STANDARD INTEGRALS

$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\quad x \neq 0, \text{ if } n < 0$

$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$

$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$

$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$

$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$

$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\quad -a < x < a$

$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$

$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE : $\ln x = \log_e x,\quad x > 0$

INDEPENDANT 2005

2005
Higher School Certificate
Trial Examination

Mathematics
Extension 2

General Instructions

- Reading time – 5minutes
- Working time – 3 hours
- Write using black or blue pen
- Board approved calculators may be used
- A table of standard integrals is provided with this paper
- All necessary working should be shown in every question

Total marks - 120

Attempt Questions 1 – 8

All questions are of equal value

This paper MUST NOT be removed from the examination room

STUDENT NUMBER/NAME: ....

Student name / number ....

Marks

Question 2 **Begin a new page**

(a)(i) Find $\int \frac{\sqrt{1-x^2}}{\sqrt{1-x}}\,dx$ **2**

(ii) Find $\int x\cos x\;dx$ **2**

(b) Evaluate in simplest exact form $\int_0^4 \frac{8-2x}{(1+x)(4+x^2)}\,dx$ **3**

(c) Use the substitution $t=\tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{2}{5+3\cos x}\,dx$, giving the answer correct to 2 significant figures. **3**

(d)(i) Use the substitution $u=\frac{\pi}{4}-x$ to show that **3**

$$\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\;dx=\int_0^{\frac{\pi}{4}} \ln\left(\frac{2}{1+\tan x}\right)dx$$

(ii) Hence find the exact value of $\int_0^{\frac{\pi}{4}} \ln(1+\tan x)\;dx$ **2**

Student name / number ....

Marks

Question 3 **Begin a new page**

(a) Solve the equation $z^2+2\bar{z}+6=0$, giving the solutions in the form $z=a+ib$ where a and b are real. **4**

(b) $z_1=2\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right)$ and $z_2=2i$ are two complex numbers.

(i) On an Argand diagram draw the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ to represent z_1 and z_2 respectively. Also draw the vectors representing z_1+z_2 and z_2-z_1. **2**

(ii) Find the exact values of $\arg(z_1+z_2)$ and $\arg(z_2-z_1)$. **2**

(c)(i) Express $z=2\sqrt{3}-2i$ in modulus-argument form. **2**

(ii) Hence find the two values of $z^{\frac{1}{2}}$ in modulus-argument form. **2**

(d) The point P representing the complex number z moves in an Argand diagram so that $|z|=|z-4+2i|$.

(i) Show that the locus of P has equation $2x-y-5=0$. **2**

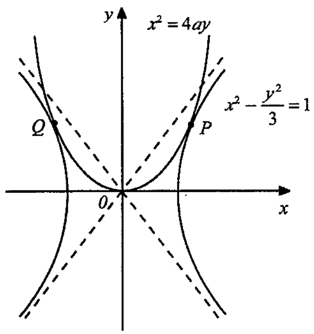
(ii) Hence find the minimum value of $|z|$. **1**

Student name / number ....

Marks

Question 4 **Begin a new page**

(a)



The parabola $x^2 = 4ay$ $(a > 0)$ touches the hyperbola $x^2 - \frac{y^2}{3} = 1$ at the points P and Q.

(i) Copy the diagram showing clearly for the hyperbola the intercepts made on the x axis, the coordinates of the foci, the directrices with their equations and the equations of the asymptotes. **4**

(ii) Find the value of a and the coordinates of P and Q. **3**

(b) $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ are two variable points on the rectangular hyperbola $xy = c^2$ which move so that the points P, Q and $S\left(c\sqrt{2}, c\sqrt{2}\right)$ are always collinear. The tangents to the hyperbola at P and Q intersect at the point R.

(i) Show that the tangent to the hyperbola $xy = c^2$ at the point $T\left(ct, \frac{c}{t}\right)$ has equation $x + t^2 y = 2ct$. **2**

(ii) Hence show that R has coordinates $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$. **2**

(iii) Show that $p + q = \sqrt{2}\,(1 + pq)$. **2**

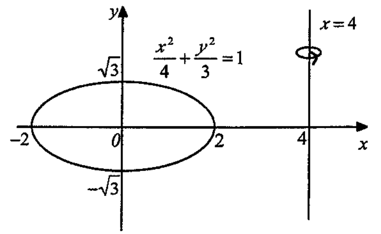
(iv) Hence find the equation of the locus of R. **2**

Student name / number ....

Marks

Question 5 **Begin a new page**

(a) A mould for a model space station is made by rotating the ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ through one complete revolution about the line $x = 4$. All measurements are in centimetres.



(i) Use the method of cylindrical shells to show that the volume V cm^3 of the toy space station is given by **3**

$$V = 2\sqrt{3}\,\pi \int_{-2}^{2} (4 - x)\sqrt{4 - x^2}\; dx$$

(ii) Hence calculate this volume correct to the nearest cubic centimetre. **3**

(b)(i) Show that $1 - (\cos n\theta + i\sin n\theta) = -2i\sin\frac{n\theta}{2}\left(\cos\frac{n\theta}{2} + i\sin\frac{n\theta}{2}\right)$. **2**

(ii) Find the sum of the series $z + z^2 + z^3 + \ldots + z^n$ for $z \neq 1$. **1**

(iii) If $z = \cos\theta + i\sin\theta$, show that for $\sin\frac{\theta}{2} \neq 0$,

$$\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta = \frac{\sin\frac{n\theta}{2}\cos\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}$$ **4**

(iv) Hence solve the equation $\cos\theta + \cos 2\theta + \cos 3\theta = 0$, $0 \le \theta \le 2\pi$. **2**

Student name / number ....

