## STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\quad x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a},\quad a > 0,\quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\quad x > 0$

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NSW INDEPENDENT 2005

## 2005

Higher School Certificate

## Trial Examination

# Mathematics

## Extension 1

### General Instructions

- Reading time – 5minutes
- Working time – 2 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided with this paper
- All necessary working should be shown in every question

**Total marks - 84**

Attempt Questions 1 – 7

All questions are of equal value

**This paper MUST NOT be removed from the examination room**

STUDENT NUMBER/NAME: ……………………………

Marks

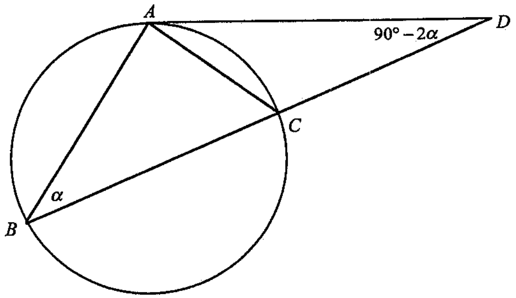
**Question 1** **Begin a new page**

(a) Evaluate $\sum_{k=1}^{4}(k!)^2$ **2**

(b) Find the acute angle between the lines $2x-y-1=0$ and $x-2y+1=0$. Give your answer correct to the nearest degree. **3**

(c) The equation $2x^3-6x+1=0$ has roots $\alpha$, $\beta$ and $\gamma$. Evaluate $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$. **3**

(d)



Triangle $ABC$ is inscribed in a circle. The tangent to the circle at $A$ meets $BC$ produced at $D$. $\angle ABC=\alpha$ and $\angle ADC=90°-2\alpha$, where $0°<\alpha<45°$.

(i) Copy the diagram

(ii) Give a reason why $\angle DAC=\angle ABC$. **1**

(iii) Show that $BC$ is a diameter of the circle. **3**

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Marks

**Question 2** **Begin a new page**

(a) Find the value of $\lim_{x\to 0}\frac{\sin 3x}{2x}$ **2**

(b)(i) Show that $\cos(A+B)+\cos(A-B)=2\cos A\cos B$. **1**

(ii) Hence evaluate $\cos 45°\cos 15°$ in simplest exact form. **2**

(c) $A(x,7)$ and $B(y,-11)$ are two points. $M(2,-2)$ is the midpoint of $AB$. $N(4,-5)$ divides $AB$ internally in the ratio 2 : 1.

(i) Write down two equations in $x$ and $y$. **2**

(ii) Hence find the values of $x$ and $y$. **1**

(d) $P(2p,p^2)$ and $Q(2q,q^2)$ are two points on the parabola $x^2=4y$. $M$ is the midpoint of $PQ$.

(i) Show that $(p-q)^2=2(p^2+q^2)-(p+q)^2$. **1**

(ii) If $P$ and $Q$ move on the parabola so that $p-q=1$, show that the locus of $M$ is the parabola $x^2=4y-1$ and find its focus. **3**

**Marks**

### Question 3

**Begin a new page**

(a) Consider the function $f(x)=\dfrac{x^2}{x^2-1}$.

(i) Find the domain of $f(x)$ and evaluate $\lim\limits_{x\to\infty} f(x)$. **2**

(ii) Show that $f(x)$ is an even function. **1**

(iii) Find the coordinates and nature of the stationary point on the curve $y=f(x)$. **2**

(iv) Sketch the graph of the curve $y=f(x)$ showing the coordinates of the stationary point and the equations of any asymptotes. **2**

(v) Explain why the function $f(x)$ does not have an inverse. **1**

(b) Use Mathematical Induction to show that $5^n+12n-1$ is divisible by $16$ for all positive integers $n\ge 1$. **4**

### Question 4

**Begin a new page**

(a) Consider the function $y=2\sin^{-1}\frac{x}{3}$.

