NSW INDEPENDENT SCHOOLS

2001
Higher School Certificate
Trial Examination

Mathematics
Extension 2

General Instructions

- Reading time – 5minutes
- Working time – 3 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided on the last page
- All necessary working should be shown in every question
- Write your student number and/or name at the top of every page

Total marks (120)

Attempt Questions 1 – 8

All questions are of equal value

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME: ....

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\quad x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\quad x > 0$

Student name / number ....

Marks

Question 1 — Begin a new page

(a) Consider the functions $f(x)=|x|+1$ and $g(x)=\dfrac{6}{|x|}$.

(i) Solve the equation $f(x)=g(x)$. **2**

(ii) Sketch the graphs of $y=f(x)$ and $y=g(x)$ on the same set of axes. **2**

(iii) Solve the inequality $g(x)>f(x)$. **1**

(b) Consider the function $f(x)=1-2\cos x$, $-\pi \le x \le \pi$.

(i) Sketch the graph of $y=f(x)$. **2**

(ii) Use your graph in (i) to sketch the graph of $y=\{f(x)\}^2$. **2**

(iii) Use your graph in (i) to sketch the graph of $y=\dfrac{1}{f(x)}$. **2**

(c) Consider the function $y=\sin^{-1}(e^x)$.

(i) Find the domain and range of the function. **2**

(ii) Sketch the graph of the function showing clearly the coordinates of any endpoints and the equations of any asymptotes. **2**

Question 2 — Begin a new page

(a) (i) Find $\displaystyle\int \frac{e^x}{e^x+1}\,dx$. **1**

(ii) Find $\displaystyle\int \frac{e^x}{(e^x+1)^2}\,dx$. **2**

(b) Find $\displaystyle\int \frac{x^2}{(1-x^2)^{\frac{3}{2}}}\,dx$ using the substitution $x=\sin\theta$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$. **3**

(c) Evaluate $\displaystyle\int_0^{\frac{2\pi}{3}} \frac{1}{5+4\cos x}\,dx$ using the substitution $t=\tan\frac{x}{2}$. **3**

(d) (i) Use the substitution $x=u^2$, $u>0$, to show that $\displaystyle\int_4^{16}\frac{\sqrt{x}}{x-1}\,dx = 4+2\ln 3-\ln 5$. **4**

(ii) Hence use integration by parts to evaluate $\displaystyle\int_4^{16}\frac{\ln(x-1)}{\sqrt{x}}\,dx$ in simplest exact form. **2**

Student name / number ....

Marks

Question 3 — Begin a new page

(a) (i) Find the values of real numbers a and b such that $(a+ib)^2=5-12i$. **2**

(ii) Hence or otherwise solve the equation $ix^2+3x+(3-i)=0$. **2**

(b) It is given that $z=\cos\theta+i\sin\theta$ where $0<\arg z<\frac{\pi}{2}$.

(i) Show that $z+1=2\cos\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)$ and express $z-1$ in modulus / argument form. **3**

(ii) Hence show that $\mathrm{Re}\left(\dfrac{z-1}{z+1}\right)=0$. **1**

(c) The fixed complex number α is such that $0<\arg\alpha<\frac{\pi}{2}$. In an Argand diagram α is represented by the point A while $i\alpha$ is represented by the point B. z is a variable complex number which is represented by the point P.

(i) Draw a diagram showing A, B and the locus of P if $|z-\alpha|=|z-i\alpha|$. **1**

(ii) Draw a diagram showing A, B and the locus of P if $\arg(z-\alpha)=\arg(i\alpha)$. **1**

(iii) Find in terms of α the complex number represented by the point of intersection of the two loci in (i) and (ii). **1**

(d) z_1 and z_2 are complex numbers.

(i) Show that $|z_1||z_2|=|z_1 z_2|$. **1**

(ii) By taking $z_1=2+3i$ and $z_2=4+5i$, express 533 (the product of 13 and 41) as a sum of squares of two positive integers. **1**

(iii) By taking other values for z_1 and z_2, express 533 as a sum of squares of two other positive integers. **2**

Student name / number ....

Marks

Question 4 Begin a new page

Consider the ellipse $\dfrac{x^2}{16} + \dfrac{y^2}{12} = 1$.

(a) (i) Find the eccentricity of the ellipse. **1**
(ii) Find the coordinates of the foci and the equations of the directrices of the ellipse. **2**
(iii) Sketch the graph of the ellipse showing clearly all of the above features and the intercepts on the coordinate axes. **2**

(b) (i) Use differentiation to derive the equations of the tangent and the normal to the ellipse at the point $P(2,3)$. **3**
(ii) The tangent and normal to the ellipse at P cut the y axis at A and B respectively. Find the coordinates of A and B. **1**

(c) (i) Show that AB subtends a right angle at the focus S of the ellipse. **2**
(ii) Show that the points A, P, S and B are concyclic. **1**
(iii) Find the centre and radius of the circle which passes through the points A, P, S and B. **3**

Question 5 Begin a new page

(a) The equation $x^3 + 3px - 1 = 0$, where p is real, has roots α, β and γ.

(i) By first replacing x by $\sqrt{x}$, or otherwise, show that the monic cubic equation, with coefficients in terms of p, whose roots are α^2, β^2 and γ^2 is **2**
$x^3 + 6px^2 + 9p^2x - 1 = 0$.
(ii) Hence or otherwise obtain the monic cubic equation, with coefficients in terms of p, whose roots are $\dfrac{\beta\gamma}{\alpha}$, $\dfrac{\gamma\alpha}{\beta}$ and $\dfrac{\alpha\beta}{\gamma}$. **3**

(b) (i) Show that the general solution of the equation $\cos 5\theta = -1$ is given by **2**
$\theta = (2n+1)\frac{\pi}{5}$, $n = 0, \pm1, \pm2, \ldots$.
Hence solve the equation $\cos 5\theta = -1$ for $0 \le \theta \le 2\pi$.
(ii) Use De Moivre's Theorem to show that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$. **3**
(iii) Find the exact trigonometric roots of the equation $16x^5 - 20x^3 + 5x + 1 = 0$. **2**
(iv) Hence find the exact values of $\cos\frac{\pi}{5} + \cos\frac{3\pi}{5}$ and $\cos\frac{\pi}{5} . \cos\frac{3\pi}{5}$ and factorise $16x^5 - 20x^3 + 5x + 1$ into irreducible factors over the rational numbers. **3**

Student name / number ....