Marks

Question 6 **Begin a new page**

(a) A particle of mass m is projected vertically upwards with speed $U\text{ms}^{-1}$ and returns to its starting point with speed $V\text{ms}^{-1}$. The resistance to its motion in Newtons has magnitude $\frac{mv}{10}$ when its speed is $v\text{ms}^{-1}$. The acceleration due to gravity is 10ms^{-2}.

(i) For the ascent of the particle, show that its acceleration $a\text{ms}^{-2}$ is given by $a=-10-\frac{v}{10}$. Hence show that the distance travelled x metres is given by **4**

$$x=10(U-v)-1000\ln\left(\frac{100+U}{100+v}\right)$$

(ii) For the descent, show that its acceleration $a\text{ms}^{-2}$ is given by $a=10-\frac{v}{10}$. **4**
Hence show that the distance fallen (from the point of maximum height) x metres is given by

$$x=-10v+1000\ln\left(\frac{100}{100-v}\right)$$

(iii) Find the greatest height reached and hence show that **3**

$$U+V=100\ln\left(\frac{100+U}{100-V}\right)$$

(b) The equation $x^3+2x+1=0$ has roots α, β and γ.

(i) Find the monic cubic equation with roots α^{-1}, β^{-1} and γ^{-1}. **2**

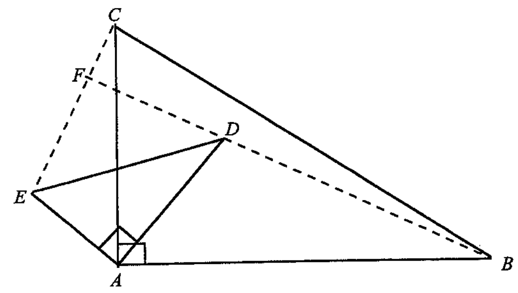
(ii) Find the monic cubic equation with roots α^{-2}, β^{-2} and γ^{-2}. **2**

Student name / number ....

Marks

Question 7 **Begin a new page**

(a)



Triangles ABC and ADE are each right angled at A, and $\Delta ABC ||| \Delta ADE$.
BD produced meets CE at F.

(i) Copy the diagram.

(ii) Show that $\Delta BDA ||| \Delta CEA$. **3**

(iii) Hence show that $ADFE$ is a cyclic quadrilateral. **2**

(iv) Deduce that $BF \perp CE$. **2**

(b) A sequence T_n is given by $T_1=1$ and $T_n=\sqrt{3+2T_{n-1}}$ for $n=2, 3, 4, \ldots$

(i) Use Mathematical Induction to show that $T_n<3$ for all positive integers $n\geq 1$. **3**

(ii) Hence show that $T_{n+1}>T_n$ for all positive integers $n\geq 1$. **2**

(iii) Show that $T_{n+2}-T_{n+1}=\frac{T_{n+2}^2-T_{n+1}^2}{T_{n+2}+T_{n+1}}$. Hence show that $T_{n+2}-T_{n+1}<T_{n+1}-T_n$ for all positive integers $n\geq 1$. **3**

Student name / number ....

Marks

Question 8 **Begin a new page**

(a)(i) Show that for real numbers a and b, $a^2 + b^2 \geq 2ab$. **1**

(ii) Hence show that for real numbers a, b and c,

$a^2 + b^2 + c^2 \geq ab + bc + ca$ and $a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a + b + c)$. **3**

(iii) The equation $x^3 + px^2 + qx + r = 0$, where p, q and r are real, has three real roots a, b and c. Show that $q^2 \geq 3pr$. **3**

(b) A fair die is thrown 5 times. Find the probabilities that

(i) the 5 scores are all different numbers **2**

(ii) the 5 scores are consecutive numbers **2**

(iii) the 5 scores include exactly 3 different numbers, one of which occurs 3 times. **2**

(iv) the product of the 5 scores is an even number. **2**

Trial HSC Mathematics Extension 2 2005 : Marking Guidelines

Question 1

(a) Outcomes Assessed: H5, E6

Marking Guidelines

Criteria	Marks
i. • finds the derivative	1
• finds the coordinates of the stationary points	1
ii. • finds the gradient of the tangent	1
• states values of k	1
iii. (α) • correct shape showing coordinates of stationary points	1
(β) • correct shape, including turning point at the origin	1
• shows coordinates of other stationary points	1
(γ) • correct shape	1
• shows coordinates of stationary points	1

Answer

i. $y = xe^{-\frac{1}{2}x^2}$

$\frac{dy}{dx} = 1.e^{-\frac{1}{2}x^2} + x.e^{-\frac{1}{2}x^2}.(-x)$

$= (1 - x^2)e^{-\frac{1}{2}x^2}$

$\frac{dy}{dx} = 0 \Rightarrow x = \pm 1$

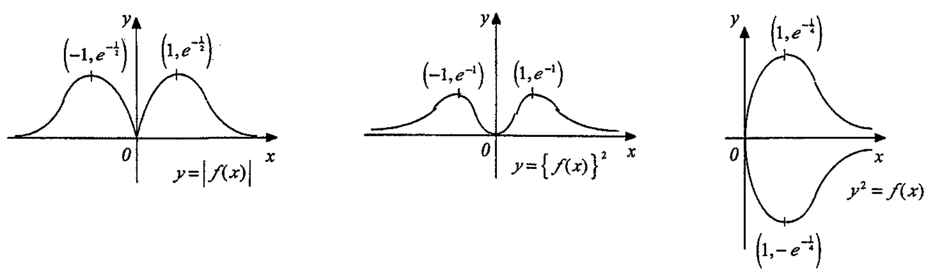
Stationary points are $A\left(-1, -e^{-\frac{1}{2}}\right)$, $B\left(1, e^{-\frac{1}{2}}\right)$.

ii. $x = 0 \Rightarrow \frac{dy}{dx} = 1$

Tangent at O has gradient 1.