(i) Find the domain and range of the function. **2**

(ii) Sketch the graph of the function showing clearly the coordinates of the endpoints. **1**

(iii) The region in the first quadrant bounded by the curve $y=2\sin^{-1}\frac{x}{3}$, the $y$ axis and the line $y=\pi$, is rotated through one complete revolution about the $y$ axis. Find the volume of the solid formed, giving your answer in simplest exact form. **3**

(b)(i) Show that the equation $e^x+x=0$ has a real root $\alpha$ such that $-1<\alpha<0$. **2**

(ii) On the same diagram, draw the graphs of $y=e^x$ and $y=-x$. Hence deduce that the equation $e^x+x=0$ has exactly one real root. **2**

(iii) If $a$ is taken as an initial approximation to this real root $\alpha$, use Newton's method to show that the next approximation $a_1$ is given by $a_1=\dfrac{(a-1)e^a}{e^a+1}$. Hence if the initial approximation is taken as $a=-0\cdot 5$, find the next approximation for $\alpha$ correct to 1 decimal place. **2**

**Marks**

### Question 5

**Begin a new page**

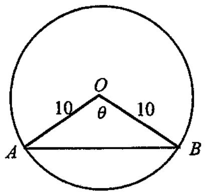
(a) Use the substitution $u=x+1$ to evaluate $\displaystyle\int_0^{15}\frac{x}{\sqrt{x+1}}\,dx$ **4**

(b) There are 4 multiple choice questions in a test. For each question there is a probability $\frac{1}{3}$ that Bob answers the question correctly.

(i) Find the exact probability that Bob answers exactly 2 of the 4 questions correctly. **2**

(ii) Find the exact probability that the fourth question Bob attempts is the second that he answers correctly. **2**

(c)



The chord $AB$ of a circle of radius 10 cm subtends an angle $\theta$ radians at the centre $O$ of the circle.

(i) Show that the perimeter $P$ cm of the minor segment cut off by the chord $AB$ is given by $P=10\theta+20\sin\frac{\theta}{2}$. 2

(ii) If $\theta$ is increasing at a rate of $0\cdot 02$ radians per second, find the rate at which $P$ is increasing when $\theta=\frac{2\pi}{3}$. 2

**Marks**

**Question 6** **Begin a new page**

(a) The number $N$ of individuals in a population at time $t$ years is given by $N = 100 + Ae^{-kt}$ for some constants $A > 0$, $k > 0$. The initial population size is 500, and when the population size is 300, it is decreasing at a rate of 20 individuals per year.

(i) Show that $\frac{dN}{dt} = -k(N - 100)$. **1**

(ii) Show that $A = 400$ and $k = 0{\cdot}1$. **2**

(iii) On separate diagrams sketch the graphs of $N$ as a function of $t$ and $\frac{dN}{dt}$ as a function of $t$. **2**

(iv) Find the first year during which the population size falls below 120. **1**

(b) A particle is moving in a straight line with Simple Harmonic Motion. At time $t$ seconds it has displacement $x$ metres from a fixed point $O$ on the line, velocity $v\,\text{ms}^{-1}$, and acceleration $a\,\text{ms}^{-2}$ given by $a = -4x + 4$. Initially the particle is 2 m to the right of $O$ and is moving away from $O$ with speed $2\sqrt{3}\ \text{ms}^{-1}$.

(i) Use integration to show that $v^2 = -4x^2 + 8x + 12$. **2**

(ii) Hence find the centre and amplitude of the motion. **2**

(iii) If $x = 1 + 2\cos(2t + \alpha)$ for some $0 < \alpha < 2\pi$, find the exact value of $\alpha$. **2**

**Marks**

**Question 7** **Begin a new page**

(a) A particle is projected from a point $O$ with speed $V\,\text{ms}^{-1}$ at an angle $\alpha$ above the horizontal where the acceleration due to gravity is $g\,\text{ms}^{-2}$. At time $t$ seconds, its horizontal and vertical displacements from $O$ are $x$ metres and $y$ metres respectively where $x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{1}{2}gt^2$.

