NSW INDEPENDENT SCHOOLS

TRIAL EXAMINATION

1998

MATHEMATICS

2/3 UNIT (COMMON)

Time Allowed - Three hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are supplied.
- Board-approved calculators may be used.
- Each question attempted is to be handed in separately clearly marked Question 1, Question 2 etc..
- Write your Student Number/Name on every page
- ***The question paper must be handed to the supervisor at the end of the examination.***

STUDENT NUMBER / NAME.......................................

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Question 1

(a) Factorise completely $3t^2-18t-81$. **2**

(b) Evaluate correct to 3 significant figures : $\dfrac{28.36-17.24}{(6.1\times 5.2)^2}$ **2**

(c) Simplify $\dfrac{1}{x-y}-\dfrac{1}{x+y}$ **2**

(d) Find a primitive of $(2x-1)^3$ **1**

(e) Given $f(x)=\begin{cases} x-2 \text{ for } x\le 2 \\ (x-2)^2 \text{ for } x>2 \end{cases}$ **2**

Find (i) $f(-1)$

(ii) $f(a+2)$ where $a>0$

(f) Express $\dfrac{11}{3\sqrt{3}-1}$ with rational denominator. **2**

(g) Solve $|2x-1|<3$ **2**

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Question 2 *(Start a new page)*

(a) Differentiate with respect to x **5**

(i) $\sqrt{2x-3}$

(ii) x^3e^{-x}

(iii) $\cos(8-3x)$

(b) The value of AMP shares jumped to $45.00 when put on the market but then fell quickly. The rate of decrease slowed gradually until the price stabilised around $19.00 four days later. Draw a graph of the share value as a function of time showing this fall in price from $45.00 to $19.00. **2**

(c) Evaluate $\int_0^{\frac{\pi}{3}} \sin 2x \, dx$ **3**

(d) Find $\int \frac{x^2+1}{x} dx$ **2**

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Question 3 *(Start a new page)*

(a) $A(-3,1)$, $B(2,-1)$ and $C(-1,6)$ are three points on the number plane. Draw a diagram showing these points. **9**

(i) Find the gradient of BC.

(ii) M is the midpoint of BC. Find the coordinates of M. Mark it on your diagram.

(iii) The line l is the perpendicular bisector of BC. Show that the line l has equation $3x - 7y + 16 = 0$.

(vi) Find the equation of the line AB.

(v) Find the coordinates of the point of intersection of l and AB.

(vi) What does the result in (v) tell you about the triangle ABC.

(b) Sketch the graph of the function $y = x^2 - x - 6$ showing the points of intersection with the axes. Using your graph, or otherwise, solve the inequality $x^2 - x - 6 \geq 0$ **3**

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Question 4 *(Start a new page)*

(a) Water is dripping from a tap into a bucket which is 42 cm high. After 1 minute the depth of the water is 2 mm, after 2 minutes the depth is 6 mm, after 3 minutes it is 12 mm, after 4 minutes 20 mm and so on. **5**

(i) Show that the **increase** in the depth of water in the bucket each minute forms an Arithmetic Progression.

(ii) What is the increase in depth in the 10th minute?

(iii) When is the bucket full?

(b) Shade on a number plane the region $x^2+(y-4)^2\le 16$ **2**

(c) Find the equation of the normal to the curve $y=e^{\frac{x}{2}}$ at the point $(2,e)$. **3**

