INTEGRATION - THE MEGA ASSIGNMENT

1) Evaluate $\displaystyle\int_0^5 \frac{2}{\sqrt{x+4}}\,dx$.¤

2) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos^4\theta}\,d\theta$.¤

3) Find $\displaystyle\int \frac{1}{x^2+2x+3}\,dx$.¤

4) Find $\displaystyle\int \frac{4t-6}{(t+1)(2t^2+3)}\,dt$.¤

5) Evaluate $\displaystyle\int_0^{\frac{\pi}{3}} x\sec^2 x\,dx$.¤

6) Evaluate $\displaystyle\int_1^3 \frac{4}{(2+x)^2}\,dx$.¤

7) Find $\displaystyle\int \sec^2\theta\tan\theta\,d\theta$.¤

8) Find $\displaystyle\int \frac{5t^2+3}{t(t^2+1)}\,dt$.¤

9) Using integration by parts, or otherwise, find $\displaystyle\int x\tan^{-1}x\,dx$.¤

10) Using the substitution $x = 2\sin\theta$, or otherwise, calculate $\displaystyle\int_{-1}^{\sqrt{3}} \frac{x^2}{\sqrt{4-x^2}}\,dx$.¤

11) i. Show that $\displaystyle\int_0^{\frac{\pi}{2}} (\sin x)^{2k}\cos x\,dx = \frac{1}{2k+1}$, where k is a positive integer.

ii. By writing $(\cos x)^{2n} = (1-\sin^2 x)^n$, show that $\displaystyle\int_0^{\frac{\pi}{2}} (\cos x)^{2n+1}\,dx = \sum_{k=0}^{n} \frac{(-1)^k}{2k+1}\binom{n}{k}$.

iii. Hence, or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \cos^5 x\,dx$.¤

12) i. Let $f(x) = \text{Ln}x - ax + b$, for $x > 0$, where a and b are real numbers and $a > 0$. Show that $y = f(x)$ has a single turning point which is a maximum.

ii. The graphs of $y = \text{Ln}x$ and $y = ax - b$ intersect at points A and B. Using the result of part (i), or otherwise, show that the chord AB lies below the curve $y = \text{Ln}x$.

iii. Using integration by parts, or otherwise, show that $\displaystyle\int_1^k \text{Ln}x\,dx = k\,\text{Ln}k - k + 1$.

iv. Use the trapezoidal rule on the intervals with integer endpoints 1, 2, 3,..., k to show that $\displaystyle\int_1^k \text{Ln}x\,dx$ is approximately equal to $\frac{1}{2}\text{Ln}k + \text{Ln}[(k-1)!]$.

v. Hence deduce that $k! < e\sqrt{k}\left(\frac{k}{e}\right)^k$.¤

13) Find $\displaystyle\int \frac{dx}{x(\ln x)^2}$.¤

answer

14) Find $\int x e^x dx$.¤

15) Show that $\int_1^4 \frac{6t + 23}{(2t - 1)(t + 6)} dt = \ln 70$.¤

16) Find $\frac{d}{dx}(x \sin^{-1}x)$, and hence find $\int \sin^{-1}x \, dx$.¤

17) Using the substitution $t = \tan\frac{x}{2}$, or otherwise, calculate $\int_0^{\frac{\pi}{2}} \frac{dx}{5 + 3\sin x + 4\cos x}$.¤

18) i. Show that, if $0 < x < \frac{\pi}{2}$, then $\frac{\sin(2m + 1)x}{\sin x} - \frac{\sin(2m - 1)x}{\sin x} = 2\cos(2mx)$.

ii. Show that, for any positive integer m, $\int_0^{\frac{\pi}{2}} \cos(2mx)dx = 0$.

iii. Deduce that, if m is any positive integer, $\int_0^{\frac{\pi}{2}} \frac{\sin(2m + 1)x}{\sin x} dx = \int_0^{\frac{\pi}{2}} \frac{\sin(2m - 1)x}{\sin x} dx$.

iv. Show that, if m = 1 then, $\int_0^{\frac{\pi}{2}} \frac{\sin(2m - 1)x}{\sin x} dx = \frac{\pi}{2}$.

v. Hence show that $\int_0^{\frac{\pi}{2}} \frac{\sin 5x}{\sin x} dx = \frac{\pi}{2}$.¤

19) Let $I_n = \int_0^{\frac{\pi}{2}} (\sin x)^n dx$, where n is an integer, $n \geq 0$.

i. Using integration by parts, show that, for $n \geq 2$, $I_n = \left(\frac{n-1}{n}\right) I_{n-2}$

ii. Deduce that $I_{2n} = \frac{2n - 1}{2n} \cdot \frac{2n - 3}{2n - 2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$ and $I_{2n+1} = \frac{2n}{2n + 1} \cdot \frac{2n - 2}{2n - 1} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1$.

