STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

INDEPENDENT TRIAL

HSC 2003

2003

Higher School Certificate

Trial Examination

Mathematics

Extension 2

General Instructions

- Reading time – 5minutes
- Working time – 3 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided with this paper
- All necessary working should be shown in every question

Total marks - 120

Attempt Questions 1 – 8

All questions are of equal value

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME: ....

Student name / number ....

Question 1 **Begin a new page** Marks

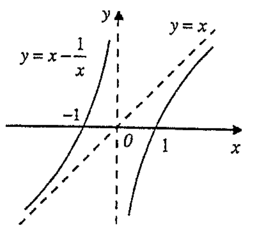
(a)(i) Find the coordinates and the nature of the stationary points on the curve $y = x^3 + 6x^2 + 9x + k$ where k is real. **2**

(ii) Hence find the set of values of k for which the equation $x^3 + 6x^2 + 9x + k = 0$ has three real and different roots. **2**

(b)(i) Find the domain and range of the function $f(x) = \tan^{-1} e^x$. Sketch the curve $y = f(x)$ showing any intercepts on the coordinate axes and the equations of any asymptotes. **3**

(ii) Show that $f'(x) = \frac{1}{2}\sin 2y$. **2**

(c)



The diagram shows the graph of the function $f(x) = x - \frac{1}{x}$.

On separate diagrams sketch the following curves, showing for each any intercepts on the coordinate axes and the equations of any asymptotes:

(i) $y = |f(x)|$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y^2 = f(x)$ **2**

Student name / number ....

Question 2 **Begin a new page** Marks

(a) (i) Find $\int (\sec x + \tan x)^2\, dx$. **2**

(ii) Find $\int \frac{1-x}{1-\sqrt{x}}\, dx$. **2**

(b) Use the substitution $u = e^x + 1$ to find $\int \frac{e^{2x}}{(e^x+1)^2}\, dx$. **2**

(c) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{\cos x + 2\sin x + 3}\, dx$, giving the answer correct to three significant figures. **4**

(d)(i) Find the exact value of $\int_0^{\frac{1}{2}} \frac{1}{1-x^2}\, dx$. **2**

(ii) If $I_n = \int_0^{\frac{1}{2}} \frac{x^n}{1-x^2}\, dx$ for $n = 0, 1, 2, \ldots$, show that $I_{n-2} - I_n = \frac{1}{(n-1)2^{n-1}}$ for $n = 2, 3, 4, \ldots$. Hence find the exact value of $\int_0^{\frac{1}{2}} \frac{x^4}{1-x^2}\, dx$. **3**

Question 3 **Begin a new page**

(a)(i) Express the roots of the equation $z^2 + 4z + 8 = 0$ in the form $a + ib$ for real a, b. **1**

(ii) Express the roots of the equation $z^2 + 4z + 8 = 0$ in modulus argument form. **2**

(b) Find all the complex numbers $z = a + ib$, where a, b are real, such that $|z|^2 + i\bar{z} = 11 + 3i$. **4**

(c)(i) On an Argand diagram mark the points P, Q representing the complex numbers $z_1 = 4 + i$, $z_2 = 1 + 4i$ respectively. Show how to construct the point R representing $z_1 + z_2$ and explain what type of quadrilateral $OPRQ$ is, where O is the origin. **2**

(ii) Find the area of the quadrilateral $OPRQ$. **2**

(d)(i) On and Argand diagram shade the region containing all the points representing complex numbers z such that both $|z| \le 1$ and $|z - 1| \le \sqrt{2}$. **2**

(ii) Find the exact area of the shaded region. **2**

Student name / number ....

Question 4 **Begin a new page** Marks

(a) The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, where $a>b>0$, has eccentricity $e=\frac{1}{2}$. The point $P(2,3)$ lies on the ellipse.

(i) Find the values of a and b. **3**

(ii) Sketch the graph of the ellipse showing clearly the intercepts on the axes, the coordinates of the foci and the equations of the directrices. **3**

(b) $P(x)$ is a polynomial of degree at least 2 such that $P'(a)=0$. Show that when $P(x)$ is divided by $(x-a)^2$ the remainder is $P(a)$. **3**

(c)(i) The equation $x^3+px^2+qx+r=0$ (where p, q, r are non zero) has roots α, β, γ such that $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are consecutive terms in an arithmetic sequence. Show that $\beta=\frac{-3r}{q}$. **3**

(ii) The equation $x^3-26x^2+216x-576=0$ has roots α, β, γ such that $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ are consecutive terms in an arithmetic sequence. Find the values of α, β, γ. **3**

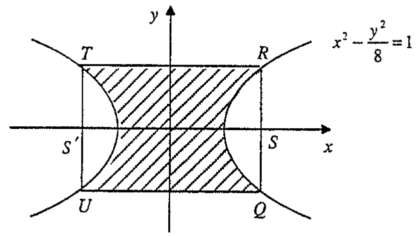
Student name / number ....

Question 5 **Begin a new page** Marks

(a) Use integration by parts and the table of standard integrals to show that **4**

$$\int\sqrt{y^2+8}\ dy=\tfrac{1}{2}y\sqrt{y^2+8}\ +\ 4\ln\left(y+\sqrt{y^2+8}\right)+c.$$

(b)



S, S' are foci of the hyperbola $x^2-\frac{y^2}{8}=1$. Focal chords RSQ and $TS'U$ are perpendicular to the x axis.

