INTEGRATION

REWRITE INTEGRAND

eg Divide and use PARTIAL FRACTIONS	eg Use TRIG IDENTITIES

SUBSTITUTION

$t = f(x)$ $dx = \dfrac{dt}{f'(x)}$		$x = f(t)$ $dx = f'(t)dt$	Change limits for definite integrals
e.g. $t = \tan x/2$ $dx = \dfrac{2dt}{1+t^2}$	e.g. $t =$ inner bit	e.g. $x = a\sin\theta$ etc	

SUM

$$\int u+v = \int u + \int v$$

I_0 (or I_1)

REDUCTION FORMULA

$$I_n = f(I_{n-1})$$

PARTS / Priority for u

$u =$	$dv =$	$\log x$ inverse trig
$du =$	$v =$	x^n etc
$\int u \, dv = uv - \int v \, du$		
easier integral	essentially same integral	e^x $\cos x$ $\sin x$

STANDARD INTEGRALS

$x^n \ (n \neq -1)$	$\dfrac{1}{n+1} x^{n+1}$
$\dfrac{1}{x}$	$\log x$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\tan x$	$\log \sec x$
$\sec^2 x$	$\tan x$
e^x	e^x
$\log x$	$x \log x - x$
$\dfrac{1}{x^2 + a^2}$	$\dfrac{1}{a} \tan^{-1}\left(\dfrac{x}{a}\right)$
$\dfrac{1}{x^2 - a^2}$	$\dfrac{1}{2a} \log\left(\dfrac{x-a}{x+a}\right)$
$\dfrac{1}{\sqrt{x^2 \pm a^2}}$	$\log\left(x + \sqrt{x^2 \pm a^2}\right)$
$\dfrac{1}{\sqrt{a^2 - x^2}}$	$\sin^{-1}\left(\dfrac{x}{a}\right)$