For $f(x) = kx$ to have 3 real roots, the line $y = kx$ must cut the curve in three points. Hence $0 < k < 1$.

iii.



(b) Outcomes Assessed: HE4, E6

Marking Guidelines

Criteria	Marks
i. • differentiates both terms	1
• simplifies to show required result	1
ii. • deduces the value of the expression	1
iii. • shows f is odd	1
iv. • correct shape	1
• shows equations of asymptotes	1

Answers

i. $\frac{d}{dx}\left\{\tan^{-1}e^x + \tan^{-1}e^{-x}\right\} = \frac{e^x}{1+e^{2x}} + \frac{-e^{-x}}{1+e^{-2x}}$

$= \frac{e^x}{1+e^{2x}} - \frac{e^{-x}.e^{2x}}{e^{2x}+e^{-2x}.e^{2x}}$

$= \frac{e^x}{1+e^{2x}} - \frac{e^x}{e^{2x}+1}$

$= 0$

ii. Since the function is continuous,

$\tan^{-1}e^x + \tan^{-1}e^{-x} = c$, for some constant c.

But $\tan^{-1}e^0 + \tan^{-1}e^{-0} = \frac{\pi}{4} + \frac{\pi}{4}$.

$\therefore \tan^{-1}e^x + \tan^{-1}e^{-x} = \frac{\pi}{2}$.

iii. $f(x) = \tan^{-1}e^x - \frac{\pi}{4}$

$f(-x) = \tan^{-1}e^{-x} - \frac{\pi}{4}$

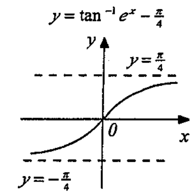
$= \left(\frac{\pi}{2} - \tan^{-1}e^x\right) - \frac{\pi}{4}$

$= -\left(\tan^{-1}e^{-x} - \frac{\pi}{4}\right)$

$= -f(x)$

Hence f is an odd function.

iv.



Question 2

(a) Outcomes Assessed: H8, E8

Marking Guidelines

Criteria	Marks
i. • simplifies integrand	1
• writes primitive	1
ii. • uses integration by parts	1
• writes primitive	1

Answers

i. $\int \frac{\sqrt{1-x^2}}{\sqrt{1-x}}\, dx = \int \sqrt{1+x}\, dx$

$= \frac{2}{3}(1+x)^{\frac{3}{2}} + c$

ii. $\int x\cos x\, dx = x\sin x - \int \sin x\, dx$

$= x\sin x + \cos x + c$

(b) Outcomes Assessed: E8

Marking Guidelines

Criteria	Marks
• expresses the integrand in partial fraction form	1
• finds the primitive	1
• evaluates in simplest exact form	1

Answer

$\frac{8-2x}{(1+x)(4+x^2)} \equiv \frac{a}{1+x} + \frac{bx+c}{4+x^2}$

$8 - 2x \equiv a(4+x^2) + (bx+c)(1+x)$

sub. $x = -1$: $10 = 5a \Rightarrow a = 2$

equate coeffs of x^2: $0 = a + b \Rightarrow b = -2$

sub. $x = 0$: $8 = 4a + c \Rightarrow c = 0$

$$\int_0^4 \frac{8-2x}{(1+x)(4+x^2)}\,dx = \int_0^4 \frac{2}{1+x}+\frac{-2x}{4+x^2}\,dx$$
$$= \left[2\ln(1+x)-\ln(4+x^2)\right]_0^4$$
$$= 2(\ln 5-\ln 1)-(\ln 20-\ln 4)$$
$$= \ln 5$$

(c) Outcomes Assessed: HE6, E8

Marking Guidelines

Criteria	Marks
• writes dx in terms of dt and converts limits	1
• finds integrand in terms of t	1
• evaluates definite integral between appropriate limits	1

Answer

$t=\tan\frac{x}{2}$

$dt=\frac{1}{2}\sec^2\frac{x}{2}\,dx$

$=\frac{1}{2}(1+t^2)\,dx$

$dx=\frac{2}{1+t^2}\,dt$

$x=0\Rightarrow t=0$

$x=\frac{\pi}{2}\Rightarrow t=1$

$5+3\cos x$

$=5+\frac{3(1-t^2)}{1+t^2}$

$=\frac{8+2t^2}{1+t^2}$

$$\int_0^{\frac{\pi}{2}}\frac{2}{5+3\cos x}\,dx=\int_0^1\frac{2(1+t^2)}{2(4+t^2)}\cdot\frac{2}{1+t^2}\,dt$$
$$=\int_0^1\frac{2}{4+t^2}\,dt$$
$$=\left[\tan^{-1}\tfrac{t}{2}\right]_0^1$$
$$=\tan^{-1}\tfrac{1}{2}$$
$$\approx 0\cdot 46 \quad (2\ \textit{sig. fig})$$

(d) Outcomes Assessed: HE6

Marking Guidelines

Criteria	Marks
i. • finds dx in terms of du and converts limits	1
• writes integrand in terms of $\tan u$	1
• reverses limits then replaces variable of integration u by x to complete proof	1
ii. • uses log law to rearrange integrand	1
• hence evaluates required integral	1