(d) Sketch the curve $y=2-\sin 3x$ for $0\le x\le 2\pi$. **2**

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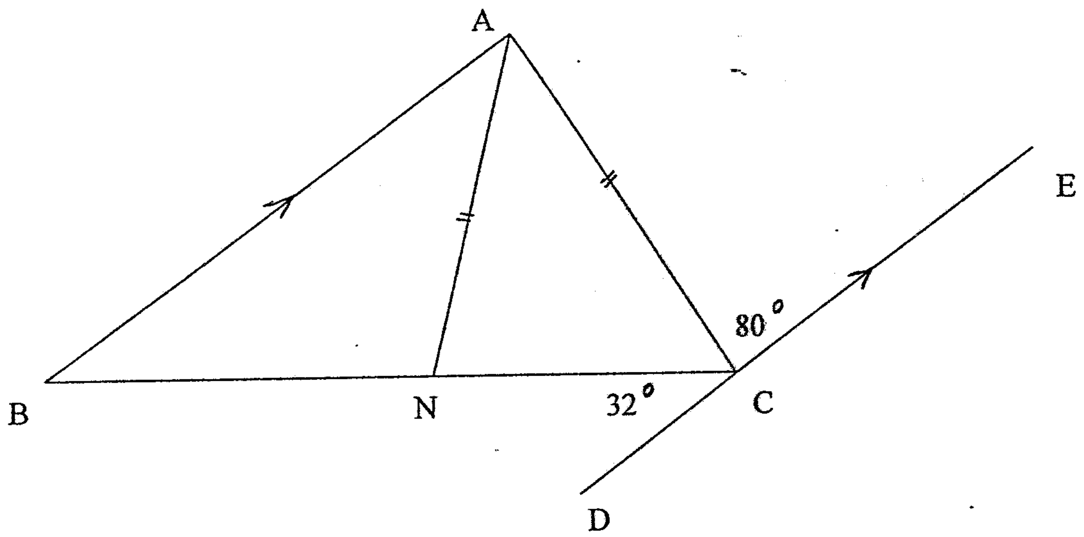
Question 5 *(Start a new page)*

(a) If $\tan\alpha = \frac{3}{5}$ and, $\pi \le \alpha \le \frac{3\pi}{2}$ find the exact values of : **3**

(i) $\cos\alpha$

(ii) $\operatorname{cosec}\alpha$

(b) **3**



In the diagram above

$AB \parallel ED$, $AN = AC$

$\angle NCD = 32^0$ and $\angle ACE = 80^0$

Find $\angle NAB$

(c) Copy and complete the table for $f(x) = \log_e x$ **3**

x	1	2	3	4	5
$f(x)$					

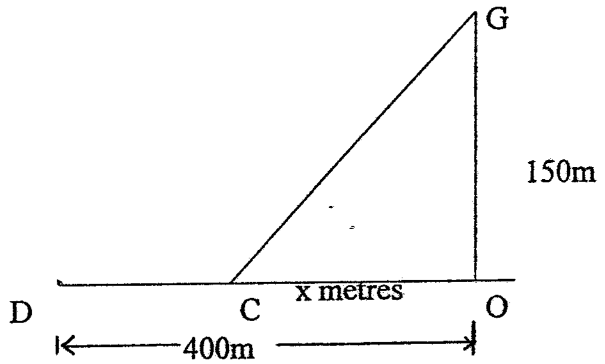
Using Simpson's Rule with the above function values, find an estimate for

$$\int_1^5 \log_e x\,dx$$

(d) Find the exact value of $\dfrac{\tan 60^o \cdot \cos 45^o}{\sin 30^o}$ (leave your answer in surd form) **3**

Question 6 *(Start a new page)*

(a) A girl, G, on a surfboard is 150metres from the nearest point *O* of a straight beach. Her destination, *D*, is 400 metres along the beach from *O*. She can paddle at 15m/s and walk at 20 m/s. The girl realises that the quickest way to get to *D* is to paddle to a point C, somewhere between O and D and then walk. This is illustrated in the diagram below. **7**



(i) If OC = x metres, show that GC is a distance of $\sqrt{22500+x^2}$ metres.

(ii) Explain why the time taken to travel from G to C could be expressed as $\frac{\sqrt{22500+x^2}}{15}$ seconds.

(iii) Show that the total time, T seconds, to travel from G to C and then to D could be expressed as :

$$T=\frac{\sqrt{22500+x^2}}{15}+\frac{400-x}{20}$$

(iv) Use calculus to find the distance her landing point C should be from O so that she completes her trip in the least amount of time.