(i) Show that the particle has a maximum height $H$ metres and a horizontal range $R$ metres given by $H = \frac{V^2\sin^2\alpha}{2g}$ and $R = \frac{V^2\sin 2\alpha}{g}$. **4**

(ii) Show that $R\tan\alpha = 4H$, and hence express $R$ in terms of $V$, $g$ and $H$. **4**

(b)(i) Expand $\left(x + \frac{1}{x}\right)^5$ in descending powers of $x$. **1**

(ii) If $x + \frac{1}{x} = a$, express $x^5 + \frac{1}{x^5}$ in terms of $a$. **3**

# *NSW INDEPENDENT TRIAL EXAMS -2005*

## Trial HSC Mathematics Extension 1 2005 Marking Guidelines

### Question 1

(a) **Outcomes Assessed: H5**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| • shows an understanding of the notation (eg writes the sum of squares) | 1 |
| • calculates the sum | 1 |

**Answer**

$$\sum_{k=1}^{4}(k!)^2 = 1^2+2^2+6^2+24^2 = 617$$

(b) **Outcomes Assessed: P4**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| • finds the gradients of both lines | 1 |
| • writes expression for $\tan\theta$ in terms of these gradients | 1 |
| • calculates the angle to the nearest degree | 1 |

**Answer**

The lines have gradients 2 and $\frac{1}{2}$. $\therefore \tan\theta = \left|\dfrac{2-\frac{1}{2}}{1+2\times\frac{1}{2}}\right| = \frac{3}{4}$ and $\theta \approx 37°$.

(c) **Outcomes Assessed: PE3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| • writes sum of reciprocals in terms of appropriate sums of products | 1 |
| • writes value of one of $\beta\gamma+\gamma\alpha+\alpha\beta,\ \alpha\beta\gamma$ | 1 |
| • writes value of the remaining expression then evaluates sum of reciprocals | 1 |

**Answer**

$$\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha\beta\gamma}=\frac{-3}{\left(-\frac{1}{2}\right)}=6$$

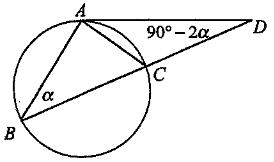
(d) **Outcomes Assessed: PE3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| ii. • quotes alternate segment theorem | 1 |
| iii.• finds either $\angle ACD$ or $\angle ACB$ in terms of $\alpha$ giving a reason | 1 |
| • explains why $\angle BAC = 90°$ | 1 |
| • deduces that $BC$ is a diameter giving a reason | 1 |

**Answer**

i.



ii. The angle between a tangent and a chord drawn to the point of contact is equal to the angle subtended by the chord in the alternate segment.

iii. $\angle ACB = \alpha + (90° - 2\alpha)$ *(exterior $\angle$ of $\Delta ACD$ is equal*
$\quad = 90° - \alpha$ *to sum of interior opp. $\angle$ s)*

$\therefore \angle BAC = 90°$ *($\angle$ sum of $\Delta ABC$ is* 180°)

$\therefore BC$ is a diameter *($\angle$ in a semi-circle is a right angle)*

### Question 2

(a) **Outcomes Assessed: H5**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| • rearranges expression into form $k\frac{\sin h}{h}$ | 1 |
| • evaluates limit | 1 |

**Answer**

$$\lim_{x\to 0}\frac{\sin 3x}{2x}=\lim_{x\to 0}\left(\frac{3}{2}\times\frac{\sin 3x}{3x}\right)=\tfrac{3}{2}\times 1=\tfrac{3}{2}$$

(b) **Outcomes Assessed: H5**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i. • expands LHS and simplifies | 1 |
| ii. • uses i. to convert product to a sum of cosines | 1 |
| • evaluates in simplest surd form | 1 |

**Answer**

i. $\cos(A+B) = \cos A\cos B - \sin A\sin B$
$\cos(A-B) = \cos A\cos B + \sin A\sin B$
$\therefore \cos(A+B)+\cos(A-B) = 2\cos A\cos B$

ii. $2\cos 45°\cos 15° = \cos(45°+15°)+\cos(45°-15°)$
$= \frac{1}{2} + \frac{\sqrt{3}}{2}$
$\therefore \cos 45°\cos 15° = \dfrac{1+\sqrt{3}}{4}$