Answer

i. $u=\frac{\pi}{4}-x$

$du=-dx$

$x=0\Rightarrow u=\frac{\pi}{4}$

$x=\frac{\pi}{4}\Rightarrow u=0$

$$\int_0^{\frac{\pi}{4}}\ln(1+\tan x)\,dx=\int_{\frac{\pi}{4}}^0\ln\left\{1+\tan\left(\tfrac{\pi}{4}-u\right)\right\}.\ -du$$
$$=\int_0^{\frac{\pi}{4}}\ln\left\{1+\frac{1-\tan u}{1+\tan u}\right\}du$$
$$=\int_0^{\frac{\pi}{4}}\ln\left(\frac{2}{1+\tan u}\right)du$$
$$=\int_0^{\frac{\pi}{4}}\ln\left(\frac{2}{1+\tan x}\right)dx$$

ii. $$\int_0^{\frac{\pi}{4}}\ln(1+\tan x)\,dx=\int_0^{\frac{\pi}{4}}\left\{\ln 2-\ln(1+\tan x)\right\}dx$$
$$2\int_0^{\frac{\pi}{4}}\ln(1+\tan x)\,dx=\int_0^{\frac{\pi}{4}}\ln 2\,dx$$
$$=\tfrac{\pi}{4}\ln 2$$
$$\therefore \int_0^{\frac{\pi}{4}}\ln(1+\tan x)\,dx=\tfrac{\pi}{8}\ln 2$$

Question 3

(a) Outcomes Assessed: E3

Marking Guidelines

Criteria	Marks
• writes equation in terms of a and b	1
• equates real and imaginary parts to form simultaneous equations	1
• solves the simultaneous equations	1
• writes the solutions for z	1

Answer

Let $z=a+ib$, (a, b real). Then

$$z^2+2\bar{z}+6=0$$
$$a^2+2iab-b^2+2a-2ib+6=0$$
$$(a^2+2a-b^2+6)+2ib(a-1)=0$$
$$\left\{(a+1)^2+5-b^2\right\}+2ib(a-1)=0$$

Equating real and imaginary parts:

$$(a+1)^2+5-b^2=0 \quad (1)$$
$$b(a-1)=0 \quad (2)$$

From (2), $b=0$ *or* $a=1$

But from (1), there is no real a for which $b=0$.

$\therefore a=1$, and from (1) $b=\pm 3$.

Hence $z=1\pm 3i$.

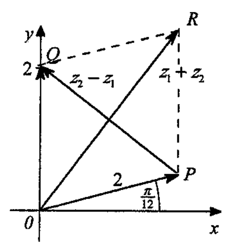
(b) Outcomes Assessed: E3

Marking Guidelines

Criteria	Marks
i • shows $\overrightarrow{OP}$, $\overrightarrow{OQ}$ with correct position and length	1
• shows z_1+z_2, z_2-z_1 with correct directions	1
ii. • finds value of $\arg(z_1+z_2)$	1
• finds value of $\arg(z_2-z_1)$	1

Answers

i.



ii. Complete the parallelogram $OPRQ$. Then the diagonals $\overrightarrow{OR}$, $\overrightarrow{PQ}$ represent z_1+z_2, z_2-z_1 respectively.

Now $|z_1|=|z_2|=2$. $\therefore OPRQ$ is a rhombus.

$\therefore \angle POR=\frac{1}{2}\left(\frac{\pi}{2}-\frac{\pi}{12}\right)=\frac{5\pi}{24}$ (*diagonal of rhombus bisects vertex angle*)

$\therefore \arg(z_1+z_2)=\frac{\pi}{12}+\frac{5\pi}{24}=\frac{7\pi}{24}$

Also $OR\perp PQ$ (*diagonals of rhombus meet at right angles*)

$\therefore \arg(z_2-z_1)=\frac{\pi}{2}+\arg(z_1+z_2)=\frac{19\pi}{24}$

(c) Outcomes Assessed: E3

Marking Guidelines

Criteria	Marks
i. • finds correct modulus	1
• finds correct argument	1
ii. • writes one value in required form	1
• writes second value in required form	1

Answers

i. $z=4\left(\frac{\sqrt{3}}{2}+(-\frac{1}{2})i\right)$

$z=4\left(\cos(-\frac{\pi}{6})+i\sin(-\frac{\pi}{6})\right)$

ii. $z^{\frac{1}{2}}=\pm 2\left(\cos(-\frac{\pi}{12})+i\sin(-\frac{\pi}{12})\right)$. The two values of $z^{\frac{1}{2}}$ are $2\left(\cos(-\frac{\pi}{12})+i\sin(-\frac{\pi}{12})\right)$ and $2\left(\cos\frac{11\pi}{12}+i\sin\frac{11\pi}{12}\right)$.

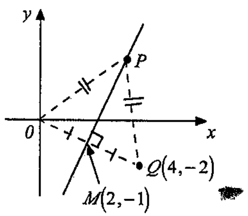
(d) Outcomes Assessed: E3

Marking Guidelines

Criteria	Marks
i. • deduces P lies on the perpendicular bisector of the join of $(0,0)$ and $(4,-2)$	1
• finds the equation of the locus	1
ii. • finds perpendicular distance from O to the locus of P	1

Answers

i.



$Q(4,-2)$ represents $(4-2i)$ in the Argand diagram.

Then $|z|=|z-(4-2i)| \Rightarrow PO=PQ$

Hence the locus of P is the perpendicular bisector of OQ.

But gradient OQ is $-\frac{1}{2}$ and midpoint of OQ is $M(2,-1)$.

Locus of P has equation $y+1=2(x-2)$

$2x-y-5=0$

ii. $\min|z|=OM=\sqrt{5}$

Question 4

(a) Outcomes Assessed: P4, E3, E4

Marking Guidelines

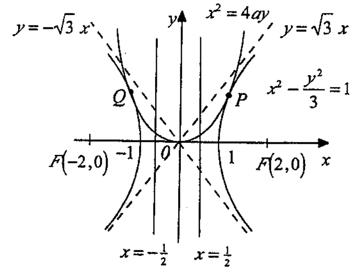
Criteria	Marks
i. • shows x intercepts	1
• finds value of e and shows coordinates of foci	1
• shows directrices with their equations	1
• shows equations of asymptotes	1
ii • solves simultaneously to get equation in y	1
• uses zero discriminant to find the value of a	1
• finds coordinates of P and Q	1

Answer

i.