(b) The curve $y=f(x)$ has gradient function $f'(x)=kx-2$. The curve has gradient 10 at the point (2,9) on it. **3**

Find (i) The value of k

(ii) $f(x)$

(c) Paul and Francis have decided to install a security system in their home. Paul wants to use a 4 digit number as their security code while Francis thinks a 3 letter code would be better. Assuming that repetition is allowed in both systems, which is the more secure. Give reasons for your answer. **2**

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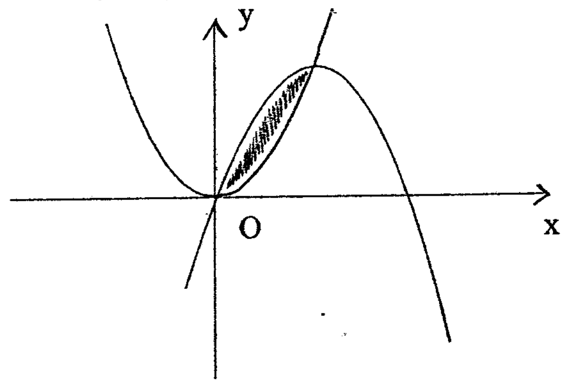
Question 7 *(Start a new page)*

(a) The graphs $y = x^2$ and $y = x(4-x)$ are shown below **4**



(i) Find the coordinates of the points of intersection of the two graphs

(ii) Hence find the area of the shaded region.

(b) Bill started a new company manufacturing plastic cups. The company produced 1 000 cups in the first week and plans to increase production by 20% each week. The maximum number of cups the factory can produce is 3583. Subsequently the company hopes to maintain this weekly production of 3583 cups per week. **6**

(i) How many cups does the company plan to produce in the 3rd week?

(ii) How many weeks will it take for the company to reach its maximum of 3583 cups per week? (Answer to the nearest week).

(iii) Show that the total number of plastic cups produced in "w" weeks is $5000\left(1.2^{w}-1\right)$, for $w \le$ the answer to part (ii).

(iv) What will the total number of plastic cups produced in 20 weeks?

(c) For which value(s) of k will the quadratic equation $x^2-(k-1)x-(2k+1)=0$ have equal roots **2**

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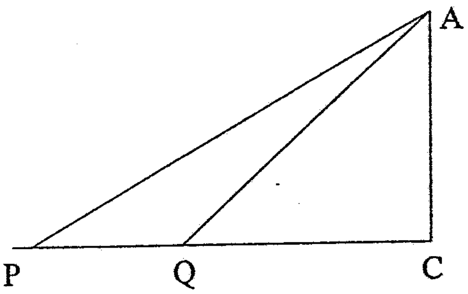
Question 8 *(Start a new page)*

(a) From a point P, a man observes that the angle of elevation to the top, A, of a cliff AC is 40^0. After walking 100 metres along a straight level road towards the cliff to point Q, he finds that the angle of elevation is now 48^0. **4**

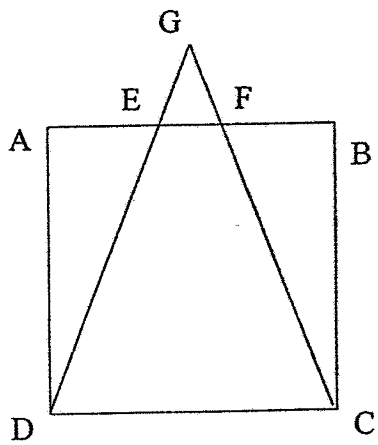
(i) Copy the diagram below and fill in the information.



(ii) Show that the distance AQ is given by $AQ = \dfrac{100\sin 40^0}{\sin 8^0}$

(iii) Calculate the height of the cliff.

(b) **5**

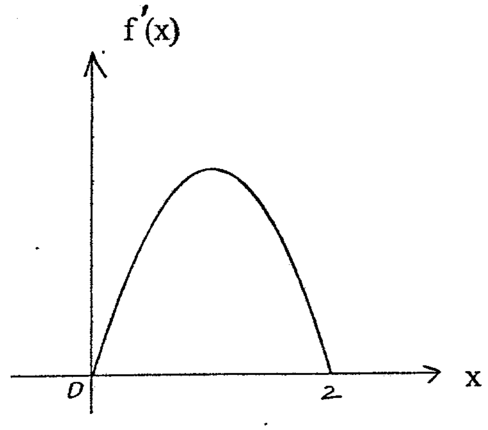


In the diagram above *ABCD* is a square. $AE = BF$. Copy the diagram onto your working paper.

(i) Prove that $\Delta AED \equiv \Delta BFC$.

(ii) hence show that ΔGEF is isosceles.

(c) **3**



The graph shows the first derivative of a function $y = f(x)$ for the domain $0 \le x \le 2$. You are also given that $f(0) = 0$. Draw a possible graph of $y = f(x)$.