(i) Find the coordinates of the foci S, S'. **2**

(ii) Find the coordinates of Q, R, T and U. **1**

(iii) Use the result in (a) above to find the area of the shaded region. **3**

(c) The base of a particular solid is the shaded region in part (b) above. Every cross section of the solid perpendicular to the y axis is an equilateral triangle with one side in the base of the solid. Find the exact volume of the solid. **5**

Question 6 **Begin a new page**

(a)(i) Use DeMoivre's theorem to show that $\cos 4\theta=\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta$. **2**

(ii) Hence find the exact value of $\cos^4\left(\frac{\pi}{12}\right)+\sin^4\left(\frac{\pi}{12}\right)$. **3**

(b) A particle of mass m kg is set in motion with speed $u\,\text{ms}^{-1}$ and moves in a straight line before coming to rest. After time t seconds the particle has displacement x metres from its starting point O, velocity $v\,\text{ms}^{-1}$ and acceleration $a\,\text{ms}^{-2}$. The resultant force acting on the particle directly opposes its motion and has magnitude $m(1+v)$ Newtons.

(i) Show that $a=-(1+v)$. **1**

(ii) Find expressions for x in terms of v, v in terms of t and x in terms of t. **6**

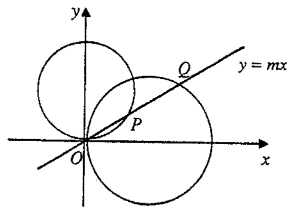
(iii) Show that $x+v+t=u$. **1**

(iv) Find the distance travelled and the time taken by the particle in coming to rest. **2**

Marks

Question 7 Begin a new page

(a)

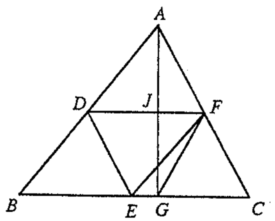


Two circles pass through the origin $O(0,0)$. The first circle has centre $(a,0)$ and the second has centre $(0,b)$ where $a>0$ and $b>0$. The line $y=mx$ through O cuts the first circle again at Q and the second again at P. M is the midpoint of PQ.

(i) Show that M has coordinates $\left(\dfrac{a+bm}{1+m^2}, \dfrac{m(a+bm)}{1+m^2}\right)$. 4

(ii) Find the equation of the locus of M as m varies and describe this locus geometrically. 3

(b)



ABC is an acute angled triangle. D, E, F are the midpoints of AB, BC, CA respectively. AG is an altitude of triangle ABC. AG and DF intersect at J.

(i) Copy the diagram.

(ii) Show that $\Delta AJF \equiv \Delta GJF$. 3

(iii) Show that $\angle FGC = \angle FCG$. 2

(iv) Show that $DEGF$ is a cyclic quadrilateral. 3

Marks

Question 8 Begin a new page

(a) $T_1, T_2, T_3, \ldots$ is an arithmetic sequence. All the terms of the sequence are positive and the common difference is d.

(i) Show that $\dfrac{1}{\sqrt{T_{n-1}}+\sqrt{T_n}} = \dfrac{\sqrt{T_n}-\sqrt{T_{n-1}}}{d}$ for $n = 2, 3, 4, \ldots$. 2

(ii) Hence show that $\dfrac{1}{\sqrt{T_1}+\sqrt{T_2}} + \dfrac{1}{\sqrt{T_2}+\sqrt{T_3}} + \ldots + \dfrac{1}{\sqrt{T_{n-1}}+\sqrt{T_n}} = \dfrac{n-1}{\sqrt{T_1}+\sqrt{T_n}}$ 3

for $n = 2, 3, 4, \ldots$.

(b)(i) For positive real numbers a, b show that $a^2+b^2 \geq 2ab$. 1

(ii) Hence show for positive real numbers a, b, c, d 2

$3(a^2+b^2+c^2+d^2) \geq 2(ab+ac+ad+bc+bd+cd)$.

(iii) Hence show that if a, b, c, d are positive real numbers such that $a+b+c+d=1$ 2

then $ab+ac+ad+bc+bd+cd \leq \frac{3}{8}$.

(c) $x_1, x_2, \ldots x_p$ are p different even integers and $y_1, y_2, \ldots y_q$ are q different odd integers. 5

Show that the sum S of all the products taken three at a time from

$(-1)^{x_1}, (-1)^{x_2}, \ldots, (-1)^{x_p}, (-1)^{y_1}, (-1)^{y_2}, \ldots, (-1)^{y_q}$

is given by $S = \frac{1}{6}(p-q)\left\{(p-q)^2 - 3(p+q) + 2\right\}$.

Independent Trial HSC 2003 Mathematics Extension 2 Answers and Marking Guidelines.

1(a) Outcomes Assessed: (i) **H6** (ii) **PE3**

Marking Guidelines

Criteria	Marks
(i) • finding minimum turning point	1
• finding maximum turning point	1
(ii) • noting that k and $k-4$ have opposite signs with explanation	1
• finding the set of values of k	1

Answer

(i) $y=x^3+6x^2+9x+k$

$$\frac{dy}{dx}=3x^2+12x+9=3(x+3)(x+1)$$
$$\frac{dy}{dx}=0 \Rightarrow x=-1,\ -3$$

$$\frac{d^2y}{dx^2}=6x+12=6(x+2)$$

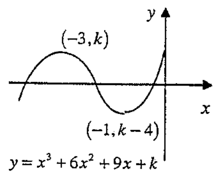
$x=-1 \Rightarrow \frac{d^2y}{dx^2}=6>0$ and $y=k-4$.