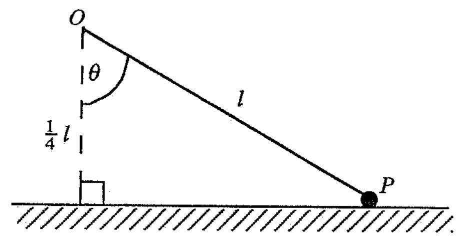
Marks

Question 6 Begin a new page

(a) A lifebelt mould is made by rotating the circle $x^2 + y^2 = 64$ through one complete revolution about the line $x = 28$, where all the measurements are in centimetres.

(i) Use the method of slicing to show that the volume $V\text{ cm}^3$ of the lifebelt is given by **5**
$$V = 112\pi\int_{-8}^{8}\sqrt{64 - y^2}\,dy.$$
(ii) Find the exact volume of the lifebelt. **2**

(b)



One end of a light inextensible string of length l is attached to a fixed point O which is at a height $\frac{1}{4}l$ above a smooth horizontal table. A particle P of mass m is attached to the other end of the string and rests on the table with the string taut. The particle is projected so that it moves in a circle on the table with constant speed v.

(i) Copy the diagram. Show the forces acting on the particle P. **1**
(ii) Show that the tension in the string has magnitude $T = \dfrac{16mv^2}{15l}$. **2**
(iii) Show that the reaction R exerted by the table on P has magnitude $R = m\left(g - \dfrac{4v^2}{15l}\right)$. **2**
(iv) Hence show that $v \le \sqrt{\dfrac{15gl}{4}}$. Explain what would happen at a higher speed. **3**

Student name / number ....

Marks

Question 7 Begin a new page

(a) A, B and C are three distinct points on a horizontal straight line that is on the same level as the foot P of a vertical tower PQ of height h. The distances AB and BC are both equal to d and the angles of elevation of the top Q of the tower from the points A, B, C are equal to α, β, γ respectively.

(i) If the line ABC passes through the foot P of the tower so that A, B, C are all on the same side of P, show that $2\cot\beta = \cot\alpha + \cot\gamma$. **3**

(ii) If the line ABC does not pass through the foot P of the tower, use the cosine rule in each of ΔABP, ΔCBP to show that $h^2\left(\cot^2\alpha - 2\cot^2\beta + \cot^2\gamma\right) = 2d^2$. **4**

(b) After t minutes the number N of bacteria in a culture is given by $N = \dfrac{a}{1+be^{-ct}}$ for some constants $a>0$, $b>0$ and $c>0$. Initially there are 300 bacteria in the culture and the number of bacteria is initially increasing at a rate of 20 per minute. As t increases indefinitely the number of bacteria in the culture approaches a limiting value of 900.

(i) Show that $\dfrac{dN}{dt} = \dfrac{c}{a}(a-N)N$. **2**

(ii) Find the values of a, b and c. **3**

(iii) Show that the maximum rate of increase in the number of bacteria occurs when $N=450$. Sketch the graph of N against t. **3**

Student name / number ....

Marks

Question 8 Begin a new page

(a) A die is biased so that in any one throw there is a constant probability p, where $p \neq 0\cdot5$, that the die shows an even number. In 8 throws of the die the probability of 2 even numbers is four times the probability of 5 even numbers. Find the value of p. **4**

(b) The equation $x^3 - 3px + q = 0$, where $p>0$, $q \neq 0$ are both real, has three distinct, non-zero real roots.

(i) Show that the graph of $y = x^3 - 3px + q$ has a relative maximum value of $q + 2p\sqrt{p}$ and a relative minimum value of $q - 2p\sqrt{p}$. **3**

(ii) Hence show that $q^2 < 4p^3$. **3**

(c) It is given that if a, b, c are any three positive real numbers, then $\dfrac{a+b+c}{3} \geq \sqrt[3]{abc}$.
If $a>0$, $b>0$ and $c>0$ are real numbers such that $a+b+c=1$, use the given result to show that

(i) $\dfrac{1}{abc} \geq 27$ **1**

(ii) $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} \geq 9$ **2**

(iii) $(1-a)(1-b)(1-c) \geq 8abc$ **2**

Question 1

(a) **Outcomes Assessed:** (i) **P4** (ii) **P4** (iii) **PE3**

Marking Guidelines

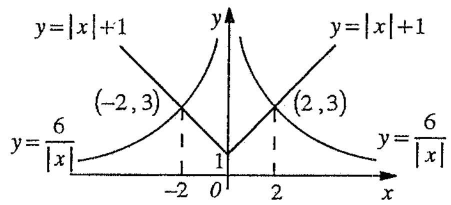
Criteria	Marks
(i) • one mark for values of $\lvert x\rvert$ • one mark for values of x	2
(ii) • one mark for graph of $y=f(x)$ • one mark for graph of $y=g(x)$ • one mark for solution of inequality	3

Answer

(i)

$$|x|+1=\frac{6}{|x|}$$
$$|x|^2+|x|-6=0$$
$$(|x|+3)(|x|-2)=0$$
$$|x|\ge 0 \Rightarrow |x|=2$$
$$\therefore\quad x=\pm 2$$

(ii)