$3=1(e^2-1)$

$e=2$



ii. At P, Q $x^2=4ay$ and $x^2-\frac{y^2}{3}=1$

$4ay-\frac{y^2}{3}=1$

$12ay-y^2=3$

$y^2-12ay+3=0$

By symmetry, the y coordinates of P and Q are equal. This quadratic equation in y must have equal real roots.

$\therefore \Delta=(-12a)^2-12=0 \quad \therefore a=\frac{1}{\sqrt{12}}=\frac{\sqrt{3}}{6}$

Then equation becomes $(y-\sqrt{3})^2=0 \quad \therefore y=\sqrt{3}$

Hence $P(\sqrt{2},\sqrt{3})$ and $Q(-\sqrt{2},\sqrt{3})$

(b) Outcomes Assessed: E3, E4

Marking Guidelines

Criteria	Marks
i. • finds the gradient of the tangent by differentiation	1
• finds the equation of the tangent	1
ii • solves simultaneous equations to find expression for one of x, y at R	1
• finds second coordinate of R	1
iii • uses P, Q, S collinear to write expression relating p, q	1
• simplifies relation to obtain required result	1
iv • attempts to eliminate p, q from equations for x, y at R	1
• completes elimination to find equation of locus of R	1

Answers

i. $x=ct \quad y=\frac{c}{t}$

$\frac{dx}{dt}=c \quad \frac{dy}{dt}=-\frac{c}{t^2}$

$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=-\frac{1}{t^2}$

Tangent at T has gradient $-\frac{1}{t^2}$ and equation

$y-\frac{c}{t}=-\frac{1}{t^2}(x-ct)$

$t^2y-ct=-x+ct$

$\therefore x+t^2y=2ct$

ii. At R, $x+p^2y=2cp \quad (1)$

$x+q^2y=2cq \quad (2)$

$(1)-(2)\Rightarrow (p^2-q^2)y=2c(p-q)$

$(p-q)(p+q)y=2c(p-q)$

$y=\frac{2c}{p+q}$

$q^2\times(1)-p^2\times(2)\Rightarrow (q^2-p^2)x=2cpq(q-p)$

$(q-p)(q+p)x=2cpq(q-p)$

$x=\frac{2cpq}{p+q}$

Hence R has coordinates $\left(\frac{2cpq}{p+q},\frac{2c}{p+q}\right)$

iii. $gradient\ PS = gradient\ QS$

$$\frac{\sqrt{2}-\frac{1}{p}}{\sqrt{2}-p}=\frac{\sqrt{2}-\frac{1}{q}}{\sqrt{2}-q}$$

$$\left(\sqrt{2}-\tfrac{1}{p}\right)\left(\sqrt{2}-q\right)=\left(\sqrt{2}-p\right)\left(\sqrt{2}-\tfrac{1}{q}\right)$$

$$2-\tfrac{1}{p}\sqrt{2}-q\sqrt{2}+\tfrac{q}{p}=2-p\sqrt{2}-\tfrac{1}{q}\sqrt{2}+\tfrac{p}{q}$$

$$\sqrt{2}\left(\tfrac{1}{q}-\tfrac{1}{p}\right)+\sqrt{2}\left(p-q\right)=\tfrac{p}{q}-\tfrac{q}{p}$$

$$(p-q)\sqrt{2}+pq(p-q)\sqrt{2}=p^2-q^2$$

$$(p-q)\sqrt{2}(1+pq)=(p-q)(p+q)$$

$$\sqrt{2}(1+pq)=p+q$$

iv. $$\frac{1+pq}{p+q}=\frac{1}{\sqrt{2}}$$

$$\frac{1}{p+q}+\frac{pq}{p+q}=\frac{1}{\sqrt{2}}$$

$$\frac{2c}{p+q}+\frac{2cpq}{p+q}=\frac{2c}{\sqrt{2}}$$

Hence the coordinates (x,y) of R satisfy $y+x=c\sqrt{2}$

(which is therefore the equation of the locus of R.

Question 5

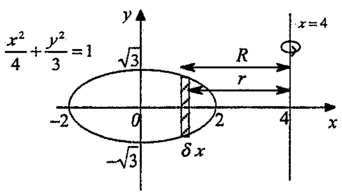
(a) Outcomes Assessed: E7

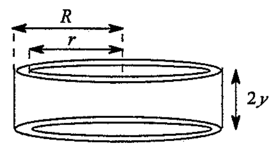
Marking Guidelines

Criteria	Marks
i • expresses volume of shell in terms of x and y	1
• finds shell volume in terms of x	1
• expresses V explicitly as limiting sum of shell volumes and hence as definite integral	1
ii • expresses V as a sum of two integrals and uses area of semicircle to evaluate one	1
• recognises second integrand is an odd function, giving value 0 for definite integral	1
• evaluates V	1

Answer

i.