$\therefore (-1, k-4)$ is a minimum turning point.

$x=-3 \Rightarrow \frac{d^2y}{dx^2}=-6<0$ and $y=k$.

$\therefore (-3, k)$ is a maximum turning point.

(ii)



If the equation has 3 real and different roots, the curve $y=x^3+6x^2+9x+k$ must cut the x axis in three distinct points. Hence k, $k-4$ must have opposite signs. $\therefore 0<k<4$.

1(b) Outcomes Assessed: (i) **P5** (ii)**HE4**

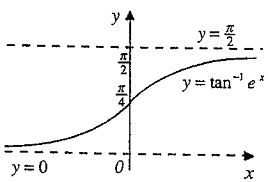
Marking Guidelines

Criteria	Marks
(i) • domain and range	1
• intercept and asymptote	1
• shape and position	1
(ii) • derivative in terms of e^x (or implicit differentiation)	1
• use of appropriate trig. identities to obtain required result	1

Answer

(i) $f(x)=\tan^{-1}e^x$ Domain: *All real* x

Range : $\{y: 0<y<\frac{\pi}{2}\}$



(ii)

$y=\tan^{-1}e^x \Rightarrow \tan y=e^x$

$$\frac{dy}{dx}=\frac{e^x}{1+(e^x)^2}=\frac{\tan y}{1+\tan^2 y}$$

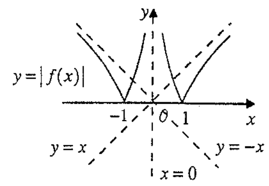
$$t=\tan y \Rightarrow \frac{dy}{dx}=\frac{1}{2}\cdot\frac{2t}{1+t^2}=\frac{1}{2}\sin 2y.$$

1(c) Outcomes Assessed: (i) **E6** (ii) **E6** (iii) **E6**

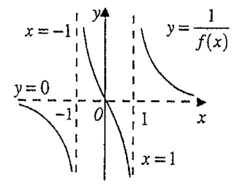
Marking Guidelines

Criteria	Marks
(i) • intercepts and asymptotes	1
• shape and position	1
(ii) • intercept and asymptotes	1
• shape and position	1
(iii)• intercepts and asymptote	1
• shape and position	1

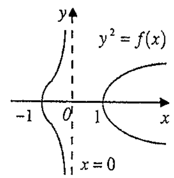
Answer (i)



(ii)



(iii)



2(a) Outcomes Assessed: (i) **H8** (ii) **H8**

Marking Guidelines

Criteria	Marks
(i) • simplification of integrand	1
• primitive function	1
(ii) • simplification of integrand	1
• primitive function	1

Answer

(i) $\int(\sec x+\tan x)^2\,dx=\int\left(2\sec^2 x-1+2\sec x\tan x\right)dx=2\tan x-x+2\sec x+c=2(\sec x+\tan x)-x+c$

(ii) $\int\frac{1-x}{1-\sqrt{x}}\,dx=\int\frac{(1-\sqrt{x})(1+\sqrt{x})}{1-\sqrt{x}}\,dx=\int\left(1+\sqrt{x}\right)dx=x+\frac{2}{3}x^{\frac{3}{2}}+c=\frac{1}{3}x\left(3+2\sqrt{x}\right)+c$

2(b) Outcomes Assessed: **HE6**

Marking Guidelines

Criteria	Marks
• integral in terms of u	1
• primitive function in terms of x	1

Answer

$u=e^x+1$

$du=e^x\,dx$

$$\int\frac{e^{2x}}{(e^x+1)^2}\,dx=\int\frac{e^x}{(e^x+1)^2}\cdot e^x\,dx=\int\frac{u-1}{u^2}\,du=\int\left(\frac{1}{u}-\frac{1}{u^2}\right)du$$

$$\int\frac{e^{2x}}{(e^x+1)^2}\,dx=\ln u+\frac{1}{u}+c=\ln\left(e^x+1\right)+\frac{1}{e^x+1}+c$$

2(c) Outcomes Assessed: HE6

Marking Guidelines

Criteria	Marks
• limits in terms of t and dx in terms of dt	1
• integral in terms of t in simplified form	1
• primitive function	1
• evaluation correct to 3 significant figures	1

Answer

$t=\tan\frac{x}{2}$

$dt=\frac{1}{2}\sec^2\frac{x}{2}\,dx$

$dt=\frac{1}{2}(1+t^2)\,dx$

$dx=\frac{2}{1+t^2}\,dt$

$x=0\Rightarrow t=0$

$x=\frac{\pi}{2}\Rightarrow t=1$

$$\cos x+2\sin x+3=\frac{1-t^2+4t+3(1+t^2)}{1+t^2}=\frac{2(t^2+2t+2)}{1+t^2}=\frac{2\{1+(t+1)^2\}}{1+t^2}$$

$$I=\int_0^{\frac{\pi}{2}}\frac{1}{\cos x+2\sin x+3}\,dx=\int_0^1\frac{1+t^2}{2\{1+(t+1)^2\}}\cdot\frac{2}{1+t^2}\,dt=\int_0^1\frac{1}{1+(t+1)^2}\,dt=\left[\tan^{-1}(t+1)\right]_0^1$$

$$\therefore I=\tan^{-1}2-\frac{\pi}{4}\approx 0\cdot 322$$

2(d) Outcomes Assessed: (i) E8 (ii) E8

Marking Guidelines

Criteria	Marks
(i) • expressing integrand in partial fraction form	1
• evaluation of integral	1
(ii) • simplification of $I_{n-2}-I_n$	1
• evaluation of $I_{n-2}-I_n$	1
• evaluation of I_4 using the recurrence relation	1