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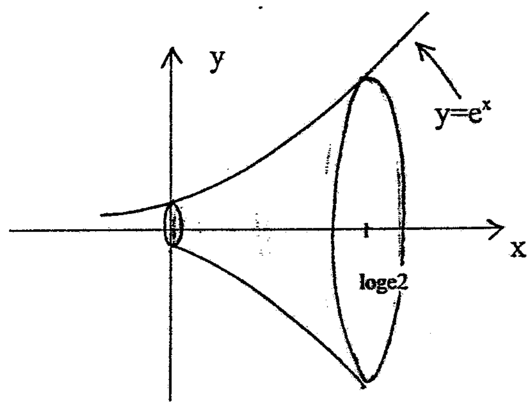
Question 9 *(Start a new page)*

(a) A die is biased such that a "6" is twice as likely to show as any other number when it is rolled. The numbers 1 to 5 are all equally likely. If the die is rolled three times find **4**

(i) the probability of scoring three sixes.

(ii) the probability of scoring no more than two sixes.

(b) Funerary vases were used by the ancient Egyptians to stores the intestines of mummies. One of these vases on display in the "Life and Death of the Pharaohs" exhibition at the Australian Museum was designed by rotating the curve $y = e^x$ about the x-axis, as shown in the diagram below. The vase is $(\log_e 2)$ m tall. **4**



Calculate the volume of the vase.

(c) (i) Given that $f(x) = \sin^2 x$, find $f'(x)$ and $f''(x)$ **4**

(ii) Show that $\dfrac{f''(x) + 2f(x)}{f'(x)} = \cot x$.

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Question 10 *(Start a new page)*

(a) The velocity of an object, in cm/sec, is given by the formula **6**

$$v = \frac{1}{t+1},\ t \geq 0.$$

(i) Given that the object is initially at $x = 4$, find its displacement as a function of time.

(ii) Express the acceleration of the object as a function of time.

(iii) Find the acceleration when the displacement is 5cm.

(iv) Describe the behaviour of the object as time becomes large.

(b) A new radioactive isotope POBIUM decays at a rate proportional to the amount P of Pobium present, i.e. $\frac{dP}{dt} = -kP$. It is known that 100g of Pobium reduces to 80g in 2 hours. **6**

(i) Show that $P = P_0 e^{-kt}$ satisfies the equation $\frac{dP}{dt} = -kP$.

(ii) Find the values of P_0 and k.

(iii) Hence find the half-life of Pobium (correct to the nearest minute).

1998 HSC TRIAL
SUGGESTED ANSWERS

1 (a) $3(t^2-6t-27)$

$=\underline{3(t-9)(t+3)}$

(b) ~~0.009879153~~ 0.0111

$=$ ~~0.00988~~

(c) $\dfrac{(x+y)-(x-y)}{x^2-y^2}$

$=\dfrac{x+y-x+y}{x^2-y^2}$

$=\underline{\dfrac{2y}{x^2-y^2}}$

(d) $\dfrac{(2x-1)^4}{4\times 2}+c$

$=\underline{\frac{1}{8}(2x-1)^4+c}$

(e) $f(-1)=-1-2$

$=\underline{-3}$

$f(a+2)=((a+2)-2)^2$

$=\underline{a^2}$

(f) $\dfrac{11}{3\sqrt{3}-1}\times\dfrac{3\sqrt{3}+1}{3\sqrt{3}+1}$

$=\dfrac{11(3\sqrt{3}+1)}{27-1}$

$=\underline{\dfrac{11(3\sqrt{3}+1)}{26}}$

(g) $2x-1<3$ | $-2x+1<3$

$2x<4$ | $-2x<2$

$x<2$ | $x>-1$

$\therefore \underline{-1<x<2}$

Q2 (a)(i) $\dfrac{d}{dx}(2x-3)^{1/2}$

$=\frac{1}{2}(2x-3)^{-1/2}\times 2$

$=\underline{\dfrac{1}{\sqrt{2x-3}}}$

(ii) $x^3\cdot -e^{-x}+e^{-x}\cdot 3x^2$

$=\underline{x^2e^{-x}(3-x)}$

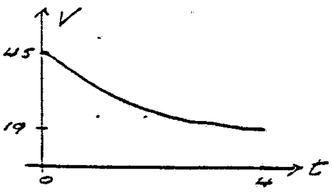
(iii) $\dfrac{d}{dx}\cos(8-3x)$

$=-3\times -\sin(8-3x)$

$=\underline{3\sin(8-3x)}$

①

(b)