(c) **Outcomes Assessed: P4**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i. • writes one of the two equations | 1 |
| • writes the second equation | 1 |
| ii. • finds the values of $x$ and $y$ | 1 |

**Answer**

i. $\dfrac{x+y}{2}=2 \Rightarrow x+y=4 \quad (1)$

and $\dfrac{x+2y}{3}=4 \Rightarrow x+2y=12 \quad (2)$

ii. $(2)-(1) \Rightarrow y=8$
sub. $y=8$ in $(1) \Rightarrow x=-4$

(d) **Outcomes Assessed: P4, PE3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i. • establishes result by rearrangement and simplification | 1 |
| ii. • finds coordinates of $M$ | 1 |
| • uses (i) to find equation of locus of $M$ | 1 |
| • finds the focus of this parabola | 1 |

**Answer**

i. $2(p^2+q^2)-(p+q)^2 = 2p^2+2q^2-\{p^2+q^2+2pq\}$
$= p^2+q^2-2pq$
$= (p-q)^2$

ii. At $M$, $x=\frac{2p+2q}{2}=p+q$ and $y=\frac{p^2+q^2}{2}$.

Then using (i), $(p-q)^2=4y-x^2$

$\therefore\ p-q=1 \Rightarrow 1=4y-x^2$

Hence the locus of $M$ is parabola $x^2=4y-1$.

$x^2=4\left(y-\frac{1}{4}\right)$ has vertex $\left(0,\frac{1}{4}\right)$

and focal length 1.

Hence focus has coordinates $\left(0,\frac{5}{4}\right)$.

**Question 3**

(a) **Outcomes Assessed: P5, H5, HE4**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i. • writes domain | 1 |
| • evaluates the limit with explanation | 1 |
| ii. • shows algebraically that $f(-x)=f(x)$ | 1 |
| iii. • finds the first derivative | 1 |
| • establishes that there is a maximum turning point at the origin | 1 |
| iv. • sketches a curve of correct shape | 1 |
| • includes required detail (equations of asymptotes, coordinates of turning point) | 1 |
| v ¥ explains why the function does not have an inverse | 1 |

**Answer**

i. Domain $\{x: x\neq\pm1\}$

$$\lim_{x\to\infty} f(x)=\lim_{x\to\infty}\frac{1}{1-\frac{1}{x^2}}=\frac{1}{1-0}=1$$

ii. $f(-x)=\frac{(-x)^2}{(-x)^2-1}=\frac{x^2}{x^2-1}=f(x)$

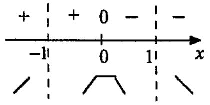
Hence $f$ is an even function

iii. $$\frac{dy}{dx}=\frac{2x.(x^2-1)-x^2.2x}{(x^2-1)^2}$$

$$=\frac{-2x}{(x^2-1)^2}$$

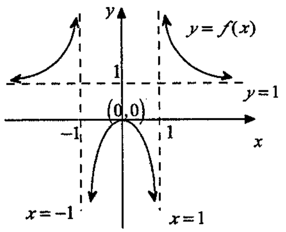
$\frac{dy}{dx}=0 \Rightarrow x=0$

*Sign of* $\frac{dy}{dx}$



$(0,0)$ is a maximum turning point.

iv.



v. There is a horizontal line which cuts the graph more than once. ($y=2$ cuts the graph twice). Hence there is a $y$ value which corresponds to more than one $x$ value ($y=2$ corresponds to two different $x$ values).

(b) **Outcomes Assessed: HE2**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| • establishes truth of statement for $n=1$ | 1 |
| • attempts to show that truth of any statement in the sequence forces truth of the next | 1 |
| • writes expression in $k$ from $S(k+1)$ in terms of expression in $k$ from $S(k)$ | 1 |
| • explains why the resulting expression in $k$ is a multiple of 16 if $S(k)$ is true | 1 |

**Answer**

Let $S(n)$, $n=1,2,3,...$ be the sequence of statements $5^n+12n-1=16m$ *for some integer m.*

*Consider* $S(1)$: $5^1+12\times1-1=16=16\times1$ Hence $S(1)$ is true.