Answer

(i)

$$\int_0^{\frac{1}{2}}\frac{1}{1-x^2}\,dx=\frac{1}{2}\int_0^{\frac{1}{2}}\left(\frac{1}{1-x}+\frac{1}{1+x}\right)dx=\frac{1}{2}\left[-\ln(1-x)+\ln(1+x)\right]_0^{\frac{1}{2}}=\frac{1}{2}\left[\ln\left(\frac{1+x}{1-x}\right)\right]_0^{\frac{1}{2}}=\frac{1}{2}(\ln 3-\ln 1)=\frac{1}{2}\ln 3$$

(ii)

$$I_{n-2}-I_n=\int_0^{\frac{1}{2}}\frac{x^{n-2}-x^n}{1-x^2}\,dx=\int_0^{\frac{1}{2}}\frac{x^{n-2}(1-x^2)}{1-x^2}\,dx=\int_0^{\frac{1}{2}}x^{n-2}\,dx=\frac{1}{n-1}\left[x^{n-1}\right]_0^{\frac{1}{2}}=\frac{1}{(n-1)2^{n-1}}$$

Hence

$$\left.\begin{array}{l}I_0-I_2=\frac{1}{2}\\ I_2-I_4=\frac{1}{24}\end{array}\right\}\Rightarrow I_0-I_4=\frac{13}{24}$$

$$\therefore I_4=I_0-\frac{13}{24}=\frac{1}{2}\ln 3-\frac{13}{24}$$

$$\therefore\int_0^{\frac{1}{2}}\frac{x^4}{1-x^2}\,dx=\frac{1}{2}\ln 3-\frac{13}{24}$$

3(a) Outcomes Assessed: (i) E3 (ii) E3

Marking Guidelines

Criteria	Marks
(i) • both roots in required form	1
(ii)• modulus of roots	1
• arguments of roots	1

Answer

(i)

$z^2+4z+8=0$

$z^2+4z+4=-4$

$(z+2)^2=-4$

$(z+2)=\pm 2i$

$z=-2\pm 2i$

(ii)

$|z|=\sqrt{8}=2\sqrt{2}$

$z=2\sqrt{2}\left(-\frac{1}{\sqrt{2}}\pm\frac{1}{\sqrt{2}}i\right)$

Hence roots are

$2\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right),\quad 2\sqrt{2}\left\{\cos\left(-\frac{3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right\}$

3(b) Outcomes Assessed: E3

Marking Guidelines

Criteria	Marks
• equation in term of a and b	1
• equating real and imaginary parts	1
• solution of simultaneous equations for a and b	1
• values of solutions for z	1

Answer

$z=a+ib$

$|z|^2+i\bar{z}=11+3i$

$a^2+b^2+i(a-ib)=11+3i$

$(a^2+b^2+b)+ia=11+3i$

Equating real and imaginary parts :

$a=3,\quad 9+b^2+b=11$

$\therefore b^2+b-2=0$

$(b+2)(b-1)=0$

$\left.\begin{array}{l}a=3\\ b=1\end{array}\right\}$ or $\left.\begin{array}{l}a=3\\ b=-2\end{array}\right\}$

$z=3+i$ or $z=3-2i$

3(c) Outcomes Assessed: (i) E3 (ii) E3

Marking Guidelines

Criteria	Marks
(i) • construction of point R	1
• classification of $OPRQ$ with reasons	1
(ii)• lengths of both diagonals	1
• area of $OPRQ$	1

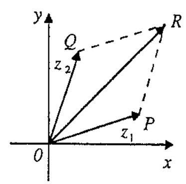
Answer

(i)

$P(4,1)$, $Q(1,4)$ represent $z_1=4+i$, $z_2=1+4i$ respectively.

Complete the parallelogram $OPRQ$ to show the point R representing the sum $z_1+z_2=5+5i$.

Since $OP=OQ=\sqrt{17}$, $OPRQ$ is a rhombus.



(ii)

$PQ=3\sqrt{2}$

$R(5,5)\Rightarrow OR=5\sqrt{2}$.

Hence $OPRQ$ has area

$\frac{1}{2}\times 3\sqrt{2}\times 5\sqrt{2}=15$ sq. units.

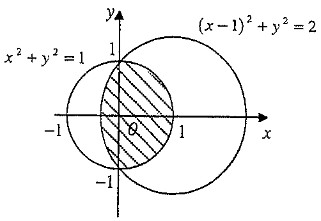
3(d) **Outcomes Assessed:** (i) E3 (ii) E3

Marking Guidelines

Criteria	Marks
(i) • sketch of both circles	1
• region shaded	1
(ii)• area of semi-circle or area of segment	1
• sum of area of semi-circle and segment	1

Answer

(i)



(ii)

Shaded region is composed of semicircle radius 1, and segment of circle radius $\sqrt{2}$ cut off by chord subtending right angle at centre.

Area is $\frac{1}{2}\pi+\frac{1}{2}.2\left(\frac{\pi}{2}-\sin\frac{\pi}{2}\right)=\pi-1$ square units.