(iii)

$$\frac{6}{|x|}>|x|+1 \Rightarrow -2<x<0 \text{ or } 0<x<2$$

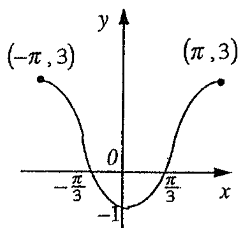
(b) **Outcomes Assessed:** (i) **H5** (ii) **E6** (iii) **E6**

Marking Guidelines

Criteria	Marks
(i) • one mark for coordinates of endpoints and intercepts • one mark for graph	2
(ii) • one mark for coordinates of endpoints and intercepts • one mark for graph	2
(iii) • one mark for coordinates of endpoints, intercept and equations of asymptotes • one mark for graph	2

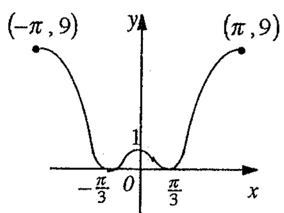
Answer

(i)



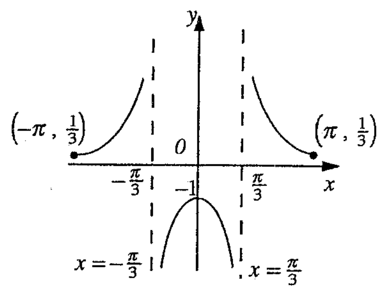
$y=1-2\cos x$

(ii)



$y=(1-2\cos x)^2$

(iii)



$$y=\frac{1}{1-2\cos x}$$

Marking Guidelines

Criteria	Marks
(i) • one mark for domain • one mark for range	2
(ii) • one mark for coordinates of endpoint and equation of asymptote • one mark for graph	2

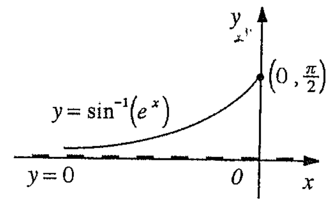
Answer

(i)

$$y=\sin^{-1}(e^x)$$

Domain	*Range*
$-1\le e^x\le 1$	$0<e^x\le 1$
$0<e^x\le 1$	$0<\sin^{-1}(e^x)\le\frac{\pi}{2}$
$\{x:x\le 0\}$	$\{y:0<y\le\frac{\pi}{2}\}$

(ii)



Question 2

(a) **Outcomes Assessed:** (i) **H5** (ii) **HE6**

Marking Guidelines

Criteria	Marks
(i) • one mark for answer	1
(ii) • one mark for expression in form $\int\frac{1}{u^2}\,du$ or recognition of pattern $\int f' f^{-2}\ dx$ • one mark for final answer	2

Answer

(i) $\displaystyle\int\frac{e^x}{e^x+1}\,dx=\ln(e^x+1)+c$

(ii) $u=e^x+1 \Rightarrow du=e^x\,dx$

$$\int\frac{e^x}{(e^x+1)^2}\,dx=\int\frac{1}{u^2}\,du=-\frac{1}{u}+c=-\frac{1}{e^x+1}+c$$

(b) **Outcomes Assessed:** **HE6**

Marking Guidelines

Criteria	Marks
• one mark for expression in form $\int\tan^2\theta\ d\theta$ • one mark for expression in form $\tan\theta-\theta+c$ • one mark for final answer	3

Answer

$x=\sin\theta,\ -\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$

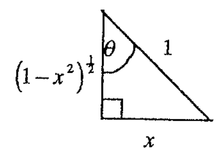
$dx=\cos\theta\ d\theta$

$(1-x^2)^{\frac{3}{2}}=(\cos^2\theta)^{\frac{3}{2}}=\cos^3\theta$

$$\frac{x^2}{(1-x^2)^{\frac{3}{2}}}=\frac{\sin^2\theta}{\cos^3\theta}=\frac{\tan^2\theta}{\cos\theta}$$

$$\int\frac{x^2}{(1-x^2)^{\frac{3}{2}}}\,dx=\int\frac{\tan^2\theta}{\cos\theta}\cos\theta\,d\theta$$

$$\int\tan^2\theta\ d\theta=\int(\sec^2\theta-1)\ d\theta$$
$$=\tan\theta-\theta+c$$
$$=\frac{x}{(1-x^2)^{\frac{1}{2}}}-\sin^{-1}x+c$$



Marking Guidelines

Criteria	Marks
• one mark for expression in form $\int_0^{\sqrt{3}} \frac{2}{1+t^2}\,dt$ • one mark for expression in form $\frac{2}{3}\left[\tan^{-1}\left(\frac{t}{3}\right)\right]_0^{\sqrt{3}}$ • one mark for final answer	3

Answer

$$t=\tan\tfrac{x}{2}$$

$$dt=\tfrac{1}{2}\sec^2\tfrac{x}{2}\,dx \qquad x=0 \Rightarrow t=0$$

$$2\,dt=\left(1+t^2\right)dx \qquad x=\tfrac{2\pi}{3} \Rightarrow t=\sqrt{3}$$

$$dx=\frac{2}{1+t^2}\,dt$$

$$5+4\cos x=5+\frac{4\left(1-t^2\right)}{1+t^2}=\frac{9+t^2}{1+t^2}$$

$$\int_0^{\frac{2\pi}{3}} \frac{1}{5+4\cos x}\,dx=\int_0^{\sqrt{3}} \frac{1+t^2}{9+t^2}\,\frac{2}{1+t^2}\,dt$$

$$=\tfrac{2}{3}\int_0^{\sqrt{3}} \frac{3}{9+t^2}\,dt$$

$$=\tfrac{2}{3}\left[\tan^{-1}\tfrac{t}{3}\right]_0^{\sqrt{3}}$$

$$=\tfrac{2}{3}\left(\tfrac{\pi}{6}-0\right)$$

$$=\tfrac{\pi}{9}$$

(d) **Outcomes Assessed:** (i) **HE6, E8** (ii) **E8**

Marking Guidelines

Criteria	Marks
(i) • one mark for expression in form $\int_2^4 \frac{2u^2}{u^2-1}\,du$ • one mark for expression in form $\int_2^4\left(2+\frac{1}{u-1}-\frac{1}{u+1}\right)du$ • one mark for expression in form $\left[2u+\ln\left\|\frac{u-1}{u+1}\right\|\right]_2^4$ or equivalent • one mark for final answer	4
(ii) • one mark for expression in form $\left[2\sqrt{x}\ln(x-1)\right]_4^{16}-2\int_4^{16}\frac{\sqrt{x}}{x-1}\,dx$ • one mark for final answer	2