*If* $S(k)$ *is true*: $5^k+12k-1=16m$ for some integer $m$ **

*Consider* $S(k+1)$: $5^{k+1}+12(k+1)-1=5.5^k+12k+11$

$=5(5^k+12k-1)+16-48k$

$=5(16m)+16(1-3k)$ *if $S(k)$ is true (using **)*

$=16\{5m+(1-3k)\}$ *where m, k and hence*

*$\{5m+(1-3k)\}$ are integers.*

Hence if $S(k)$ is true, then $S(k+1)$ is true. But $S(1)$ is true, hence $S(2)$ is true, and then $S(3)$ is true and so on. Hence by Mathematical Induction $S(n)$ is true for all integers $n\geq1$.

**Question 4**

(a) **Outcomes Assessed: H8, HE4**

**Marking Guidelines**

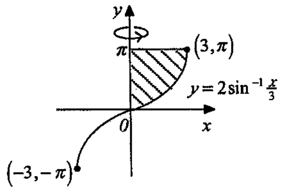
| Criteria | Marks |
|---|---|
| i. • writes domain | 1 |
| • writes range | 1 |
| ii. • sketches showing endpoints consistent with domain and range | 1 |
| iii. • writes integral for $V$ with integrand in terms of $y$ | 1 |
| • uses trig. identity to write integrand in terms of $\cos y$ | 1 |
| • evaluates definite integral to find exact value of $V$ | 1 |

**Answer**

i. Domain $\{x: -3\leq x\leq 3\}$

Range $\{y: -\pi\leq y\leq\pi\}$

ii.



iii. $V=\pi\int_0^{\pi} x^2 \ \ dy=\pi\int_0^{\pi}\left(3\sin\frac{y}{2}\right)^2 \ \ dy$

$V=\frac{9\pi}{2}\int_0^{\pi} 2\sin^2\frac{y}{2} \ \ dy$

$=\frac{9\pi}{2}\int_0^{\pi}(1-\cos y) \ \ dy$

$=\frac{9\pi}{2}\left[y-\sin y\right]_0^{\pi}$

$=\frac{9\pi^2}{2}$ *Ans.* $\frac{9\pi^2}{2}$ cubic units

(b) **Outcomes Assessed: P5, PE3**

**Marking Guidelines**

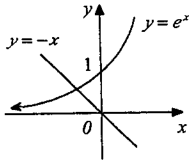
| Criteria | Marks |
|---|---|
| i. • shows $f(x)$ changes sign in interval $-1<x<0$ | 1 |
| • notes that $f$ is continuous then makes required deduction | 1 |
| ii. • sketches both graphs on same diagram | 1 |
| • relates single intersection point to fact that equation has exactly one real root | 1 |
| iii. • applies Newton's method to find expression for $a_1$ in terms of $a$ | 1 |
| • substitutes for $a$ and calculates $a_1$ to 1 decimal place | 1 |

**Answer**

i. $f(x)=e^x+x$ is continuous and $f(-1)=\frac{1}{e}-1<0$ while $f(0)=1>0$.

Hence $f(x)=0$ for some $-1<x<0$. $\therefore e^x+x=0$ has a real root $\alpha$ such that $-1<\alpha<0$.

ii.