4(a) **Outcomes Assessed:** (i) E4 (ii) E3

Marking Guidelines

Criteria	Marks
(i) • relation between a^2, b^2 using $e=\frac{1}{2}$	1
• relation between a^2, b^2 using coordinates of P	1
• values of a and b	1
(ii) • ellipse with intercepts on axes	1
• foci with coordinates	1
• directrices with equations	1

Answer

(i) $e=\frac{1}{2}\Rightarrow b^2=a^2\left(1-\frac{1}{4}\right)=\frac{3}{4}a^2$

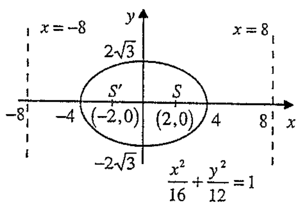
$P(2,3)$ on ellipse $\Rightarrow \dfrac{4}{a^2}+\dfrac{9}{b^2}=1$

$\therefore \dfrac{4}{a^2}+\dfrac{12}{a^2}=1$

$\therefore a^2=16,\ b^2=12$

$\therefore a=4,\ \ b=2\sqrt{3}$

(ii)



4(b) **Outcomes Assessed:** E4

Marking Guidelines

Criteria	Marks
• write expression for $P(x)$ using division transformation with divisor $(x-a)^2$	1
• obtain expression for $P'(a)$	1
• use $P'(a)=0$ to show division by $(x-a)^2$ leaves required remainder.	1

Answer

Divisor has degree 2, hence remainder has degree <2. Let the remainder be $cx+d$, c and d constants.
Using the division transformation, $P(x)=(x-a)^2Q(x)+cx+d$, where $Q(x)$ is a polynomial.
Now $P'(x)=2(x-a)Q(x)+(x-a)^2Q'(x)+c \ \Rightarrow\ c=P'(a)=0$
Then $P(x)=(x-a)^2Q(x)+d \ \Rightarrow\ d=P(a)$. Hence remainder is $P(a)$.

4(c) **Outcomes Assessed:** (i) E4 (ii) E4

Marking Guidelines

Criteria	Marks
(i) • deduction that $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{3}{\beta}$ using property of A.P	1
• expressing $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$ in terms of coefficients of equation	1
• deducing required result	1
(ii)• finding the value of β	1
• values of $\alpha+\gamma,\ \alpha\gamma$	1
• values of $\alpha,\ \lambda$.	1

Answer

(i) $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ in AP $\Rightarrow \frac{1}{\beta}-\frac{1}{\alpha}=\frac{1}{\gamma}-\frac{1}{\beta} \ \Rightarrow\ \frac{2}{\beta}=\frac{1}{\alpha}+\frac{1}{\gamma}$. Then $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{3}{\beta}$

But $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{\beta\gamma+\gamma\alpha+\alpha\beta}{\alpha\beta\gamma}=-\frac{q}{r}$. Hence $\frac{3}{\beta}=-\frac{q}{r}$. $\therefore \beta=\frac{-3r}{q}$

(ii) $x^3-26x^2+216x-576=0$ such that $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ in AP $\Rightarrow \beta=\frac{-3r}{q}=\frac{3\times 576}{216}=8$

Then $\alpha+\gamma=26-8=18$ and $\alpha\gamma=576\div 8=72$.

Hence $\alpha,\ \gamma$ zeros of $x^2-18x+72=(x-12)(x-6)$.

$\alpha,\ \beta,\ \gamma$ are $6,8,12$ or $12,8,6$ respectively.

5(a) **Outcomes Assessed:** E8

Marking Guidelines

Criteria	Marks
• integration by parts	1
• rearrangement of integrand into appropriate form	1
• identifying appropriate entry in table of integrals	1
• primitive function	1

Answer

$$\int\sqrt{y^2+8}\,dy=y\sqrt{y^2+8}-\int y.\tfrac{1}{2}\left(y^2+8\right)^{-\frac{1}{2}}.2y\,dy$$

$$=y\sqrt{y^2+8}-\int\left(y^2+8-8\right).\left(y^2+8\right)^{-\frac{1}{2}}dy$$

$$=y\sqrt{y^2+8}-\int\left\{\left(y^2+8\right)^{\frac{1}{2}}-8\left(y^2+8\right)^{-\frac{1}{2}}\right\}dy$$

$$=y\sqrt{y^2+8}-\int\sqrt{y^2+8}\,dy+8\int\frac{1}{\sqrt{y^2+8}}dy$$

$$\therefore 2\int\sqrt{y^2+8}\,dy=y\sqrt{y^2+8}+8\int\frac{1}{\sqrt{y^2+8}}dy$$

$$=y\sqrt{y^2+8}+8\ln\left(y+\sqrt{y^2+8}\right)+c_1$$

$$\therefore \int\sqrt{y^2+8}\,dy=\tfrac{1}{2}y\sqrt{y^2+8}+4\ln\left(y+\sqrt{y^2+8}\right)+c$$

5(b) Outcomes Assessed: (i) **E4** (ii) **E4** (iii) **H8**

Marking Guidelines

Criteria	Marks
(i) • value of eccentricity e	1
• coordinates of both foci	1
(ii) • coordinates of all 4 points	1
(iii) • expression for area in integral form	1
• writing primitive function and substituting limits	1
• value of definite integral in simplified form	1

Answer

(i) $x^2 - \frac{y^2}{8} = 1$ $\quad b^2 = a^2(e^2-1) \Rightarrow e^2 - 1 = 8 \quad \therefore e = 3$. $\quad$ Foci are $S(3,0)$, $S'(-3,0)$.