Answer

(i)

$$x=u^2,\quad u>0$$

$$dx=2u\,du$$

$$\frac{\sqrt{x}}{x-1}=\frac{u}{u^2-1}$$

$$x=4 \Rightarrow u=2$$

$$x=16 \Rightarrow u=4$$

$$I=\int_4^{16}\frac{\sqrt{x}}{x-1}\,dx=\int_2^4\frac{2u^2}{u^2-1}\,du=\int_2^4\frac{2\left(u^2-1\right)+2}{u^2-1}\,du$$

$$I=\int_2^4\left(2+\frac{2}{u^2-1}\right)du=\int_2^4\left(2+\frac{1}{u-1}-\frac{1}{u+1}\right)du$$

$$\therefore I=\left[2u+\ln\left|\frac{u-1}{u+1}\right|\right]_2^4=2(4-2)+\ln\tfrac{3}{5}-\ln\tfrac{1}{3}$$

$$\therefore I=4+\ln 3-\ln 5+\ln 3=4+2\ln 3-\ln 5$$

(ii)

$$\int_4^{16}\frac{\ln(x-1)}{\sqrt{x}}\,dx=\left[2\sqrt{x}\ln(x-1)\right]_4^{16}-2\int_4^{16}\frac{\sqrt{x}}{x-1}\,dx$$

$$=2\left(4\ln 15-2\ln 3\right)-2\left(4+2\ln 3-\ln 5\right)$$

$$=2\left(4\ln 5+2\ln 3\right)-2\left(4+2\ln 3-\ln 5\right)$$

$$=10\ln 5-8$$

Question 3

(a) **Outcomes Assessed:** (i) **E3** (ii) **P4, E3**

Marking Guidelines

Criteria	Marks
(i) • one mark for equating real and imaginary parts • one mark for values of a and b	2
(ii) • one mark for use of quadratic formula • one mark for final answer	2

Answer

(i)

$$(a+ib)^2=5-12i \Rightarrow \left(a^2-b^2\right)+2ab\,i=5-12i$$

$$\therefore a^2-b^2=5 \quad\text{and}\quad ab=-6$$

$$a^4-a^2b^2=5a^2 \Rightarrow a^4-5a^2-36=0$$

$$\left(a^2+4\right)\left(a^2-9\right)=0 \quad \therefore a^2>0 \Rightarrow a^2=9$$

$$\therefore \begin{cases} a=3 \\ b=-2 \end{cases} \text{ or } \begin{cases} a=-3 \\ b=2 \end{cases}$$

(ii)

$$ix^2+3x+(3-i)=0$$

$$\Delta=9-4i(3-i)=5-12i$$

$$x=\frac{-3\pm\sqrt{\Delta}}{2i}$$

$$=-\tfrac{i}{2}\{-3\pm(3-2i)\}$$

$$\therefore x=-1 \quad\text{or}\quad x=1+3i$$

(b) **Outcomes Assessed:** (i) **P4, E3** (ii) **E2, E3**

Marking Guidelines

Criteria	Marks
(i) • one mark for expression for $z+1$ • one mark for modulus of $z-1$ • one mark for argument of $z-1$	3
(ii) • one mark for answer	1

Answer

(i)

$$z+1=1+\cos\theta+i\sin\theta$$

$$=2\cos^2\tfrac{\theta}{2}+i\left(2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\right)$$

$$=2\cos\tfrac{\theta}{2}\left(\cos\tfrac{\theta}{2}+i\sin\tfrac{\theta}{2}\right)$$

$$z-1=-(1-\cos\theta)+i\sin\theta$$

$$=-2\sin^2\tfrac{\theta}{2}+i\left(2\sin\tfrac{\theta}{2}\cos\tfrac{\theta}{2}\right)$$

$$=2\sin\tfrac{\theta}{2}\left(-\sin\tfrac{\theta}{2}+i\cos\tfrac{\theta}{2}\right)$$

$$=2\sin\tfrac{\theta}{2}\left\{\cos\left(\tfrac{\pi}{2}+\tfrac{\theta}{2}\right)+i\sin\left(\tfrac{\pi}{2}+\tfrac{\theta}{2}\right)\right\}$$

(ii)

Then $\left|\frac{z-1}{z+1}\right|=\tan\frac{\theta}{2}$ and $\arg\left(\frac{z-1}{z+1}\right)=\left(\frac{\pi}{2}+\frac{\theta}{2}\right)-\frac{\theta}{2}=\frac{\pi}{2} \Rightarrow \frac{z-1}{z+1}=i\tan\frac{\theta}{2} \quad \therefore \operatorname{Re}\left(\frac{z-1}{z+1}\right)=0$

3(c) **Outcomes Assessed:** (i) E3 (ii) E3 (iii) E3

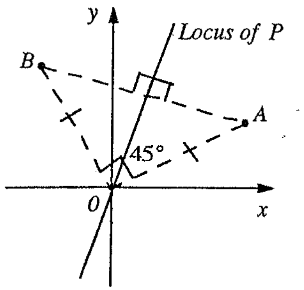
Marking Guidelines

Criteria	Marks
(i) • one mark for answer	1
(ii) • one mark for answer	1
(iii) • one mark for answer	1

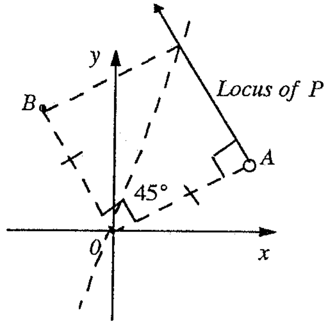
Answer

Let $z = x + iy$, x, y real

(i) Locus of P is perpendicular bisector of AB.