iii. Explain why $I_k > I_{k+1}$.

iv. Hence, using the fact that $I_{2n-1} > I_{2n}$ and $I_{2n} > I_{2n+1}$, show that

$$\frac{\pi}{2}\left(\frac{2n}{2n+1}\right) < \frac{2^2.4^2...(2n)^2}{1.3^2.5^2...(2n-1)^2(2n+1)} < \frac{\pi}{2}$$.¤

20) Find

i. $\int \frac{4x - 12}{x^2 - 6x + 13} dx$

ii. $\int \frac{1}{x^2 - 6x + 13} dx$.¤

21) Evaluate $\int_{\frac{3}{2}}^{3} \sqrt{9 - u^2} du$.¤

22) Evaluate $\int_1^2 \frac{11 - 2t}{(2t - 1)(3 - t)} dt$.¤

23) Evaluate $\int_0^{\pi} e^{2x} \sin x \, dx$.¤

24) i. Given that $\sin x > \frac{2x}{\pi}$ for $0 < x < \frac{\pi}{2}$, explain why $\int_0^{\frac{\pi}{2}} e^{-\sin x} dx < \int_0^{\frac{\pi}{2}} e^{\frac{-2x}{\pi}} dx$.

ii. Show that $\int_{\frac{\pi}{2}}^{\pi} e^{-\sin x} dx = \int_0^{\frac{\pi}{2}} e^{-\sin x} dx$.

iii. Hence show that $\int_0^{\pi} e^{-\sin x} dx < \frac{\pi}{e}(e - 1)$.¤

25) Evaluate $\int_3^8 \frac{x}{(x + 1) - \sqrt{x + 1}} dx$ by using the substitution $x + 1 = u^2$.¤

26) Evaluate $\int_0^{\frac{\pi}{2}} \frac{d\theta}{2+\cos\theta}$ by using the substitution $t = \tan\frac{\theta}{2}$.¤

27) Evaluate $\int_0^{\frac{1}{2}} \sin^{-1}x \, dx$.¤

28) i. Find real numbers a, b and c such that $\frac{4x+3}{(x^2+1)(x+2)} = \frac{ax+b}{x^2+1} + \frac{c}{x+2}$.

ii. Hence find $\int \frac{4x+3}{(x^2+1)(x+2)} dx$.¤

29) Suppose k is a constant greater than 1. Let $f(x) = \frac{1}{1+(\tan x)^k}$ where $0 \le x \le \frac{\pi}{2}$. [You may assume $f\left(\frac{\pi}{2}\right) = 0$.]

i. Show that $f(x) + f\left(\frac{\pi}{2} - x\right) = 1$ for $0 \le x \le \frac{\pi}{2}$.

ii. Sketch $y = f(x)$ for $0 \le x \le \frac{\pi}{2}$.
[There is no need to find $f'(x)$ but assume $y = f(x)$ has a horizontal tangent at $x = 0$. Your graph should exhibit the property of (b)(i).]

iii. Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+(\tan x)^k}$.¤

30) Find:

i. $\int \tan\theta \sec^2\theta \, d\theta$.

ii. $\int \frac{2x+6}{x^2+6x+1} dx$.¤

31) Evaluate $\int_{\frac{3}{2}}^{\frac{5}{2}} \frac{dx}{\sqrt{(x-1)(3-x)}}$ by using the substitution $u = x - 2$.¤

32) Evaluate $\int_0^1 \frac{5(1-t)}{(t+1)(3-2t)} dt$.¤

33) i. Find $\int xe^{x^2} dx$.

ii. Evaluate $\int_0^1 2x^3 e^{x^2} dx$.¤

34) Each of the following statements is either true or false. Write TRUE or FALSE for each statement and give brief reasons for your answers. (You are not asked to find the primitive functions).

i. $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^7\theta \, d\theta = 0$

ii. $\int_0^{\pi} \sin^7\theta \, d\theta = 0$

iii. $\int_{-1}^{1} e^{-x^2} dx = 0$

iv. $\int_0^{\frac{\pi}{2}} (\sin^8\theta - \cos^8\theta) \, d\theta = 0$

v. For n = 1, 2, 3, ..., $\int_0^1 \frac{dt}{1+t^n} \le \int_0^1 \frac{dt}{1+t^{n+1}}$.¤

35) Find:

i. $\int \frac{t-1}{t^3}\,dt$;

ii. $\int \frac{e^x}{e^{2x}+9}\,dx$, using the substitution $u = e^x$.¤

36) i. Evaluate $\int_0^1 \frac{5}{(2t+1)(2-t)}\,dt$.