(c) $\left[-\frac{1}{2}\cos 2x\right]_0^{\pi/3}$

$=-\frac{1}{2}\left(\cos\frac{2\pi}{3}-\cos 0\right)$

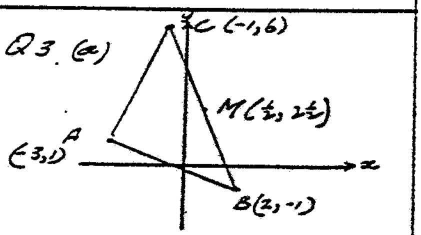
$=-\frac{1}{2}\left(-\frac{1}{2}-1\right)$

$=\underline{3/4}$

(d) $\int\left(x+\frac{1}{x}\right)dx$

$=\underline{\dfrac{x^2}{2}+\ln x+c}$

Q3 (a)



(i) $m_{BC}=\dfrac{6--1}{-1-2}=\underline{\dfrac{-7}{3}}$

(ii) $M=\left(\dfrac{-1+2}{2},\dfrac{6-1}{2}\right)$

$=\underline{\left(\frac{1}{2},2\frac{1}{2}\right)}$

(iii) $m_2=+\dfrac{3}{7}$

$\therefore y-2\frac{1}{2}=\frac{3}{7}\left(x-\frac{1}{2}\right)$

$7y-17\frac{1}{2}=3x-1\frac{1}{2}$

$\underline{3x-7y+16=0}$

(iv) $\dfrac{y+1}{x-2}=\dfrac{1+1}{-3-2}$

$-5y-5=2x-4$

$\underline{2x+5y+1=0}$

(v) $6x-14y+32=0$

$6x+15y+3=0$

$29y-29=0$

$y=1$

$\therefore 2x+5+1=0$

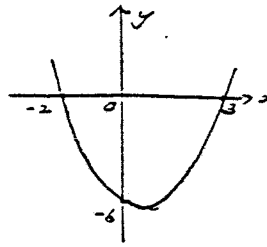
$x=-3$

$\underline{\text{meet at } A(-3,1)}$

(vi) $\underline{\text{isosceles}}$ as $\perp$ bisector passes through the vertex

2U 1998 Solutions

(b) $y=(x-3)(x+2)$



$\underline{x\geq 3 \text{ OR } x\leq -2}$

Q4 (a)(i) $T_1=2$, $T_2=4$, $T_3=6$, $T_4=8$.

$\underline{AP}$, $a=2$, $\underline{d=2}$.

(ii) $T_{10}=2+9\times 2$

$=\underline{20\text{ mm.}}$

(iii) $S_n=\frac{n}{2}(2a+(n-1)d)$

$420=\frac{n}{2}(4+(n-1)2)$

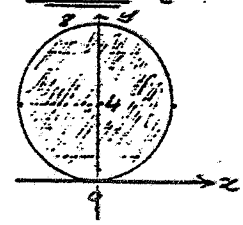
$=n(n-1)$

$n^2-n-420=0$

$(n-21)(n+20)=0$

$\therefore \underline{n=21}$ (as $n\neq -20$,

(b)



(c) $y'=\frac{1}{2}e^{x/2}$

$m_1=\frac{1}{2}e^1$

$m_2=-\dfrac{2}{e}$

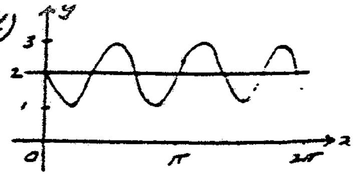
Normal is

$y-e=-\frac{2}{e}(x-2)$

$ey-e^2=-2x+4$

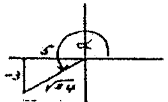
$\underline{2x-ey-e^2-4=0}$

(d)



These suggested answers/marking schemes are issued as a guide only -offerred as an assistance in constructing your own marking format (individual teachers/schools find many other acceptable responses)

Q5(a)