(ii) Locus is ray from A parallel to $\overrightarrow{OB}$



(iii) If P is the point of intersection of these loci, $OAPB$ is a square and the diagonal $\overrightarrow{OP}$ represents the sum of α and $i\alpha$. Hence P represents $(1+i)\alpha$.

(d) **Outcomes Assessed:** (i) E3 (ii) E3 (iii) E2, E3

Marking Guidelines

Criteria	Marks
(i) • one mark for answer	1
(ii) • one mark for answer	1
(iii) • one mark for choice of z_1, z_2 • one mark for answer	2

Answer

(i)

$z_1 = a+ib,\ z_2 = c+id \Rightarrow z_1 z_2 = (ac-bd) + i(ad+bc)$

$|z_1 z_2|^2 = (ac-bd)^2 + (ad+bc)^2 = a^2c^2 - 2acbd + b^2d^2 + a^2d^2 + 2adbc + b^2c^2$

$\therefore |z_1 z_2|^2 = (a^2+b^2)(c^2+d^2) = |z_1|^2 . |z_2|^2$ $\qquad \therefore |z_1| . |z_2| = |z_1 z_2|$

(ii)

$z_1 = 2+3i \qquad \Rightarrow |z_1|^2 = 4+9 = 13$

$z_2 = 4+5i \qquad \Rightarrow |z_2|^2 = 16+25 = 41$

$z_1 . z_2 = -7+22i \Rightarrow |z_1 . z_2|^2 = 7^2 + 22^2$

$\therefore 533 = 13 \times 41 = 7^2 + 22^2$

(iii)

For example :

$z_1 = 3+2i,\ z_2 = 5-4i,\ z_1 . z_2 = 23-2i$

$|z_1|^2 = 13,\ |z_2|^2 = 41,\ |z_1 . z_2|^2 = 23^2 + 2^2$

$\therefore 533 = 13 \times 41 = 23^2 + 2^2$

Question 4

(a) **Outcomes Assessed:** (i) E4 (ii) E4 (iii) E3

Marking Guidelines

Criteria	Marks
(i) • one mark for eccentricity	1
(ii) • one mark for coordinates of foci • one mark for equations of directrices	2
(iii) • one mark for graph with intercepts • one mark for showing foci and directrices	2

Answer

(i)

$\frac{x^2}{16} + \frac{y^2}{12} = 1$

$12 = 16(1-e^2)$

$e^2 = \frac{1}{4}$, $e = \frac{1}{2}$

(ii)

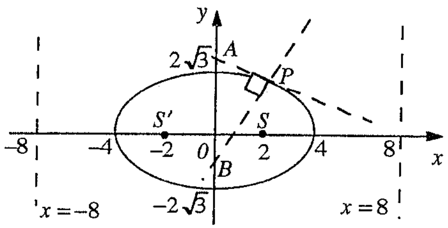
foci $(\pm ae, 0)$

$S'(-2,0)$, $S(2,0)$

directrices $x = \pm \frac{a}{e}$

$x = -8$, $x = 8$

(iii)



(b) **Outcomes Assessed:** (i) E4 (ii) E4

Marking Guidelines

Criteria	Marks
(i) • one mark for expression $\frac{dy}{dx} = -\frac{3x}{4y}$ • one mark for equation of tangent • one mark for equation of normal	3
(ii) • one mark for coordinates of A and B	1

Answer

(i)

$\frac{x^2}{16} + \frac{y^2}{12} = 1 \Rightarrow \frac{2x}{16} + \frac{2y}{12}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{x}{8} \div \frac{y}{6} = -\frac{3x}{4y}$

at $P(2,3)$ $\quad \frac{dy}{dx} = -\frac{1}{2}$

(ii) (iii)

Tangent at $P(2,3)$ has gradient $-\frac{1}{2}$ and equation

$y-3 = -\frac{1}{2}(x-2) \Rightarrow x+2y-8=0 \Rightarrow A(0,4)$

Normal at $P(2,3)$ has gradient 2 and equation

$y-3 = 2(x-2) \Rightarrow 2x-y-1=0 \Rightarrow B(0,-1)$

4(c) **Outcomes Assessed:** (i) P4, E2 (ii) PE3, E2 (iii) P4, PE3

Marking Guidelines

Criteria	Marks
(i) • one mark for the gradients of AS and BS • one mark for showing $AS \perp BS$	2
(ii) • one mark for showing points A, P, S and B are concyclic	1
(iii) • one mark for noting AB is diameter • one mark for centre of circle • one mark for radius of circle	3

Answer

(i)

$S(2,0)$ $A(0,4)$ $B(0,-1)$

$grad\ AS . grad\ BS = -2 \times \frac{1}{2} = -1$

$\therefore A\hat{S}B = 90°$

(ii)

$A\hat{P}B = 90°$ (*tangent* $\perp$ *normal*)

AB subtends equal angles of 90° at P and S.