ii. By using the substitution $t = \tan\frac{\theta}{2}$ and (i), evaluate $\int_0^{\frac{\pi}{2}} \frac{d\theta}{3\sin\theta + 4\cos\theta}$.¤

37) Let $I_n = \int_0^{\frac{\pi}{2}} \sin^n x\,dx$, where n is a non-negative integer.

i. Show that $I_n = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}x\cos^2 x\,dx$ when $n \geq 2$.

ii. Deduce that $I_n = \frac{n-1}{n} I_{n-2}$ when $n \geq 2$.

iii. Evaluate I_4.¤

38) Find the exact values of:

i. $\int_2^3 \frac{x+1}{\sqrt{x^2+2x+5}}\,dx$

ii. $\int_0^{\sqrt{2}} \sqrt{4-x^2}\,dx$ ¤

39) Find:

i. $\int \frac{dx}{(x+1)(x^2+2)}$;

ii. $\int \cos^3 x\,dx$, by writing $\cos^3 x = (1-\sin^2 x)\cos x$, or otherwise.¤

40) Let $I_n = \int_0^x (1+t^2)^n\,dt, n = 1,2,3,\ldots$

Use integration by parts to show that $I_n = \frac{1}{2n+1}(1+x^2)^n x + \frac{2n}{2n+1} I_{n-1}$.

Hint: Observe that $(1+t^2)^{n-1} + t^2(1+t^2)^{n-1} = (1+t^2)^n$.¤

41) Evaluate:

i. $\int_0^{\sqrt{3}} \frac{2}{x^2+9}\,dx$;

ii. $\int_1^3 x^2 \ln x\,dx$;

iii. $\int_{\sqrt{3}}^2 \sqrt{4-x^2}\,dx$.¤

42) i. Write $\frac{4x^2-5x-7}{(x-1)(x^2+x+2)}$ in the form $\frac{A}{x-1} + \frac{Bx+C}{x^2+x+2}$.

ii. Hence evaluate $\int_{-1}^0 \frac{4x^2-5x-7}{(x-1)(x^2+x+2)}\,dx$.¤

43) Find the exact value of:

i. $\int_0^1 \frac{2x}{1+2x}\,dx$;

ii. $\int_1^e \frac{(\log_e x)^2}{x}\,dx$;

iii. $\int_0^{\frac{1}{2}} \cos^{-1}x\,dx$.¤

44) Use the substitution $t = \tan\frac{\theta}{2}$ to find the exact value of $\int_0^{\frac{\pi}{2}} \frac{d\theta}{\sin\theta + 2}$.¤

45) Prove that:

a. $\int_1^2 \frac{t+1}{t^2}\,dt = \frac{1}{2} + \log_e 2$;

b. $\int_4^6 \frac{4\,dx}{(x-1)(x-3)} = 2\log_e\left(\frac{9}{5}\right)$.¤

46) a. Use the substitution $x = \frac{2}{3}\sin\theta$ to prove that $\int_0^{\frac{2}{3}} \sqrt{4-9x^2}\,dx = \frac{\pi}{3}$.

b. Hence, or otherwise, find the area enclosed by the ellipse $9x^2 + y^2 = 4$.¤

47) a. Given that $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$, prove that $I_n = \left(\frac{n-1}{n}\right)I_{n-2}$ where n is and integer and $n \geq 2$.

b. Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^5 x\,dx$.¤

48) Evaluate $\int_0^1 \frac{x}{\sqrt{(2-x)}}\,dx$.¤

49) Use integration by parts to show that $\int_0^1 \tan^{-1}x\,dx = \frac{\pi}{4} - \frac{1}{2}\log_e 2$.¤

50) Find numbers A, B, C such that $\frac{x^2}{4x^2-9} \equiv A + \frac{B}{2x-3} + \frac{C}{2x+3}$.

Hence evaluate $\int_0^1 \frac{x^2}{4x^2-9}\,dx$.¤

51) Using the substitution $t = \tan(\frac{1}{2}\theta)$, or otherwise, show that $\int_0^{\frac{\pi}{3}} \frac{1}{1+\sin\theta}\,d\theta = \sqrt{3} - 1$.¤

52) Find:

a. $\int_0^{3\pi} x\cos x\,dx$;

b. $\int_0^1 \frac{1}{x^2+4x+5}\,dx$.¤

53) Find real numbers A, B, C such that $\frac{x}{(x-1)^2(x-2)} \equiv \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$.