One intersection point, hence equation $e^x = -x$ has exactly one real root.

iii.

$$f'(x) = e^x + 1$$

$$\therefore a_1 = a - \frac{e^a + a}{e^a + 1}$$

$$= \frac{a\left(e^a + 1\right) - \left(e^a + a\right)}{e^a + 1}$$

$$= \frac{\left(a - 1\right)e^a}{e^a + 1}$$

If $a = -0 \cdot 5$

$$a_1 = \frac{-1 \cdot 5\, e^{-0 \cdot 5}}{e^{-0 \cdot 5} + 1}$$

$$\approx -0 \cdot 6$$

**Question 5**

(a) **Outcomes Assessed: HE6**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| • writes $dx$ in terms of $du$ and converts $x$ limits to $u$ limits | 1 |
| • converts integrand to a function of $u$ | 1 |
| • finds primitive as a function of $u$ | 1 |
| • substitutes limits and evaluates definite integral | 1 |

**Answer**

$u = x + 1$

$du = dx$

$x = 0 \Rightarrow u = 1$

$x = 15 \Rightarrow u = 16$

$$\frac{x}{\sqrt{x+1}} = \frac{u-1}{\sqrt{u}}$$

$$= u^{\frac{1}{2}} - u^{-\frac{1}{2}}$$

$$\int_0^{15} \frac{x}{\sqrt{x+1}}\,dx = \int_1^{16} \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right)\,du$$

$$= \left[\tfrac{2}{3}u^{\frac{3}{2}} - 2u^{\frac{1}{2}}\right]_1^{16}$$

$$= \tfrac{2}{3}\left(16^{\frac{3}{2}} - 1\right) - 2\left(16^{\frac{1}{2}} - 1\right)$$

$$= 36$$

(b) **Outcomes Assessed: HE3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i. • writes numerical expression for required probability | 1 |
| • evaluates in simplest fraction form | 1 |
| ii. • writes numerical expression for required probability | 1 |
| • evaluates in simplest fraction form | 1 |

**Answer**

i. $P\left(2\ correct\right) = {}^4C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2 = \frac{8}{27}$

ii. Bob answers exactly 1 correct from first 3 attempted, then answers the last attempted correctly, with probability ${}^3C_1\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^2 \times \frac{1}{3} = \frac{4}{27}$.

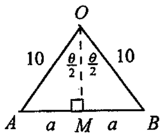
(c) **Outcomes Assessed: H5, HE5**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i. • finds length of the arc $AB$ in terms of $\theta$ | 1 |
| • finds length of the chord $AB$ in terms of $\theta$ and hence the perimeter $P$ | 1 |
| ii. • finds $\frac{dP}{dt}$ in terms of $\theta$ and $\frac{d\theta}{dt}$ | 1 |
| • evaluates rate of increase of $P$ | 1 |

**Answer**

i.



Let $M$ be a point on $AB$ such that $OM \perp AB$.
Then $M$ is the midpoint of $AB$ (*radius perpendicular to a chord bisects that chord*)
Let $AM = BM = a$.
Also $\angle MOA = \angle MOB = \frac{\theta}{2}$ ( *corresp.* $\angle$*s equal where* $\Delta MOA \equiv \Delta MOB\,(RHS)$)

Then $AB = 2a = 2\left(10\sin\frac{\theta}{2}\right) = 20\sin\frac{\theta}{2}$

Hence $P = arc\ AB + AB = 10\theta + 20\sin\frac{\theta}{2}$

ii. $\frac{dP}{dt} = \frac{dP}{d\theta} \times \frac{d\theta}{dt} = \left(10 + 10\cos\frac{\theta}{2}\right)\frac{d\theta}{dt}$

Hence when $\theta = \frac{2\pi}{3}$, $\frac{dP}{dt} = 10\left(1 + \cos\frac{\pi}{3}\right) \times 0 \cdot 02 = 0 \cdot 3$

$\therefore$ Perimeter is increasing at a rate $0 \cdot 3$ cms$^{-1}$

**Question 6**

(a) **Outcomes Assessed: HE3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i. • differentiates | 1 |
| ii • substitutes $t = 0$, $N = 500$ into expression for $N$ to find $A$ | 1 |
| • substitutes $\frac{dN}{dt} = -20$, $N = 300$ into expression for $\frac{dN}{dt}$ to find $k$ | 1 |
| iii. • sketches graph of $N$ as a function of $t$ showing $N$ intercept and asymptote | 1 |
| • sketches derivative as a function of $t$ showing intercept and asymptote | 1 |
| iv. • finds value of $t$ when $N = 120$ | 1 |