$\therefore A, P, S, B$ are concyclic

(iii)

Diameter AB

centre $\left(0, \frac{3}{2}\right)$

radius $\frac{5}{2}$

(a) **Outcomes Assessed:** (i) **E4** (ii) **E2, E4**

Marking Guidelines

Criteria	Marks
(i) • one mark for the equation in the form $\sqrt{x}\,(x+3p)=1$ • one mark for the final equation	2
(ii) • one mark for rewriting roots in form $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$ • one mark for replacing x by $\frac{1}{x}$ • one mark for the final equation	3

Answer

(i)

$x^3+3px-1=0$ has roots α, β, γ

$(\sqrt{x})^3+3p(\sqrt{x})-1=0$ has roots $\alpha^2, \beta^2, \gamma^2$

$\sqrt{x}\,(x+3p)=1 \quad \Rightarrow \quad x(x^2+6px+9p^2)=1$

Required equation is $\quad x^3+6px^2+9p^2x-1=0$

(ii)

$\alpha\beta\gamma=1 \quad \Rightarrow \quad \frac{\beta\gamma}{\alpha}=\frac{\alpha\beta\gamma}{\alpha^2}=\frac{1}{\alpha^2}$

Similarly $\frac{\gamma\alpha}{\beta}=\frac{1}{\beta^2}, \quad \frac{\alpha\beta}{\gamma}=\frac{1}{\gamma^2}$

$x^3+6px^2+9p^2x-1=0$ has roots $\alpha^2, \beta^2, \gamma^2$.

$\left(\frac{1}{x}\right)^3+6p\left(\frac{1}{x}\right)^2+9p^2\left(\frac{1}{x}\right)-1=0$ ** has roots

$\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2} \qquad$ i.e. $\frac{\beta\gamma}{\alpha}, \frac{\gamma\alpha}{\beta}, \frac{\alpha\beta}{\gamma}$

Multiplying ** by $-x^3$, equation with roots

$\frac{\beta\gamma}{\alpha}, \frac{\gamma\alpha}{\beta}, \frac{\alpha\beta}{\gamma}$ becomes

$$x^3-9p^2x^2-6px-1=0$$

5(b) **Outcomes Assessed:** (i) **H5** (ii) **E2, E3** (iii) **E2, E4** (iv) **E2, E4**

Marking Guidelines

Criteria	Marks
(i) • one mark for general solution • one mark for particular solution	2
(ii) • one mark for expression for $\text{Re}(\cos\theta+i\sin\theta)^5$ in terms of $\cos\theta$, $\sin\theta$ • one mark for expression for $\text{Re}(\cos\theta+i\sin\theta)^5$ in terms of $\cos\theta$ • one mark for final answer	3
(iii) • one mark for noting that $x=\cos\theta$ where $\cos 5\theta=-1$ • one mark for solution	2
(iv) • one mark for value of $\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}$ • one mark for value of $\cos\frac{\pi}{5}.\cos\frac{3\pi}{5}$ • one mark for factorisation	3

Answer

(i)

$\cos 5\theta=-1 \Rightarrow 5\theta=(2n+1)\pi$

$\theta=(2n+1)\frac{\pi}{5}$, $n=0, \pm1, \pm2, \ldots$

$0\le\theta\le 2\pi \Rightarrow \theta=\frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$

(ii) Using the binomial expansion,

$\text{Re}\left\{(\cos\theta+i\sin\theta)^5\right\}$

$=\cos^5\theta+10\cos^3\theta\,(i\sin\theta)^2+5\cos\theta\,(i\sin\theta)^4$

$=\cos^5\theta-10\cos^3\theta\sin^2\theta+5\cos\theta\sin^4\theta$

$=\cos^5\theta-10\cos^3\theta\,(1-\cos^2\theta)+5\cos\theta\,(1-\cos^2\theta)^2$

$=16\cos^5\theta-20\cos^3\theta+5\cos\theta$

Using De Moivre's Theorem,

$(\cos\theta+i\sin\theta)^5=\cos 5\theta+i\sin 5\theta$

Hence $\quad \cos 5\theta=16\cos^5\theta-20\cos^3\theta+5\cos\theta$

(iii) $16x^5-20x^3+5x+1=0$

has solutions $x=\cos\theta$ where $\cos 5\theta=-1$.

$x=\cos\frac{\pi}{5}, \cos\frac{3\pi}{5}, \cos\pi, \cos\frac{7\pi}{5}, \cos\frac{9\pi}{5}$

$x=\cos\frac{\pi}{5}, \cos\frac{\pi}{5}, \cos\frac{3\pi}{5}, \cos\frac{3\pi}{5}, -1$

(iv)

$\sum\alpha=0 \Rightarrow 2\left(\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}\right)-1=0$

$\therefore \cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=\frac{1}{2}$

Product of roots is $-\frac{1}{16}$

$\therefore -\left(\cos\frac{\pi}{5}.\cos\frac{3\pi}{5}\right)^2=-\frac{1}{16}$

$\therefore \cos\frac{\pi}{5}.\cos\frac{3\pi}{5}=-\frac{1}{4}$

$\left(\text{since } \cos\frac{\pi}{5}>0, \ \cos\frac{3\pi}{5}<0\right)$

Then $\cos\frac{\pi}{5}, \cos\frac{3\pi}{5}$ are roots of the equation $4x^2-2x-1=0$. Hence

$16x^5-20x^3+5x+1=(x+1)(4x^2-2x-1)^2$

Question 6

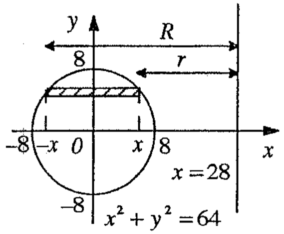
(a) **Outcomes Assessed:** (i) **E2, E7** (ii) **H8**

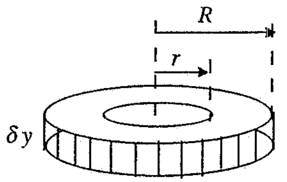
Marking Guidelines

Criteria	Marks
(i) • one mark for identifying slice as annular prism, thickness δy • one mark for inner radius r in terms of y • one mark for outer radius R in terms of y • one mark for simplified value of δV in terms of y • one mark for expression for V	5
(ii) • one mark for using area of semi circle, or appropriate integration process • one mark for final answer	2

Answer

(i)