$+ c$ for indefinite integral

AREAS & VOLUMES

AREAS

$$\int_a^b y\, dx$$

y

a given b given

$$\int_a^c y\, dx - \int_c^b y\, dx$$

calculate c

y

a given

b given
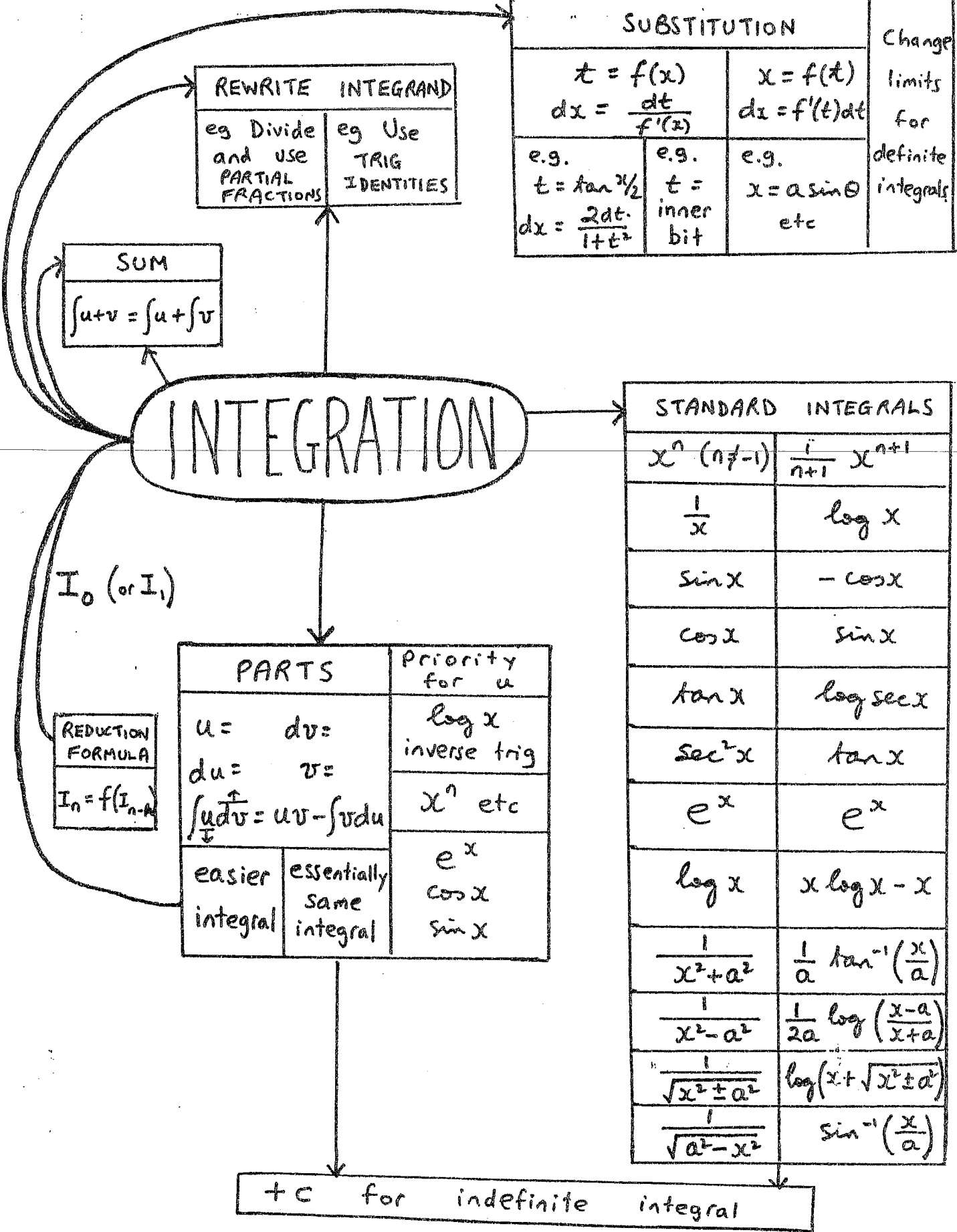
N.B. areas below x-axis correspond to negative integrals
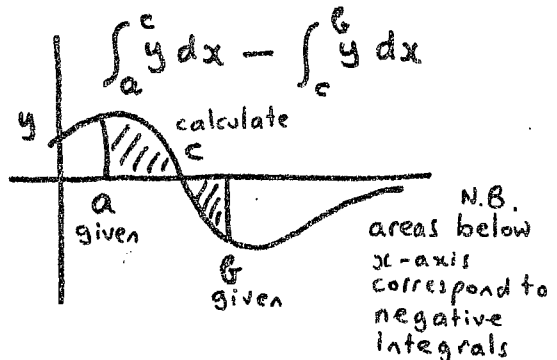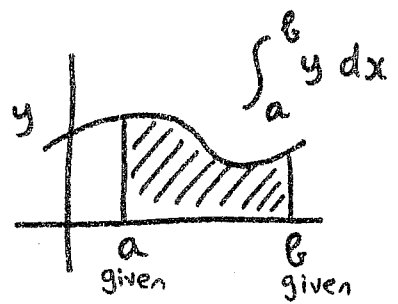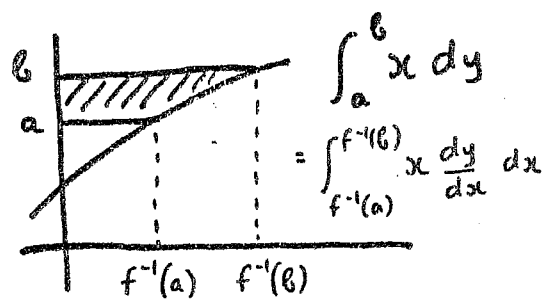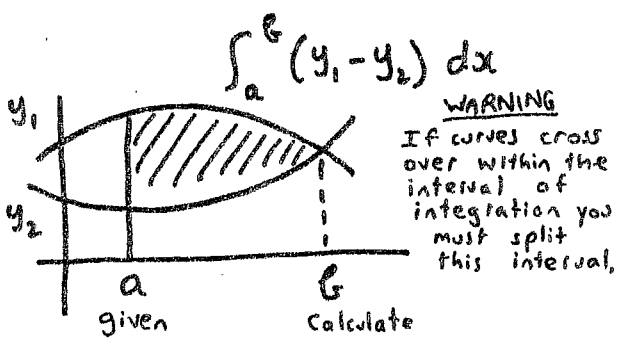
DO NOT TAKE ABSOLUTE VALUES AT THE END. You should know _in advance_ whether your integral will be $+$ or $-$.