$R=4-x$

$r=4-x-\delta x$

$y=\sqrt{3}\sqrt{1-\frac{x^2}{4}}$

$\;\;=\frac{\sqrt{3}}{2}\sqrt{4-x^2}$

Volume of shell is

$$\delta V=\pi\left(R^2-r^2\right).2y$$
$$=2\pi y(R+r)(R-r)$$
$$=2\pi y\{2(4-x)+\delta x\}\,\delta x$$

Ignoring second order terms:

$$\delta V=2\sqrt{3}\,\pi(4-x)\sqrt{4-x^2}\,\delta x$$
$$V=\lim_{\delta x\to 0}\sum_{x=-2}^{x=2}2\sqrt{3}\,\pi(4-x)\sqrt{4-x^2}\,\delta x$$
$$=2\sqrt{3}\,\pi\int_{-2}^{2}(4-x)\sqrt{4-x^2}\;dx$$

ii $$V=8\sqrt{3}\,\pi\int_{-2}^{2}\sqrt{4-x^2}\;dx-2\sqrt{3}\,\pi\int_{-2}^{2}x\sqrt{4-x^2}\;dx$$

$=8\sqrt{3}\,\pi\left(\tfrac{1}{2}\pi.2^2\right)-2\sqrt{3}\,\pi\,(0)$ *(first integral obtained from area of semi circle, radius 2)*

Volume is $16\sqrt{3}\,\pi^2\text{ cm}^3\approx 274\text{cm}^3$ *(second integral is 0 since integrand is an odd function)*

(b) Outcomes Assessed: H5, E3

Marking Guidelines

Criteria	Marks
i. • uses formula for $1-\cos 2A$ in terms of A	1
• uses formula for $\sin 2A$ in terms of A	1
ii • uses sum of a GP to find expression for sum of powers of z	1
iii • recognises sum of cosines is real part of sum of powers of z	1
• uses (i) to express RHS of (ii) in terms of $\theta, \frac{\theta}{2}, \frac{n\theta}{2}$	1
• simplifies this expression and finds real part	1
• equates real parts and simplifies to deduce required result	1
iv • identifies, tests θ in domain such that $\sin\frac{\theta}{2}=0$; lists other θ for which $\sin\frac{3\theta}{2}=0$	1
• lists θ in domain for which $\cos 2\theta=0$	1

Answer

i. $$1-(\cos n\theta+i\sin n\theta)=(1-\cos n\theta)-i\sin n\theta$$
$$=2\sin^2\tfrac{n\theta}{2}-i\left(2\sin\tfrac{n\theta}{2}\cos\tfrac{n\theta}{2}\right)$$
$$=-2i\sin\tfrac{n\theta}{2}\left(\cos\tfrac{n\theta}{2}+i\sin\tfrac{n\theta}{2}\right)$$

ii. $z+z^2+z^3+\ldots+z^n=\dfrac{z(1-z^n)}{1-z}$ for $z\neq 1$ (sum of a GP with common ratio z)

iii. $z^n=\cos n\theta+i\sin n\theta$. Hence $\mathrm{Re}\left(z+z^2+z^3+\ldots+z^n\right)=\cos\theta+\cos 2\theta+\cos 3\theta+\ldots+\cos n\theta$

Also using (i)

$$\frac{z(1-z^n)}{1-z}=\frac{(\cos\theta+i\sin\theta)\left\{-2i\sin\frac{n\theta}{2}\left(\cos\frac{n\theta}{2}+i\sin\frac{n\theta}{2}\right)\right\}}{-2i\sin\frac{\theta}{2}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)}$$
$$=\frac{\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right)\left\{\sin\frac{n\theta}{2}\left(\cos\frac{n\theta}{2}+i\sin\frac{n\theta}{2}\right)\right\}}{\sin\frac{\theta}{2}}$$

$$\therefore \mathrm{Re}\left\{\frac{z(1-z^n)}{1-z}\right\}=\frac{\sin\frac{n\theta}{2}\left(\cos\frac{\theta}{2}\cos\frac{n\theta}{2}-\sin\frac{\theta}{2}\sin\frac{n\theta}{2}\right)}{\sin\frac{\theta}{2}}=\frac{\sin\frac{n\theta}{2}\cos\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}$$

Hence equating real parts of equation in (ii):

$$\cos\theta+\cos 2\theta+\cos 3\theta+\ldots+\cos n\theta=\frac{\sin\frac{n\theta}{2}\cos\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}\quad\text{for }\sin\tfrac{\theta}{2}\neq 0\quad(z\neq 1)$$

iv. $\sin\frac{\theta}{2}=0$ for $\theta=0, 2\pi$, and these are not solutions of the equation.

For $\sin\frac{\theta}{2}\neq 0$, $\cos\theta+\cos 2\theta+\cos 3\theta=0\Rightarrow \sin\frac{3\theta}{2}\cos\frac{4\theta}{2}=0$

$\therefore \sin\frac{3\theta}{2}=0$ *or* $\cos 2\theta=0$, $0<\theta<2\pi$

$\frac{3\theta}{2}=\pi, 2\pi$ *or* $2\theta=\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2}$

$\theta=\frac{2\pi}{3},\frac{4\pi}{3},\frac{\pi}{4},\frac{3\pi}{4},\frac{5\pi}{4},\frac{7\pi}{4}$

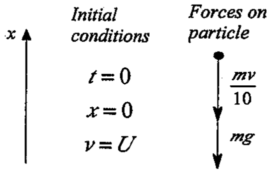
Question 6

(a) Outcomes Assessed: E5

Marking Guidelines

Criteria	Marks
i • uses Newton's second law to obtain expression for a	1
• writes a as $v\frac{dv}{dx}$ then finds expression for $\frac{dx}{dv}$ in terms of v	1
• finds primitive function	1
• uses initial conditions to evaluate constant of integration to give x as a function of v	1
ii award marks as for part (i)	4
iii • finds expression for greatest height H in terms of U	1
• uses distance fallen to express H in terms of V	1
• finds relation between U and V	1

Answer i. During ascent:



By Newton's Second Law

$$ma = -mg - \frac{mv}{10}$$

$$\therefore a = -10 - \frac{v}{10}$$

$$v\frac{dv}{dx} = -10 - \frac{v}{10}$$

$$\frac{dv}{dx} = -\frac{10}{v} - \frac{1}{10}$$

$$= -\frac{100+v}{10v}$$

$$\frac{dx}{dv} = -\frac{10v}{100+v}$$

$$= -10\left\{\frac{100+v-100}{100+v}\right\}$$

$$= -10 + \frac{1000}{100+v}$$

$$x = -10v + 1000\ln(100+v) + c$$

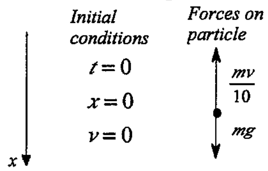
$$\left.\begin{array}{l} t=0 \\ x=0 \\ v=U \end{array}\right\} \Rightarrow \begin{array}{l} 0 = -10U + 1000\ln(100+U) + c \\ x = -10(v-U) + 1000\ln\left(\frac{100+v}{100+U}\right) \end{array}$$

$$\therefore x = 10(U-v) - 1000\ln\left(\frac{100+U}{100+v}\right)$$

iii. At greatest height H, $v = 0$. Using (i):

$$H = 10U - 1000\ln\left(\frac{100+U}{100}\right)$$

ii. During descent:



By Newton's Second Law

$$ma = mg - \frac{mv}{10}$$

$$\therefore a = 10 - \frac{v}{10}$$

$$v\frac{dv}{dx} = 10 - \frac{v}{10}$$

$$\frac{dv}{dx} = \frac{10}{v} - \frac{1}{10}$$

$$= \frac{100-v}{10v}$$

$$\frac{dx}{dv} = \frac{10v}{100-v}$$

$$= -10\left\{\frac{100-v-100}{100-v}\right\}$$

$$= -10 + \frac{1000}{100-v}$$

$$x = -10v - 1000\ln(100-v) + c$$

$$\left.\begin{array}{l} t=0 \\ x=0 \\ v=0 \end{array}\right\} \Rightarrow \begin{array}{l} 0 = 0 - 1000\ln(100) + c \\ x = -10v - 1000\ln\left(\frac{100-v}{100}\right) \end{array}$$

$$\therefore x = -10v + 1000\ln\left(\frac{100}{100-v}\right)$$

Hence on descent, $x = H$, $v = V$. Using (ii):

$$H = -10V + 1000\ln\left(\frac{100}{100-V}\right)$$

Hence

$$10U - 1000\ln\left(\frac{100+U}{100}\right) = -10V + 1000\ln\left(\frac{100}{100-V}\right)$$

$$10(U+V) = 1000\left\{\ln\left(\frac{100+U}{100}\right) + \ln\left(\frac{100}{100-V}\right)\right\}$$

$$U + V = 100\ln\left(\frac{100+U}{100-V}\right)$$

(b) Outcomes Assessed: E4

Marking Guidelines

Criteria	Marks
i • writes equation in $\frac{1}{x}$ with roots α^{-1}, β^{-1}, γ^{-1}	1
• rearranges as a monic cubic equation	1
ii • writes equation in $x^{\frac{1}{2}}$ with roots α^{-2}, β^{-2}, γ^{-2}	1
• rearranges as a monic cubic equation	1

Answer

i. $\alpha,\ \beta,\ \gamma$ satisfy $x^3 + 2x + 1 = 0$

$\therefore \alpha^{-1},\ \beta^{-1},\ \gamma^{-1}$ satisfy $\left(\frac{1}{x}\right)^3 + 2\left(\frac{1}{x}\right) + 1 = 0$

$1 + 2x^2 + x^3 = 0$

$\therefore x^3 + 2x^2 + 1 = 0$ has roots $\alpha^{-1},\ \beta^{-1},\ \gamma^{-1}$

ii. $\alpha^{-2},\ \beta^{-2},\ \gamma^{-2}$ satisfy $\left(x^{\frac{1}{2}}\right)^3 + 2\left(x^{\frac{1}{2}}\right)^2 + 1 = 0$

Then $x^{\frac{3}{2}} = -(2x+1)$ gives $x^3 = (2x+1)^2$.

Hence $x^3 - 4x^2 - 4x - 1 = 0$ has roots $\alpha^{-2},\ \beta^{-2},\ \gamma^{-2}$

Question 7

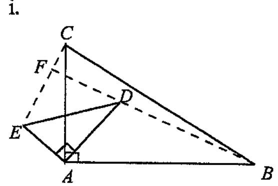
(a) Outcomes Assessed: PE2, PE3

Marking Guidelines

Criteria	Marks
ii • identifies equal angles *BAD, CAE* with reason	1
• states that including sides of specified triangles are in proportion	1
• justifies this using properties of given similar triangles *ABC, ADE.*	1
iii • deduces equality of angles *ADB, AEC* giving a reason	1
• applies test for cyclic quadrilateral	1
iv • explains why angle *EFD* is supplementary with angle *DAE*	1
• deduces that angle *EFD* is a right angle	1

Answer

i.



ii.

$\Delta ABC \parallel\!| \Delta ADE$

$\therefore \frac{AC}{AE} = \frac{AB}{AD}$ (*corresp. sides of similar* Δs *are in proportion*) **

In ΔBDA, ΔCEA

$\frac{AB}{AC} = \frac{AD}{AE}$ (*rearranging* **)

$\angle BAD = \angle CAE$ (*both complementary with* $\angle CAD$)

$\therefore \Delta BDA \parallel\!| \Delta CEA$ (*one pair of equal angles and including sides in proportion*)

iii. $\angle ADB = \angle AEC$ (*corresp.* $\angle s$ *of similar* Δs *are equal*)

$\therefore$ *ADFE* is a cyclic quadrilateral (*one exterior* $\angle$ *equal to interior opp.* $\angle$)

iv. $\therefore \angle DFE = 180° - \angle DAE$ (*opp.* $\angle s$ *of cyclic quad. are supplementary*)

$\therefore \angle DFE = 90°$ and hence $BF \perp CE$.