$R=28+\sqrt{64-y^2}$

$r=28-\sqrt{64-y^2}$

Volume of slice is

$$\delta V=\pi\left(R^2-r^2\right)\delta y$$
$$=\pi(R+r)(R-r)\,\delta y$$
$$=\pi.56.2\sqrt{64-y^2}\,.\,\delta y$$
$$V=\lim_{\delta y\to 0}\sum_{y=-8}^{8}112\pi\sqrt{64-y^2}\,.\,\delta y$$
$$=112\pi\int_{-8}^{8}\sqrt{64-y^2}\;dy$$

(ii) $\int_{-8}^{8}\sqrt{64-y^2}\;dy=\frac{1}{2}\pi.8^2=32\pi$ (*Area of semicircle radius* 8) $\Rightarrow V=3584\pi^2$

Exact volume of lifebelt is $3584\pi^2$ cm^3

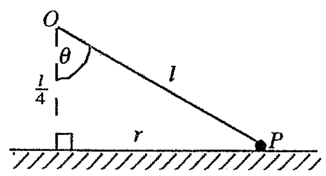
(b) **Outcomes Assessed:** (i) **E5** (ii) **E2, E5** (iii) **E2, E5** (iv) **E2, E5**

Marking Guidelines

Criteria	Marks
(i) • one mark for showing tension, reaction and weight forces on a diagram	1
(ii) • one mark for using horizontal component of resultant force to give $T\sin\theta=\frac{mv^2}{r}$ • one mark for expression for T	2
(iii) • one mark for using vertical component of resultant force to give $T\cos\theta+R=mg$ • one mark for expression for R	2
(iv) • one mark for noting $R\ge 0$ • one mark for obtaining inequality for v • one mark for stating particle would lift off the table	3

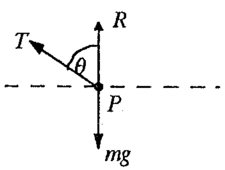
Answer

(i)



$\cos\theta=\frac{1}{4},\quad \sin\theta=\frac{\sqrt{15}}{4},\quad r=l\sin\theta$

Forces on P



Resultant force on P has magnitude $\frac{mv^2}{r}$ *and is directed horizontally towards the centre of the circle of motion.*

6(b)

(ii) Horizontal component of resultant force on P has magnitude $\frac{mv^2}{r}$.

$$T\sin\theta=\frac{mv^2}{r}$$
$$T=\frac{mv^2}{l\sin^2\theta}=\frac{16mv^2}{15l}$$

(iii) Vertical component of resultant force on P is zero.

$$T\cos\theta+R=mg$$
$$R=mg-\tfrac{1}{4}T=m\left(g-\frac{4v^2}{15l}\right)$$

(iv) Particle in contact with table provided $R\ge 0$

$$\therefore\ \frac{4v^2}{15l}\le g\ \Rightarrow\ v\le\sqrt{\frac{15gl}{4}}$$

Particle would lift off table for higher speeds.

Question 7

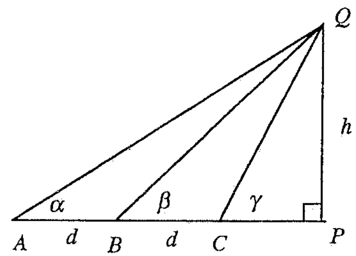
(a) **Outcomes Assessed:** (i) **P4, E2** (ii) **P4, E2**

Marking Guidelines

Criteria	Marks
(i) • one mark for expressions for $AB,\ BP\ CP$ • one mark for noting that $AP-BP=BP-CP$ • one mark for final expression	3
(ii) • one mark for appropriate statement of cosine rule in ΔABP • one mark for appropriate statement of cosine rule in ΔCBP • one mark for noting that $\cos\angle CBP=-\cos\angle ABP$ • one mark for final expression	4

Answer

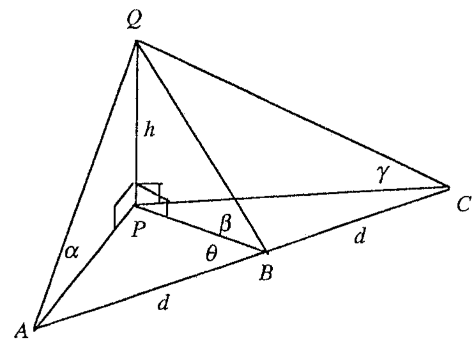
(i)



$$d=AP-BP=h\cot\alpha-h\cot\beta$$
$$d=BP-CP=h\cot\beta-h\cot\gamma$$
$$h\cot\alpha-h\cot\beta=h\cot\beta-h\cot\gamma$$
$$\therefore\ 2\cot\beta=\cot\alpha+\cot\gamma$$

(Similarly if A is closest to P by interchanging $A\leftrightarrow C,\ \alpha\leftrightarrow\gamma$.)

(ii)



In ΔABP,

$$h^2\cot^2\alpha=h^2\cot^2\beta+d^2-2dh\cot\beta\cos\theta$$

In ΔCBP,

$$h^2\cot^2\gamma=h^2\cot^2\beta+d^2-2dh\cot\beta\cos(\pi-\theta)$$
$$h^2\left(\cot^2\alpha+\cot^2\gamma\right)=2h^2\cot^2\beta+2d^2+0$$

since $\cos(\pi-\theta)=-\cos\theta$

$$\therefore\ h^2\left(\cot^2\alpha-2\cot^2\beta+\cot^2\gamma\right)=2d^2$$

7(b) **Outcomes Assessed:** (i) HE3, E2 (ii) P4, H3 (iii) H6, HE3

Marking Guidelines

Criteria	Marks
(i) • one mark for expression for $\frac{dN}{dt}$ in terms of t • one mark for final expression	2
(ii) • one mark for value of a • one mark for value of b • one mark for value of c	3
(iii) • one mark for showing maximum rate of increase when $N=450$ • one mark for correct shape of graph of N against t, with inflection at $N=450$ • one mark for showing initial value of N and asymptote for limiting value of N	3