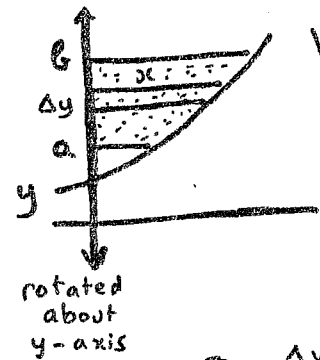$$\int_a^b (y_1 - y_2)\, dx$$

WARNING
If curves cross over within the interval of integration you must split this interval.

y_1

y_2

a given b calculate

$$\int_a^b x\, dy$$

b

a

$$= \int_{f^{-1}(a)}^{f^{-1}(b)} x\, \frac{dy}{dx}\, dx$$

$f^{-1}(a)$ $f^{-1}(b)$

VOLUMES

$\Delta V = \pi y^2 \Delta x$

$$V = \pi \int_a^b y^2\, dx$$

y

a Δx b

rotated about x-axis

$\Delta V = \pi x^2 \Delta y$

$$V = \pi \int_a^b x^2\, dy$$

b

Δy

a

y

rotated about y-axis

$\Delta V = (\pi y_1^2 - \pi y_2^2) \Delta x$

$$V = \pi \int_a^b (y_1^2 - y_2^2)\, dx$$

y_1

y_2

$y_1 - y_2$

a Δx b

rotated about x-axis

$\Delta V = \pi (y - c)^2 \Delta x$

$$V = \pi \int_a^b (y - c)^2\, dx$$

y

$y - c$

c

a Δx b

rotated about $y = c$
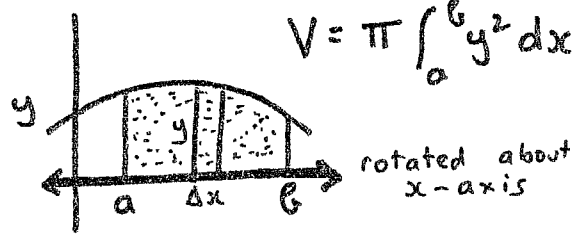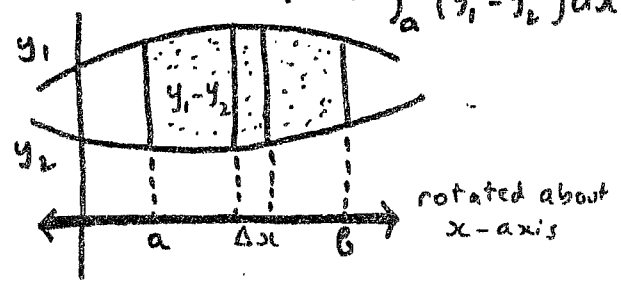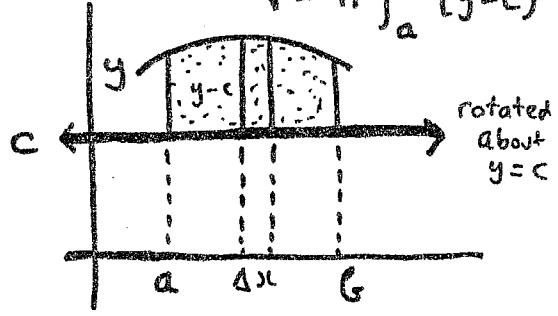
	INTEGRAL	METHOD	ANSWER (+c)
1	$\int \dfrac{\sqrt{x}\,dx}{x^2-1}$		$\dfrac{1}{2}\log_e\left(\dfrac{1-\sqrt{x}}{1+\sqrt{x}}\right)+\tan^{-1}\sqrt{x}$
2	$\int x^2\sin^{-1}x\,dx$		$\dfrac{1}{3}\left(x^3\sin^{-1}x - \int\dfrac{x^3\,dx}{\sqrt{1-x^2}}\right)$
3	$\int \dfrac{x^3\,dx}{\sqrt{1-x^2}}$		$-\dfrac{\sqrt{1-x^2}}{3}\,(x^2+2)$
4	$\int \dfrac{\sin x\,dx}{4+\cos^2 x}$		$-\dfrac{1}{2}\,\tan^{-1}\left(\dfrac{\cos x}{2}\right)$
5	$\int \dfrac{dx}{1-\sin x+\cos x}$		$-\log_e\left[1-\tan\left(\dfrac{x}{2}\right)\right]$
6	$\int \dfrac{x\,dx}{16-x^4}$		$\dfrac{1}{16}\log_e\left(\dfrac{4+x^2}{4-x^2}\right)$
7	$\int \dfrac{dx}{x^2+8x+20}$		$\dfrac{1}{2}\tan^{-1}\left(\dfrac{x+4}{2}\right)$
8	$\int \sin\sqrt{x}\,dx$		$2\sin\sqrt{x}-2\sqrt{x}\cos\sqrt{x}$
9	$\int x\tan^2 x\,dx$		$x\tan x+\log_e\cos x-\dfrac{1}{2}x^2$
10	$\int \dfrac{dx}{e^x+1}$		$x-\log_e(e^x+1)$