(b) Outcomes Assessed: P4, HE2

Marking Guidelines

Criteria	Marks
i • understands induction process and uses appropriate sequence of statements	1
• shows first statement is true	1
• uses recurrence formula to establish that if $S(k)$ is true, then $S(k+1)$ is true	1
ii • uses recurrence relation and result from (i) to show $T_{n+1} > \sqrt{3T_n}$	1
• uses result from (i) again to deduce required inequality.	1
iii • uses difference of squares expansion or factorisation to establish first result	1
• uses recurrence formula to show numerator is $2(T_{n+1}-T_n)$	1
• uses $T_n > 1$ for $n = 2, 3, 4, \ldots$ to deduce required inequality	1

Answer

i. Let $S(n)$, $n = 1, 2, 3, \ldots$ be the sequence of statements $T_n < 3$.

Consider $S(1)$: $T_1 = 1 < 3$ Hence $S(1)$ is true.

If $S(k)$ *is true*: $T_k < 3$

Consider $S(k+1)$: $T_{k+1} = \sqrt{3+2T_k} < \sqrt{3+6} = 3$ if $S(k)$ is true.

Hence $S(k+1)$ is true if $S(k)$ is true. But $S(1)$ is true, hence $S(2)$ is true, and then $S(3)$ is true and so on. Hence by Mathematical Induction, $S(n)$ is true for all positive integers $n \geq 1$.

ii. $T_{n+1} = \sqrt{3+2T_n}$

$> \sqrt{T_n + 2T_n}$ (since $3 > T_n$)

$= \sqrt{3T_n}$

$> \sqrt{T_n \cdot T_n}$ (since $3 > T_n$)

$= T_n$

$\therefore T_{n+1} > T_n$

iii. $(T_{n+2} - T_{n+1})(T_{n+2} + T_{n+1}) = T_{n+2}^{\ 2} - T_{n+1}^{\ 2}$

$$\therefore T_{n+2} - T_{n+1} = \frac{T_{n+2}^{\ 2} - T_{n+1}^{\ 2}}{T_{n+2} + T_{n+1}} = \frac{(3+2T_{n+1}) - (3+2T_n)}{T_{n+2} + T_{n+1}} = \frac{2(T_{n+1} - T_n)}{T_{n+2} + T_{n+1}}$$

Clearly $T_{n+2} > T_{n+1} > T_n > \ldots > T_1 = 1$ from (ii)

Hence for $n = 1, 2, 3, \ldots$, $T_{n+2} > 1$, $T_{n+1} > 1$ and

$T_{n+2} + T_{n+1} > 2$, giving $\dfrac{2}{T_{n+2} + T_{n+1}} < 1$.

$\therefore T_{n+2} - T_{n+1} < T_{n+1} - T_n$ for $n = 1, 2, 3, \ldots$

Question 8

(a) Outcomes Assessed: PE2, PE3

Marking Guidelines

Criteria	Marks
i. • expresses $a^2 + b^2$ as a square added to $2ab$ to produce result	1
ii • applies result to each pair chosen from a, b, c then adds to show first inequality	1
• makes appropriate replacements for a, b, c and substitutes	1
• simplifies to obtain second inequality	1
iii • expands expression for q^2 in terms of a, b, c.	1
• expresses pr in terms of a, b, c.	1
• uses second inequality from (ii) to show required result	1

Answer

i. $a^2 + b^2 = (a-b)^2 + 2ab$

But $(a-b)^2 \geq 0$. $\therefore a^2 + b^2 \geq 2ab$

ii. $(a^2+b^2) + (b^2+c^2) + (c^2+a^2) \geq 2(ab+bc+ca)$

$2(a^2+b^2+c^2) \geq 2(ab+bc+ca)$

$a^2+b^2+c^2 \geq ab+bc+ca$

Then by making the replacements

$a \to ab,\ b \to bc,\ c \to ca$

$(ab)^2 + (bc)^2 + (ca)^2 \geq (ab)(bc) + (bc)(ca) + (ca)(ab)$

$a^2b^2 + b^2c^2 + c^2a^2 \geq abc(a+b+c)$

iii. $q^2 = (ab+bc+ca)^2$

$= a^2b^2 + b^2c^2 + c^2a^2 + 2\{(ab)(bc) + (bc)(ca) + (ca)(ab)\}$

$= (a^2b^2 + b^2c^2 + c^2a^2) + 2abc(a+b+c)$

$\geq abc(a+b+c) + 2abc(a+b+c)$

$= 3abc(a+b+c)$

But $p = -(a+b+c)$ and $r = -abc$.

$\therefore abc(a+b+c) = pr$, and then $q^2 \geq 3pr$.

(b) Outcomes Assessed: H5, PE3

Marking Guidelines

Criteria	Marks
i • realises there are ${}^6P_5$ such ordered sets of scores, 6^5 outcomes without restriction	1
• calculates probability	1
ii • realises there are 2 sets of 5 consecutive numbers, each with 5! orders	1
• calculates probability	1
iii • finds that there are 60 such sets of scores	1
• realises that there are 20 arrangements of each such set and calculates the probability	1
iv • realises that at least one score must be an even number	1
• calculates this probability	1

Answer

i. $\dfrac{{}^6P_5}{6^5} = \dfrac{5}{54}$

ii. $\dfrac{2 \times 5!}{6^5} = \dfrac{5}{162}$

iii. $\dfrac{6 \times {}^5C_2 \times \left(\dfrac{5!}{3!}\right)}{6^5} = \dfrac{60 \times 20}{6^5} = \dfrac{25}{162}$

iv. $1 - P(\textit{all the scores are odd}) = 1 - \dfrac{3^5}{6^5} = \dfrac{31}{32}$