(ii) $x = \pm 3 \Rightarrow y^2 = 64 \quad \therefore Q(3,-8),\ R(3,8),\ T(-3,8),\ U(-3,-8)$

(iii) Using symmetry, area is A sq.units where

$$A = 4\int_0^8 \sqrt{1+\frac{y^2}{8}}\,dy = \sqrt{2}\int_0^8 \sqrt{y^2+8}\,dy$$

$$A = \sqrt{2}\left[\tfrac{1}{2}y\sqrt{y^2+8}\right]_0^8 + 4\sqrt{2}\left[\ln\left(y+\sqrt{y^2+8}\right)\right]_0^8$$
$$= \sqrt{2}\left(4\times 6\sqrt{2}\right) + 4\sqrt{2}\left\{\ln\left(8+6\sqrt{2}\right) - \ln\left(2\sqrt{2}\right)\right\}$$
$$= 48 + 4\sqrt{2}\ln\left(3+2\sqrt{2}\right)$$

5(c) Outcomes Assessed: (i) **E7**

Marking Guidelines

Criteria	Marks
• area of cross section of typical equilateral triangle	1
• derivation of volume as a definite integral	1
• simplification of integrand	1
• primitive function and substitution of limits	1
• exact volume in simplest form	1

Answer

Cross section is equilateral triangle with side length $2x$, where $x^2 - \frac{y^2}{8} = 1$.

Hence area of cross section is

$$\tfrac{1}{2}(2x)^2\sin 60° = 2\left(1+\frac{y^2}{8}\right)\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{8}\left(y^2+8\right).$$

Volume of slice is $\delta V = \frac{\sqrt{3}}{8}\left(y^2+8\right)\delta y$.

Hence volume of solid is V where

$$V = 2\lim_{\delta y \to 0}\sum_{y=0}^{8}\frac{\sqrt{3}}{8}\left(y^2+8\right)\delta y$$
$$\therefore V = \frac{\sqrt{3}}{4}\int_0^8\left(y^2+8\right)dy$$
$$= \frac{\sqrt{3}}{4}\left[\tfrac{1}{3}y^3+8y\right]_0^8$$
$$= \frac{\sqrt{3}}{4}\left(\tfrac{512}{3}+64\right)$$

Volume is $\frac{176\sqrt{3}}{3}$ cu. units

6(a) Outcomes Assessed: (i) **E3** (ii) **H5**

Marking Guidelines

Criteria	Marks
(i) • De Moivre's theorem to simplify $(\cos\theta + i\sin\theta)^4$	1
• using binomial expansion equating real parts to obtain required result	1
(ii) • expressing $\cos^4\theta + \sin^4\theta$ in terms of $\cos 4\theta$, $\sin 2\theta$	1
• substitution $\theta = \frac{\pi}{12}$	1
• exact value in simplest form	1

Answer

(i) Using De Moivre's theorem, $\cos 4\theta = \mathrm{Re}(\cos\theta + i\sin\theta)^4$.

Using the binomial theorem, $\mathrm{Re}(\cos\theta + i\sin\theta)^4 = \cos^4\theta + {}^4C_2\cos^2\theta(i\sin\theta)^2 + (i\sin\theta)^4$

Hence $\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

(ii)

$$\cos^4\theta + \sin^4\theta = \cos 4\theta + 6\cos^2\theta\sin^2\theta = \cos 4\theta + \tfrac{3}{2}\sin^2 2\theta$$

$$\cos^4\tfrac{\pi}{12} + \sin^4\tfrac{\pi}{12} = \cos\tfrac{\pi}{3} + \tfrac{3}{2}\sin^2\tfrac{\pi}{6} = \tfrac{1}{2} + \tfrac{3}{8} = \tfrac{7}{8}$$

6(b) Outcomes Assessed: (i) **E5** (ii) **E5** (iii) **E5** (iv) **E5**

Marking Guidelines

Criteria	Marks
(i) • use Newton's 2nd law to show result	1
(ii) • expression for $\frac{dx}{dv}$ in terms of v	1
• integration to find x in terms of v	1
• expression for $\frac{dt}{dv}$ in terms of v	1
• integration to find t in terms of v	1
• expression for v in terms of t	1
• expression for x in terms of t	1
(iii) • showing required result	1
(iv) • distance travelled	1
• time taken	1

Answer

(i) By Newton's 2nd law, $m\ddot{x} = -m(1+v)$. Hence $a = -(1+v)$.

(ii)

$$v\frac{dv}{dx} = -(1+v)$$
$$\frac{dv}{dx} = -\frac{1+v}{v}$$
$$\frac{dx}{dv} = -\frac{v}{1+v}$$
$$\frac{dx}{dv} = -1 + \frac{1}{1+v}$$
$$x = -v + \ln(1+v) + c$$

$t=0,\ x=0,\ v=u \Rightarrow$ $\quad 0 = -u + \ln(1+u) + c$, $\quad x = u - v + \ln\left(\frac{1+v}{1+u}\right)$

$$\frac{dv}{dt} = -(1+v)$$
$$\frac{dt}{dv} = -\frac{1}{1+v}$$
$$t = -\ln(1+v)A$$

$t=0,\ v=u \Rightarrow$ $\quad (1+u)A = 1$, $\quad t = -\ln\left(\frac{1+v}{1+u}\right)$

$$e^{-t} = \frac{1+v}{1+u}$$
$$v = (1+u)e^{-t} - 1$$

$$x = u - v + \ln\left(\frac{1+v}{1+u}\right)$$
$$x = u - (1+u)e^{-t} + 1 - t$$

(iii) $x = u - v + \ln\left(\frac{1+v}{1+u}\right) = u - v - t \qquad \therefore\ x + v + t = u$

(iv) $v = 0 \Rightarrow x = u - \ln(1+u)$. Hence particle travels $\{u - \ln(1+u)\}$ metres in coming to rest.
$v = 0 \Rightarrow t = \ln(1+u)$. Hence particle takes $\ln(1+u)$ seconds to come to rest.