Answer

(i)

$$N=\frac{a}{1+be^{-ct}} \Rightarrow be^{-ct}=\frac{a}{N}-1$$

$$\frac{dN}{dt}=-a\left(1+be^{-ct}\right)^{-2}\left(-bce^{-ct}\right)$$

$$=ac\,\frac{N^2}{a^2}\left(\frac{a}{N}-1\right)$$

$$=\frac{c}{a}(a-N)N$$

(ii)

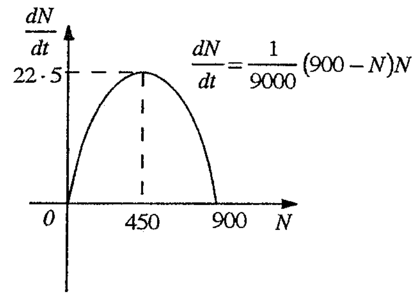
$$N=\frac{a}{1+be^{-ct}}, \quad \frac{dN}{dt}=\frac{c}{a}(a-N)N$$

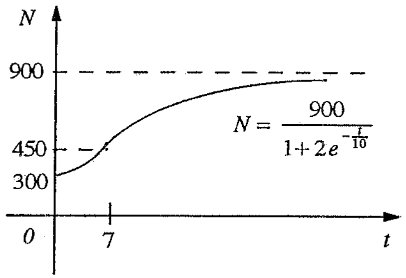
$N\to 900$ as $t\to\infty \Rightarrow a=900$

$t=0$, $N=300$, $\frac{dN}{dt}=20$ $\Rightarrow$ $\frac{a}{1+b}=300$ $\therefore b=2$; $\frac{c}{a}(a-300)300=20$ $\therefore c=\frac{1}{10}$

(iii)

Graph of $\frac{dN}{dt}$ against N is a concave down parabola.





Hence maximum rate when $N=450$.

Question 8

(a) **Outcomes Assessed:** HE3

Marking Guidelines

Criteria	Marks
• one mark for expressions $P(n \text{ even})={}^8C_n\,p^n(1-p)^{8-n}$, $n=2,5$ • one mark for equation in p, $(1-p)$ with numerical values of binomial coefficients • one mark for simplifying equation • one mark for final answer	4

Answer

$P(2 \text{ evens})={}^8C_2\,p^2(1-p)^6$

$P(5 \text{ evens})={}^8C_5\,p^5(1-p)^3$

$$\frac{P(2 \text{ evens})}{P(5 \text{ evens})}=4$$

$$4=\frac{28}{56}\,\frac{(1-p)^3}{p^3} \Rightarrow 8p^3=(1-p)^3$$

$$\therefore 2p=1-p \Rightarrow p=\tfrac{1}{3}$$

8(b) **Outcomes Assessed:** (i) H5 (ii) PE3, E2

Marking Guidelines

Criteria	Marks
(i) • one mark for stationary values of x • one mark for relative maximum value • one mark for relative minimum value	3
(ii) • one mark for relative maximum value positive and relative minimum value negative • one mark for product negative • one mark for inequality	3

Answer

(i)

$$y=x^3-3px+q$$

$$\frac{dy}{dx}=3x^2-3p=3(x^2-p)$$

$$\frac{dy}{dx}=0 \Rightarrow x=\pm\sqrt{p}$$

$$\frac{d^2y}{dx^2}=6x \Rightarrow \begin{cases}\frac{d^2y}{dx^2}>0 \text{ for } x=\sqrt{p}\\ \frac{d^2y}{dx^2}<0 \text{ for } x=-\sqrt{p}\end{cases}$$

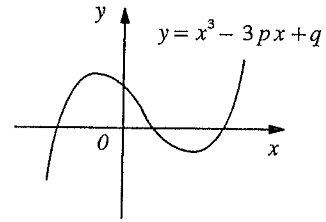
When $x=\sqrt{p}$, graph has relative minimum

$$p\sqrt{p}-3p\sqrt{p}+q = q-2p\sqrt{p}$$

When $x=-\sqrt{p}$, graph has relative maximum

$$-p\sqrt{p}+3p\sqrt{p}+q = q+2p\sqrt{p}$$

(ii) Graph has 3 distinct, non-zero x intercepts and $y\to+\infty$ as $x\to+\infty$



Hence $q+2p\sqrt{p}>0$, $q-2p\sqrt{p}<0$

$$\therefore (q+2p\sqrt{p})(q-2p\sqrt{p})<0$$

$$\therefore q^2-4p^3<0 \Rightarrow q^2<4p^3$$

(c) **Outcomes Assessed:** (i) PE3 (ii) PE3, E2 (iii) PE3, E2

Marking Guidelines

Criteria	Marks
(i) • one mark for answer	1
(ii) • one mark for replacing a, b, c by $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ respectively • one mark for final answer	2
(iii) • one mark for use of $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge 9$ • one mark for final answer	2

Answer

(i)

$$\sqrt[3]{abc}\le\frac{a+b+c}{3}=\frac{1}{3}$$

$$abc\le\tfrac{1}{27}$$

$$\frac{1}{abc}\ge 27$$

(ii)

$$\frac{1}{3}\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\ge\sqrt[3]{\left(\frac{1}{a}\right)\left(\frac{1}{b}\right)\left(\frac{1}{c}\right)}$$

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge 3\sqrt[3]{\frac{1}{abc}}$$

$$\ge 3\sqrt[3]{27}$$

$$\therefore \frac{1}{a}+\frac{1}{b}+\frac{1}{c}\ge 9$$

(iii)

$$(1-a)(1-b)(1-c)$$

$$=1-(a+b+c)+(bc+ca+ab)-abc$$

$$=(bc+ca+ab)-abc$$

$$=abc\left\{\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-1\right\}$$

$$\ge abc\,(9-1)$$

$$\therefore (1-a)(1-b)(1-c)\ge 8\,abc$$