**Answer**

i. $N = 100 + Ae^{-kt}$

$$\frac{dN}{dt} = -kAe^{-kt}$$

$$= -k\left(N - 100\right)$$

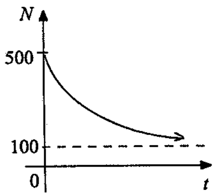
ii. When $t = 0$, $N = 500$

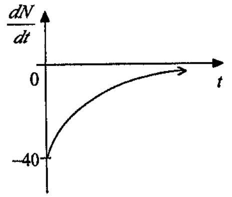
$\therefore Ae^0 = 400 \qquad \therefore A = 400$

When $N = 300$, $\frac{dN}{dt} = -20$

$\therefore -20 = -k\left(300 - 100\right) \qquad \therefore k = 0 \cdot 1$

iii.





iv. $N = 120 \Rightarrow 400e^{-0 \cdot 1t} = 20$

$$e^{-0 \cdot 1t} = \tfrac{1}{20}$$

$$-0 \cdot 1t = -\ln 20$$

$$t \approx 29 \cdot 96$$

The population first falls below 120 towards the end of the 30th year.

(b) **Outcomes Assessed:** **HE3, HE7**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i. • writes $\frac{d}{dx}\left(\frac{1}{2}v^2\right)=-4x+4$ then finds the primitive | 1 |
| • includes and evaluates the constant of integration using initial conditions | 1 |
| ii. • finds possible values of $x$ for $v^2 \geq 0$ | 1 |
| • deduces centre and amplitude of motion | 1 |
| iii. • uses initial value of $x$ to show $\cos\alpha = \frac{1}{2}$ | 1 |
| • finds $v$ as a function of $t$; uses sign of $v$ initially to deduce 4th quadrant $\alpha$ value | 1 |

**Answer**

i. $\frac{d}{dx}\left(\frac{1}{2}v^2\right)=-4x+4$

$$\frac{1}{2}v^2=-2x^2+4x+c$$

$$\left.\begin{array}{r} t=0 \\ x=2 \\ v=2\sqrt{3} \end{array}\right\} \Rightarrow \begin{array}{l} 6=-8+8+c \\ c=6 \\ \frac{1}{2}v^2=-2x^2+4x+6 \\ \therefore v^2=-4x^2+8x+12 \end{array}$$

ii. $v^2=16-4(x-1)^2$

$$v^2\geq 0 \Rightarrow (x-1)^2 \leq 4$$

$$\therefore\ -2\leq x-1\leq 2$$

$$-1\leq x\leq 3$$

Hence motion is centred 1m to the right of $O$ with amplitude 2 m

iii. $x=1+2\cos(2t+\alpha)$

$$v=-4\sin(2t+\alpha)$$

$$\left.\begin{array}{r} t=0 \\ x=2 \\ v>0 \end{array}\right\} \Rightarrow \begin{array}{l} \cos\alpha=\frac{1}{2}\ \ and \\ \sin\alpha<0 \\ \therefore \alpha=2\pi-\frac{\pi}{3}=\frac{5\pi}{3} \end{array}$$

## Question 7

(a) **Outcomes Assessed:** **HE3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i. • finds time to maximum height | 1 |
| • substitutes in expression for $y$ to find $H$ | 1 |
| • finds time of flight | 1 |
| • substitutes in expression for $x$ to find $R$ | 1 |
| ii. • simplifies $\frac{4H}{R}$ to obtain required result | 1 |
| • writes $\sin 2\alpha$ in terms of $\frac{4H}{R}$ (t-formula) in expression for range $R$ | 1 |
| • rearranges resulting formula to make $R$ the subject | 2 |