$\cos\alpha = -\frac{5}{\sqrt{34}}$

$\operatorname{cosec}\alpha = -\frac{\sqrt{34}}{3}$

(b) $A\hat{C}N + 32° + 80° = 180°$ ($\angle$ sum of st. line)

$A\hat{C}N = 68°$

$\triangle ACN$ is isos. (AC = AN)

$\therefore A\hat{N}C = 80°$ (base $\angle$s isos $\triangle$)

$N\hat{A}C + 2\times80° = 180°$ ($\angle$ sum $\triangle ANC$)

$\therefore N\hat{A}C = 20°$

$B\hat{A}C = A\hat{C}E = 80°$ (alt $\angle$s AB // ED)

$\therefore N\hat{A}B = 80° - 20° = 60°$

(c)

1	2	3	4	5
0	·693	1·099	1·386	1·609

$\int_1^5 \log_e x\,dx \doteqdot \frac{2}{6}\{0 + 1{\cdot}609 + 2\times1{\cdot}099 + 4\times(0{\cdot}693 + 1{\cdot}386)\}$

$\doteqdot 4{\cdot}041$

(e) $\frac{\sqrt3 \times 1/\sqrt2}{1/2}$

$= \sqrt3 \times \frac{1}{\sqrt2} \times 2 = \sqrt6$

(iii) $t_2 = \frac{DC}{20} = \frac{400 - x}{20}$

$\therefore T = \frac{\sqrt{22500 + x^2}}{15} + \frac{400 - x}{20}$

$T = t_1 + t_2$

(iv) $\frac{dT}{dx} = \frac{1}{2}\frac{(22500 + x^2)^{-\frac12}\cdot 2x}{15} - \frac{1}{20}$

$\frac{dT}{dx} = 0$ for Min/Max

$\therefore \frac{x}{15\sqrt{22500 + x^2}} = \frac{1}{20}$

$400x^2 = 225(22500 + x^2)$

$175x^2 = 5062500$

$x = 170{\cdot}08$

x	169	170	171
$f'(x)$	−	0	+

$\therefore$ T is min when $x \doteqdot 170$ m

(b) (i) $10 = 2k - 2$

$k = 6$

(ii) $f(x) = 3x^2 - 2x + c$

$9 = 12 - 4 + c$

$c = 1$

$\therefore f(x) = 3x^2 - 2x + 1$

(c) 4 digit $\Rightarrow 10^4$

10 000 codes

3 letters $\Rightarrow 26^3$

17576 codes

$\therefore$ 3 letters more secure

(ii) $A = \int_0^2 [x(4 - x) - x^2]\,dx$

$= \int_0^2 (4x - 2x^2)\,dx$

$= \left[2x^2 - \frac{2}{3}x^3\right]_0^2$

$= 8 - \frac{16}{3}$

$= 2\frac{2}{3}$ units^2

(b) GP $r = 1{\cdot}2$

(i) $T_3 = 1000 \times 1{\cdot}2^2$

$= 1440$

(ii) $3583 = 1000 \times 1{\cdot}2^{n-1}$

$3{\cdot}583 = 1{\cdot}2^{n-1}$

$\ln 3{\cdot}583 = (n-1)\ln 1{\cdot}2$

$n - 1 = \frac{\ln 3{\cdot}583}{\ln 1{\cdot}2}$

$n = 7{\cdot}99$

$n \doteqdot 8$ weeks

(iii) $S_n = \frac{a(r^n - 1)}{r - 1}$

$= \frac{1000(1{\cdot}2^{40} - 1)}{1{\cdot}2 - 1}$

$= 5000(1{\cdot}2^{40} - 1)$

(iv) Total

$= 5000(1{\cdot}2^8 - 1)$

$+ 12 \times 3583$

$= 59\,495$ cups

(c) $\Delta = [-(k-1)]^2 - 4[-(2k+1)]$

$= k^2 + 6k + 5$

$= (k+1)(k+5)$

Equal Roots $\Delta = 0$

$\therefore k = -1, -5$

Q6 (a)

(i) $CG^2 = 150^2 + x^2$

$CG = \sqrt{22500 + x^2}$

(ii) $t_1 = \frac{\sqrt{22500 + x^2}}{15}$

$(t = D/S)$

Q7 (a)(i) $x^2 = 4x - x^2$

$2x^2 - 4x = 0$

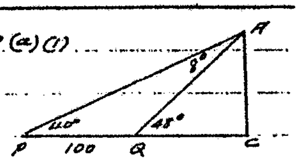
$2x(x - 2) = 0$

$x = 0, 2$

$y = 0, 4$

$(0,0)$ $(2,4)$

Q8 (a)(i)