7(a) Outcomes Assessed: (i) P4 (ii) P4

Marking Guidelines

Criteria	Marks
(i) • equations of both circles	1
• coordinates of P	1
• coordinates of Q	1
• coordinates of M	1
(ii) • m in terms of x, y coordinates of M	1
• equation of locus of M	1
• geometric description of locus of M.	1

Answer

(i) Q lies on circle centre $(a,0)$, radius a, with equation $(x-a)^2+y^2=a^2$.

At Q, $y=mx \Rightarrow (x-a)^2+m^2x^2=a^2 \Rightarrow (1+m^2)x^2-2ax=0$. $\therefore Q\left(\dfrac{2a}{1+m^2}, \dfrac{2ma}{1+m^2}\right)$

P lies on circle centre $(0,b)$, radius b, with equation $x^2+(y-b)^2=b^2$.

At P, $y=mx \Rightarrow x^2+(mx-b)^2=b^2 \Rightarrow (1+m^2)x^2-2mbx=0$. $\therefore P\left(\dfrac{2mb}{1+m^2}, \dfrac{2m^2b}{1+m^2}\right)$

Hence midpoint PQ has coordinates $\left(\dfrac{2a+2mb}{2(1+m^2)}, \dfrac{2ma+2m^2b}{2(1+m^2)}\right)$ $\therefore M\left(\dfrac{a+bm}{1+m^2}, \dfrac{m(a+bm)}{1+m^2}\right)$

(ii) At M, $(1+m^2)x=a+bm$ and $m=\dfrac{y}{x}$, $m\neq 0$.

$$\left(1+\frac{y^2}{x^2}\right)x=a+b\frac{y}{x}$$

$$x^2+y^2=ax+by$$

$$x(x-a)+y(y-b)=0$$

This is the equation of a circle on diameter OC, where O is the origin and C has coordinates $C(a,b)$.

Alternatively, circle with centre $(\frac{1}{2}a,\frac{1}{2}b)$ and radius $\frac{1}{2}\sqrt{a^2+b^2}$. The points $(a,0)$, $(0,b)$ are excluded.

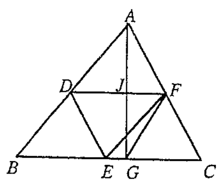
7(b) Outcomes Assessed: (i) (ii) H5 (iii) H5 (iv) PE3

Marking Guidelines

Criteria	Marks
(i) • copy diagram	0
(ii) • show $AJ=GJ$	1
• show $A\hat{J}F=G\hat{J}F$	1
• congruence proof	1
(iii) • show $GF=CF$	1
• deduce required angles equal	1
(iv) • show $DECF$ is parallelogram	1
• deduce $F\hat{C}G=E\hat{D}F$	1
• deduce quadrilateral is cyclic	1

Answer

(i)



(ii)

$DF \parallel BC$ (*line joining midpoints of two sides of ΔABC is parallel to 3rd side*)

$\therefore A\hat{J}F=A\hat{G}C$ (*corresp. $\angle$ s with $\parallel$ lines are equal*)

$\therefore A\hat{J}F=90°$ (*Altitude $AG \perp BC \Rightarrow A\hat{G}C=90°$*)

$\therefore G\hat{J}F=90°$ (*Adj. supplementary $\angle$ s add to 180°*)

$\therefore A\hat{J}F=G\hat{J}F$

Also in ΔAGC, $JF \parallel GC$ (*$DF \parallel BC$ proven above*) and F is the midpoint of AC (*given*)

$\therefore AJ=GJ$ (*line parallel to one side cuts other sides in proportion*)

Now in ΔAJF, ΔGJF

JF is common

$A\hat{J}F=G\hat{J}F$ (*proven*)

$AJ=GJ$ (*proven*)

$\therefore \Delta AJF \equiv \Delta GJF$ (*SAS*)

(iii)

$GF=AF$ (*corresp. sides of congr. Δ' s are equal*)

$AF=CF$ (*F given midpoint of AC*)

$GF=CF$

$\therefore F\hat{G}C=F\hat{C}G$ (*equal $\angle$ s opp. equal sides in ΔGFC*)

(iv)

$DE \parallel AC$ and $DF \parallel BC$ (*line joining midpoints of two sides of ΔABC is parallel to 3rd side*)

$\therefore DFCE$ is a parallelogram (*both pairs of opp. sides parallel*)

$\therefore F\hat{C}G=E\hat{D}F$ (*opp. $\angle$ s in parallelogram are equal*)

$\therefore F\hat{G}C=E\hat{D}F$ (*since $F\hat{G}C=F\hat{C}G$ proven above*)

Hence quadrilateral $DEGF$ is cyclic (*exterior $\angle$ equal to interior opposite $\angle$*)

8(a) Outcomes Assessed: (i) H5 (ii) H5

Marking Guidelines

Criteria	Marks
(i) • use defining property of AP	1
• rearrange to obtain required result	1
(ii) • use result to simplify sum	1
• rationalise numerator	1
• show required result	1