**Answer**

i. $y=Vt\sin\alpha-\frac{1}{2}gt^2$

$$\frac{dy}{dt}=V\sin\alpha-gt$$

At maximum height, $\frac{dy}{dt}=0$

$$\therefore t=\frac{V\sin\alpha}{g} \text{ when } y=H$$

$$H=\frac{V^2\sin^2\alpha}{g}-\frac{1}{2}g\frac{V^2\sin^2\alpha}{g^2}=\frac{V^2\sin^2\alpha}{2g}$$

$$y=t\left(V\sin\alpha-\frac{1}{2}gt\right)$$

When $x=R$, $y=0$ and $t\neq 0$. $\therefore t=\frac{2V\sin\alpha}{g}$

$$R=V\cos\alpha\times\frac{2V\sin\alpha}{g}=\frac{V^2\sin 2\alpha}{g}$$

ii. $\frac{4H}{R}=\frac{2V^2\sin^2\alpha}{g}\times\frac{g}{2V^2\sin\alpha\cos\alpha}$

$$\frac{4H}{R}=\frac{\sin\alpha}{\cos\alpha}=\tan\alpha$$

$$\therefore R\tan\alpha=4H$$

$$\tan\alpha=\frac{4H}{R} \Rightarrow \sin 2\alpha=\frac{2\left(\frac{4H}{R}\right)}{1+\left(\frac{4H}{R}\right)^2}$$

$$\therefore \quad R=\frac{V^2}{g}\times\frac{2\left(\frac{4H}{R}\right)}{1+\left(\frac{4H}{R}\right)^2}$$

$$1+\left(\frac{4H}{R}\right)^2=\left(\frac{V^2}{2gH}\right)\left(\frac{4H}{R}\right)^2$$

$$\left(\frac{R}{4H}\right)^2+1=\frac{V^2}{2gH}$$

$$\left(\frac{R}{4H}\right)^2=\frac{V^2}{2gH}-1$$

$$\therefore \quad R=4H\sqrt{\frac{V^2}{2gH}-1}$$

(b) **Outcomes Assessed:** **PE3**

**Marking Guidelines**

| Criteria | Marks |
|---|---|
| i. • uses binomial theorem to expand then simplifies | 1 |
| ii. • groups terms in pairs $x^n+\left(\frac{1}{x}\right)^n$ and factors $x^3+\left(\frac{1}{x}\right)^3$ | 1 |
| • expresses $x^3+\left(\frac{1}{x}\right)^3$ in terms of $\left(x+\frac{1}{x}\right)$ | 1 |
| • substitutes $x+\frac{1}{x}=a$ then rearranges to show required result | 1 |

**Answer**

i. $\left(x+\frac{1}{x}\right)^5=x^5+5x^4.\frac{1}{x}+10x^3\left(\frac{1}{x}\right)^2+10x^2\left(\frac{1}{x}\right)^3+5x\left(\frac{1}{x}\right)^4+\left(\frac{1}{x}\right)^5$

$$=x^5+5x^3+10x+10\left(\frac{1}{x}\right)+5\left(\frac{1}{x}\right)^3+\left(\frac{1}{x}\right)^5$$

ii. $\left(x+\frac{1}{x}\right)^5=x^5+\left(\frac{1}{x}\right)^5+5\left\{x^3+\left(\frac{1}{x}\right)^3\right\}+10\left(x+\frac{1}{x}\right)$

But $x^3+\left(\frac{1}{x}\right)^3=\left(x+\frac{1}{x}\right)\left\{x^2-x\frac{1}{x}+\left(\frac{1}{x}\right)^2\right\}$

$$=\left(x+\frac{1}{x}\right)\left\{x^2+2x\frac{1}{x}+\left(\frac{1}{x}\right)^2-3x\frac{1}{x}\right\}$$

$$=\left(x+\frac{1}{x}\right)\left\{\left(x+\frac{1}{x}\right)^2-3\right\}$$

Hence if $\left(x+\frac{1}{x}\right)=a$

$$a^5=x^5+\left(\frac{1}{x}\right)^5+5a\left(a^2-3\right)+10a$$

$$a^5=x^5+\left(\frac{1}{x}\right)^5+5a^3+5a$$

$$\therefore x^5+\frac{1}{x^5}=a^5-5a^3-5a$$