	INTEGRAL	METHOD	ANSWER $\quad$ (+c)
11	$\int \dfrac{dx}{\sqrt{6x-x^2}}$		$\sin^{-1}\left(\dfrac{x-3}{3}\right)$
12	$\int x^3 \cos(x^2)\ dx$		$\dfrac{1}{2}\left[x^2 \sin(x^2) + \cos(x^2)\right]$
13	$\int \dfrac{x^2\ dx}{9-x^2}$		$-\sin x + \dfrac{1}{2}\log_e\left(\dfrac{1+\sin x}{1-\sin x}\right)$
14	$\int \left(\dfrac{x+1}{x^2}\right)e^{-x}\ dx$		$-\dfrac{e^{-x}}{x}$
15	$\int x^2 \sin x\ dx$		$2x \sin x + (2-x^2)\cos x$
16	$\int x \sin^2 x\ dx$		$\dfrac{x^2}{4} - \dfrac{x \sin 2x}{4} - \dfrac{\cos 2x}{8}$
17	$\int \dfrac{90\ x^3\ dx}{(1+x^3)^7}$		$-\dfrac{6x^3+1}{(1+x^3)^6}$
18	$\int \dfrac{x\ dx}{x+2}$		$x - 2\log_e(x+2)$
19	$\int \dfrac{dx}{x(x+1)}$		$\log_e\left(\dfrac{x}{x+1}\right)$
20	$\int \dfrac{x\ dx}{\sqrt{x+2}}$		$\dfrac{2}{3}(x-4)\sqrt{x+2}$

