, using ***} \\
&= T_{k+1} + (T_{k+2} - 1)x + T_k + T_{k+1}x \quad \text{using the result from (i)} \\
&= \left(T_{k+1} + T_k\right) + \left\{\left(T_{k+2} + T_{k+1}\right) - 1\right\}x \\
&= T_{k+2} + \left(T_{k+3} - 1\right)x \\
&= RHS
\end{aligned}$$

Hence if $U(k)$ for some $k \geq 2$, then $U(k+1)$ is true. But $U(2)$ is true, hence $U(3)$ is true, and then $U(4)$ is true , and so on. Hence by Mathematical Induction, $U(n)$ is true for all positive integers $n \geq 2$.

Independent Trial HSC 2004 Mathematics Extension 2 Mapping Grid

Question	Marks	Syllabus Content	Syllabus Outcomes	Targeted Performance Bands
1(a)(i)—(iii)	4	Graphs	P5	E2—E3
1(b)(i)—(ii)	7	Graphs	HE4 , E6	E2—E3
1(c)(i)—(ii)	4	Graphs	H5 , HE5	E2—E3
2(a)(i)—(ii)	3	Integration	H3 , H5	E2—E3
2(b)	3	Integration	HE6 , E8	E2—E3
2(c)	4	Integration	HE6	E2—E3
2(d)(i)—(ii)	5	Integration	H5 , E8	E2—E3
3(a)(i)—(ii)	4	Complex Numbers	E3	E2—E3
3(b)(i)—(ii)	3	Complex Numbers	E3	E2—E3
3(c)(i)—(ii)	4	Complex Numbers	E3	E2—E3
3(d)(i)—(ii)	4	Complex Numbers	E3	E2—E3
4(a)(i)—(iii)	9	Conic Sections	E3 , E4	E2—E3
4(b)(i)—(ii)	6	Conic Sections	E3 , E4	E2—E3
5(a)(i)—(iii)	7	Integration	H8 , E7	E2—E4
5(b)(i)—(iv)	8	Complex Numbers	P4 , H5 , E3	E2—E4
6(a)(i)—(ii)	5	Polynomials	E4	E2—E3
6(b)(i)—(ii)	10	Mechanics	E5	E3—E4
7(a)	3	Harder 3 Unit	H5	E2—E3
7(b)(i)—(ii)	4	Harder 3 Unit	H5 , PE3	E2—E4
7(c)(i)—(iv)	8	Harder 3 Unit	H5 , PE3	E2—E4
8(a)	3	Harder 3 Unit	H5	E2—E3
8(b)(i)—(ii)	4	Harder 3 Unit	PE3	E3—E4
8(c)(i)—(ii)	8	Harder 3 Unit	HE2 , E9	E3—E4