(ii) $\frac{AQ}{\sin 40°} = \frac{100}{\sin 8°}$

$\therefore AQ = \frac{100 \sin 40°}{\sin 8°}$

(b)(i) In $\triangle$'s AED, BFC

AE = BF (data)

AD = BC (sides of a square)

$D\hat{A}E = C\hat{B}F = 90°$ (angles of a square)

$\therefore \triangle AED \equiv \triangle BFC$ (AAS Test)

(ii) $A\hat{E}D = B\hat{F}C$ (corr $\angle$'s in cong $\triangle$'s)

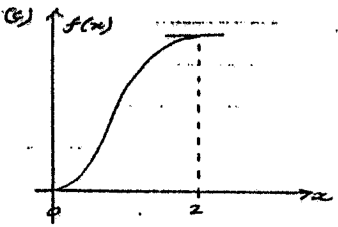
$A\hat{E}D = G\hat{E}F$ (vert. opp.)

sim $B\hat{F}C = G\hat{F}E$

$\therefore G\hat{E}F = G\hat{F}E$

$\therefore \triangle GEF$ is isosceles (2 angles equal)

(c)



Q9 (a)

(i) $P(6) = 2/7$

$\therefore P(3\text{ sixes}) = \left(\frac{2}{7}\right)^3$

$= \frac{8}{343}$

(ii) P(no more than 2 sixes)

$= 1 - P(3\text{ sixes})$

$= 1 - \frac{8}{343}$

$= \frac{335}{343}$

$V = \pi\int_0^{\ln 2} (e^x)^2\,dx$

$= \pi\left[\frac{e^{2x}}{2}\right]_0^{\ln 2}$

$= \frac{\pi}{2}(e^{2\ln 2} - e^0)$

$= \frac{\pi}{2}((e^{\ln 2})^2 - 1)$

$= \frac{\pi}{2}(2^2 - 1)$

$= \frac{3\pi}{2}$ units^3

(c) (i) $f(x) = \sin^2 x$

$f'(x) = 2\sin x \cos x$

$f''(x) = 2\sin x(-\sin x) + 2\cos x \cdot \cos x$

$= 2(\cos^2 x - \sin^2 x)$

(ii) LHS $\equiv$

$\frac{2\cos^2 x - 2\sin^2 x + 2\sin^2 x}{2\sin x \cos x}$

$= \frac{\cos x}{\sin x}$

$= \cot x$ = RHS

Q10 (a)

(i) $\frac{dx}{dt} = \frac{1}{t+1}$

$\therefore x = \ln(t+1) + c$

$4 = \ln(0+1) + c$

$c = 4$

$x = \ln(t+1) + 4$

(ii) $a = \frac{d^2x}{dt^2}$

$= -1(t+1)^{-2}\cdot 1$

$= \frac{-1}{(t+1)^2}$

(iii) $5 = \ln(t+1) + 4$

$\ln(t+1) = 1$

$t + 1 = e^1 = e$

$t = e - 1$

$\therefore a = \frac{-1}{((e-1)+1)^2}$

$= \frac{-1}{e^2}$ cm/sec^2

(iv) As a is negative and $\frac{-1}{(t+1)^2} \to 0$

$\therefore$ Object slows down at a reduced rate also $x \to \infty$

(b)(i) $\frac{dP}{dt} = -k P_0 e^{-kt}$

$= -kP$

(ii) $t = 0$ $P = 100$

$\therefore P_0 = 100$

$t = 2$ $P = 80$

$\therefore 80 = 100e^{-2k}$

$0{\cdot}8 = e^{-2k}$

$-2k = \ln(0{\cdot}8)$

$k = \frac{\ln(0{\cdot}8)}{-2}$

$k \doteqdot 0{\cdot}112$

(iii) $50 = 100e^{-0{\cdot}112t}$

$t = \frac{\ln 0{\cdot}5}{-0{\cdot}112}$

$= 6{\cdot}188\ldots$

$= 6$ hrs 11 mins