	INTEGRAL	METHOD	ANSWER $(+c)$
21	$\int \dfrac{dx}{x\sqrt{x+4}}$		$\dfrac{1}{2}\log_e\left(\dfrac{\sqrt{x+4}-2}{\sqrt{x+4}+2}\right)$
22	$\int \dfrac{dx}{x^2\sqrt{x+4}}$		$-\dfrac{\sqrt{x+4}}{4x}-\dfrac{1}{8}\int\dfrac{dx}{x\sqrt{x+4}}$
23	$\int x\sqrt{x+3}\;dx$		$\dfrac{2}{5}(x-2)(x+3)^{\frac{3}{2}}$
24	$\int \dfrac{\sqrt{x+4}}{x}dx$		$2\sqrt{x+4}+4\int\dfrac{dx}{x\sqrt{x+4}}$
25	$\int \dfrac{\sqrt{x+4}}{x^2}dx$		$-\dfrac{\sqrt{x+4}}{x}+\dfrac{1}{2}\int\dfrac{dx}{x\sqrt{x+4}}$
26	$\int \dfrac{dx}{\sqrt{x(1-x)}}$		$2\tan^{-1}\sqrt{\dfrac{x}{1-x}}$
27	$\int \dfrac{x^3}{x^2+4}\;dx$		$\dfrac{1}{2}x^2-2\log_e(x^2+4)$
28	$\int \dfrac{dx}{x(x^2+4)}$		$\dfrac{1}{8}\log_e\left(\dfrac{x^2}{x^2+4}\right)$
29	$\int \dfrac{dx}{x^2(x^2+4)}$		$-\dfrac{1}{4x}-\dfrac{1}{8}\tan^{-1}\left(\dfrac{x}{2}\right)$
30	$\int \dfrac{dx}{(x^2+4)^2}$		$\dfrac{x}{8(x^2+4)}+\dfrac{1}{16}\tan^{-1}\left(\dfrac{x}{2}\right)$

	INTEGRAL	METHOD	ANSWER $(+c)$
31	$\displaystyle\int \frac{x^3\ dx}{1-x^2}$		$-\dfrac{x^2}{2} - \dfrac{1}{2}\log_e(1-x^2)$
32	$\displaystyle\int \frac{dx}{x^2(1-x^2)}$		$-\dfrac{1}{x} + \dfrac{1}{2}\log_e\left(\dfrac{1+x}{1-x}\right)$
33	$\displaystyle\int \frac{dx}{(x^2+1)^{3/2}}$		$\dfrac{x}{\sqrt{x^2+1}}$
34	$\displaystyle\int \frac{dx}{x(x^2+1)^{3/2}}$		$\dfrac{1}{\sqrt{x^2+1}} - \log_e\left(\dfrac{1+\sqrt{x^2+1}}{x}\right)$
35	$\displaystyle\int \frac{dx}{x\sqrt{x^2-4}}$		$\dfrac{1}{2}\cos^{-1}\left(\dfrac{2}{x}\right)$
36	$\displaystyle\int \frac{dx}{x\sqrt{4-x^2}}$		$-\dfrac{1}{2}\log_e\left(\dfrac{2+\sqrt{4-x^2}}{x}\right)$
37	$\displaystyle\int \frac{x\ dx}{x^4+1}$		$\dfrac{1}{2}\tan^{-1}(x^2)$
38	$\displaystyle\int \frac{x^3\ dx}{x^4+1}$		$\dfrac{1}{4}\log_e(x^4+1)$
39	$\displaystyle\int \frac{dx}{x(1-x^6)}$		$-\dfrac{1}{6}\log_e\left(\dfrac{1-x^6}{x^6}\right)$
40	$\displaystyle\int \frac{dx}{x\sqrt{x^6-1}}$		$\dfrac{1}{3}\cos^{-1}\left(\dfrac{1}{x^3}\right)$

	INTEGRAL	METHOD	ANSWER (+c)
41	$\int \dfrac{x^2\ dx}{(1-x^2)^{3/2}}$		$\dfrac{x}{\sqrt{1-x^2}} - \sin^{-1}x$
42	$\int \dfrac{dx}{\sin 2x}$		$\dfrac{1}{2}\log_e \tan x$
43	$\int \sin^2 x\ dx$		$\dfrac{x}{2} - \dfrac{1}{4}\sin 2x$
44	$\int \dfrac{dx}{\sin^2 x}$		$-\cot x$
45	$\int \dfrac{dx}{1+\sin x}$		$-\dfrac{2}{1+\tan(x/2)}$
46	$\int \dfrac{dx}{1+\cos 2x}$		$\dfrac{1}{2}\tan x$
47	$\int \dfrac{x\ dx}{1+\cos x}$		$x\tan\left(\dfrac{x}{2}\right)+2\log_e\cos\left(\dfrac{x}{2}\right)$
48	$\int \dfrac{dx}{25-16\cos^2 x}$		$\dfrac{1}{15}\tan^{-1}\left(\dfrac{5}{3}\tan x\right)$
49	$\int \dfrac{dx}{3\sin x+4\cos x}$		$\dfrac{1}{5}\log_e \tan\left(\dfrac{x+\tan^{-1}\left(\dfrac{4}{3}\right)}{2}\right)$
50	$\int \dfrac{dx}{\sin x+\cos x+2}$		$\sqrt{2}\tan^{-1}\left(\dfrac{1+\tan(x/2)}{\sqrt{2}}\right)$