Answer

(i)

$$d=T_n-T_{n-1}=\left(\sqrt{T_{n-1}}+\sqrt{T_n}\right)\left(\sqrt{T_n}-\sqrt{T_{n-1}}\right)$$

$$\therefore \frac{1}{\sqrt{T_{n-1}}+\sqrt{T_n}}=\frac{\sqrt{T_n}-\sqrt{T_{n-1}}}{d}$$

(ii)

$$s=\frac{1}{\sqrt{T_1}+\sqrt{T_2}}+\frac{1}{\sqrt{T_2}+\sqrt{T_3}}+\ldots+\frac{1}{\sqrt{T_{n-1}}+\sqrt{T_n}}$$

$$=\frac{1}{d}\left\{\left(\sqrt{T_2}-\sqrt{T_1}\right)+\left(\sqrt{T_3}-\sqrt{T_2}\right)+\ldots\left(\sqrt{T_n}-\sqrt{T_{n-1}}\right)\right\}$$

$$=\frac{1}{d}\left(\sqrt{T_n}-\sqrt{T_1}\right)$$

$$\therefore s=\frac{1}{d}\cdot\frac{\left(\sqrt{T_n}-\sqrt{T_1}\right)\left(\sqrt{T_n}+\sqrt{T_1}\right)}{\sqrt{T_n}+\sqrt{T_1}}$$

$$=\frac{T_n-T_1}{d}\cdot\frac{1}{\sqrt{T_n}+\sqrt{T_1}}$$

$$=\frac{n-1}{\sqrt{T_n}+\sqrt{T_1}}$$

since $T_n-T_1=(n-1)d$

8(b) Outcomes Assessed: (i) PE3 (ii) PE3 (iii) PE3

Marking Guidelines

Criteria	Marks
(i) • prove result	1
(ii) • write pairwise inequalities	1
• combine to obtain required result	1
(iii) • write sum of squares in terms of sum of products taken two at a time	1
• obtain required result	1

Answer

(i)

$a^2 + b^2 - 2ab = (a-b)^2 \ge 0$

$\therefore \quad a^2 + b^2 \ge 2ab$

(ii)

$a^2 + b^2 \ge 2ab \qquad b^2 + c^2 \ge 2bc$

$a^2 + c^2 \ge 2ac \qquad b^2 + d^2 \ge 2bd$

$a^2 + d^2 \ge 2ad \qquad c^2 + d^2 \ge 2cd$

$$3\left(a^2 + b^2 + c^2 + d^2\right) \ge 2(ab + ac + ad + bc + bd + cd)$$

(iii) $a^2 + b^2 + c^2 + d^2 = (a+b+c+d)^2 - 2(ab + ac + ad + bc + bd + cd)$

But $a+b+c+d = 1$. Hence, using (ii),

$$3\left(a^2 + b^2 + c^2 + d^2\right) = 3 - 6(ab + ac + ad + bc + bd + cd)$$

$$2(ab + ac + ad + bc + bd + cd) \le 3 - 6(ab + ac + ad + bc + bd + cd)$$

$$8(ab + ac + ad + bc + bd + cd) \le 3$$

$$(ab + ac + ad + bc + bd + cd) \le \tfrac{3}{8}$$

8(c) Outcomes Assessed: E9

Marking Guidelines

Criteria	Marks
• count the number of ways of getting a product of +1	1
• count the number of ways of getting a product of –1	1
• expression for S	1
• simplification of expression for S	1
• factoring and rearranging expression to obtain required form	1

Answer

$(-1)^{x_k} = 1, \ 1 \le k \le p$ while $(-1)^{y_k} = -1, \ 1 \le k \le q$

There are ${}^pC_3 = \frac{1}{6}p(p-1)(p-2)$ terms of the form $(-1)^{x_j}(-1)^{x_k}(-1)^{x_l}$ with a sum $\frac{1}{6}p(p-1)(p-2)$

There are ${}^pC_2 \times q = \frac{1}{2}pq(p-1)$ terms of the form $(-1)^{x_j}(-1)^{x_k}(-1)^{y_l}$ with a sum $-\frac{1}{2}pq(p-1)$

There are $p \times {}^qC_2 = \frac{1}{2}pq(q-1)$ terms of the form $(-1)^{x_j}(-1)^{y_k}(-1)^{y_l}$ with a sum $\frac{1}{2}pq(q-1)$

There are ${}^qC_3 = \frac{1}{6}q(q-1)(q-2)$ terms of the form $(-1)^{y_j}(-1)^{y_k}(-1)^{y_l}$ with a sum $-\frac{1}{6}q(q-1)(q-2)$

Hence $S = \frac{1}{6}p(p-1)(p-2) - \frac{1}{2}pq(p-1) + \frac{1}{2}pq(q-1) - \frac{1}{6}q(q-1)(q-2)$

$$= \tfrac{1}{6}\left\{\left(p^3 - q^3\right) - 3\left(p^2 - q^2\right) + 2(p-q)\right\} - \tfrac{1}{2}pq(p-q)$$

$$= \tfrac{1}{6}(p-q)\left\{\left(p^2 + pq + q^2\right) - 3(p+q) + 2 - 3pq\right\}$$

$$= \tfrac{1}{6}(p-q)\left\{\left(p^2 - 2pq + q^2\right) - 3(p+q) + 2\right\}$$

$$= \tfrac{1}{6}(p-q)\left\{(p-q)^2 - 3(p+q) + 2\right\}$$