	INTEGRAL	METHOD	ANSWER (+c)
51	$\int \dfrac{dx}{4\sin^2 x - \cos^2 x}$		$\dfrac{1}{4}\log_e\left(\dfrac{2\tan x - 1}{2\tan x + 1}\right)$
52	$\int \tan^3 x\ dx$		$\frac{1}{2}\tan^2 x + \log_e \cos x$
53	$\int x\cot^2 x\ dx$		$-x\cot x + \log_e \sin x - \frac{1}{2}x^2$
54	$\int \dfrac{dx}{1 + \cot x}$		$\frac{1}{2}x - \frac{1}{2}\log_e(\sin x + \cos x)$
55	$\int x\sec^2 x\ dx$		$x\tan x + \log_e \cos x$
56	$\int (\sin^{-1} x)^2\ dx$		$x(\sin^{-1} x)^2 - 2x$ $+\ 2\sqrt{1-x^2}\cdot\sin^{-1} x$
57	$\int x\cot^{-1} x\ dx$		$\frac{1}{2}(x^2+1)\cot^{-1} x + \frac{1}{2}x$
58	$\int x^2 \sin^{-1} x\ dx$		$\dfrac{x^3}{3}\sin^{-1} x + \dfrac{x^2+2}{9}\sqrt{1-x^2}$
59	$\int \dfrac{dx}{e^{2x} + 1}$		$x - \frac{1}{2}\log_e(e^{2x} + 1)$
60	$\int \dfrac{dx}{4e^x + e^{-x}}$		$\frac{1}{2}\tan^{-1}(2e^x)$

	INTEGRAL	METHOD	ANSWER (+c)
61	$\displaystyle\int \frac{dx}{x\log_e x}$		$\log_e \log_e x$
62	$\displaystyle\int x\log_e x \, dx$		$\frac{1}{2}x^2(\log_e x - \frac{1}{2})$
63	$\displaystyle\int \frac{\log_e x}{x}\, dx$		$\frac{1}{2}(\log_e x)^2$
64	$\displaystyle\int \frac{\log_e x}{x^2}\, dx$		$-\dfrac{\log_e x}{x} - \dfrac{1}{x}$
65	$\displaystyle\int \frac{dx}{\sin x + \cos x}$		$\dfrac{1}{\sqrt{2}}\log_e\left(\dfrac{\sqrt{2}-1+\tan(\frac{1}{2}x)}{\sqrt{2}+1-\tan(\frac{1}{2}x)}\right)$
66	$\displaystyle\int \sin x . \log_e \sin x \, dx$		$\cot x \sin x - \cos x . \log_e \sin x$ $+ \log_e \tan(\frac{1}{2}x)$
67	$\displaystyle\int_0^{\pi/2} \frac{\sin x \, dx}{\sin x + \cos x}$		$\dfrac{\pi}{4}$
68	$\displaystyle\int_0^{\frac{\pi}{4}} \tan^6 x \, dx$		$\dfrac{13}{15} - \dfrac{\pi}{4}$
69	$\displaystyle\int_0^1 \frac{dx}{x^3 + 1}$		$\dfrac{1}{3}\log_e 2 + \dfrac{\pi}{3\sqrt{3}}$
70	$\displaystyle\int_1^e x^2 (\log_e x)^2 \, dx$		$\dfrac{5e^3 - 2}{27}$