Hence show that $\int_0^{\frac{1}{2}} \frac{x}{(x-1)^2(x-2)}\,dx = 2\log_e\left(\frac{3}{2}\right) - 1$.¤

54) Use the substitution $x = a - t$, where a is a constant, to prove that $\int_0^a f(x)dx = \int_0^a f(a-t)dt$. Hence, or otherwise, show that $\int_0^1 x(1-x)^{99}\,dx = \frac{1}{10\,100}$.¤

55) Show that:

a. $4\int_e^{e^4} x\log_e x\,dx = 7e^8 - e^2$;

b. $\int_e^{e^4} \frac{dx}{x\log_e x} = 2\log_e 2$.¤

56) Evaluate:

a. $\int_0^{\sqrt{3}} \frac{x+1}{x^2+1}\,dx$

b. $\int_{-4}^{4} \frac{x+6}{\sqrt{x+5}}\,dx$

c. $\int_{\frac{-\pi}{6}}^{\frac{\pi}{6}} \sin 4x' \cos 2x\,dx$.¤

57) Find $\int \cos^2 x\,dx$.†

58) Use the substitution $x = u^2$, $u > 0$, to evaluate $\int \frac{1}{x\left(\sqrt{x}+1\right)}\,dx$.†

59) $I_n = \int_0^1 \frac{x^n}{\sqrt{x+1}}\,dx$, $n = 0, 1, 2, 3, \ldots$

i. Show that $x^{n-1}\sqrt{x+1} = \frac{x^n}{\sqrt{x+1}} + \frac{x^{n-1}}{\sqrt{x+1}}$.

ii. Show that $(2n+1)I_n = 2\sqrt{2} - 2nI_{n-1}$, $n = 1, 2, 3, \ldots$

iii. Evaluate $\int_0^1 \frac{x^3}{\sqrt{x+1}}\,dx$.†

60) i. Show $\int_1^2 \frac{1}{x^2}\,\mathrm{Ln}(x+1)\,dx = \frac{1}{2}\mathrm{Ln}\frac{4}{3} + \int_1^2 \frac{1}{x(x+1)}\,dx$.

ii. Hence evaluate $\int_1^2 \frac{1}{x^2}\,\mathrm{Ln}(x+1)dx$ in simplest exact form.†

61) i. Show that $\int_0^1 x(1-x)^n dx = \frac{n!}{(n+2)!}$.

ii. Evaluate $\int_0^{\frac{\pi}{2}} \sin 2x(1-\sin x)^{10} dx$ using the substitution $u = \sin x$.†

62) i. If $\theta = \pi - \tan^{-1}\frac{3}{4}$, show $\tan\frac{\theta}{2} = 3$.

ii. Evaluate $\int_{\frac{\pi}{2}}^{\pi - \tan^{-1}\frac{3}{4}} \frac{1}{\cos\theta + 2}\,d\theta$, leaving your answer in terms of π.†

63) Find:

i. $\int \tan^2 x\,dx$

ii. $\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$.†

64) Evaluate $\int_1^6 x\sqrt{6-x}\,dx$.†

65) Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx$ using the substitution $t = \tan\frac{x}{2}$.†

66) Find:

i. $\int \text{Ln}x\,dx$

ii. $\int 2x\,\text{Ln}(x^2+1)\,dx$.†

67) i. Find $\int x\sec^2(x^2)\,dx$.

ii. Find $\int \frac{x^4}{x^2+1}\,dx$.†

68) i. Evaluate $\int_0^{\log_e 2} \frac{e^x - e^{-x}}{e^x + e^{-x}}\,dx$.

ii. Use the substitution $u = \cos x$ to evaluate $\int_0^{\frac{\pi}{3}} \frac{\sin^3 x}{\cos^2 x}\,dx$.†

69) i. Show that $(1 - \sqrt{x})^{n-1}\sqrt{x} = (1 - \sqrt{x})^{n-1} - (1 - \sqrt{x})^n$.

ii. If $I_n = \int_0^1 (1 - \sqrt{x})^n\,dx$ for $n \geq 0$ show that $I_n = \frac{n}{n+2} I_{n-1}$ for $n \geq 1$.

iii. Deduce that $\frac{1}{I_n} = \binom{n+2}{2}$ for $n \geq 0$.†

70) Find $\int \frac{e^{\tan x}}{\cos^2 x}\,dx$.†

71) Evaluate $\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$.†

72) i. Use the substitution $u = -x$ to show that $\int_{-a}^{0} f(x)\,dx = \int_0^a f(-x)\,dx$.

Deduce that $\int_{-a}^{a} f(x)\,dx = \int_0^a \{f(x) + f(-x)\}\,dx$.

ii. Hence evaluate $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} x^5 \cos x\,dx$.†

73) i. Use the substitution $x = u^2\ (u > 0)$ to show that $\int_4^9 \frac{\sqrt{x}}{x-1}\,dx = 2 + \log_e\left(\frac{3}{2}\right)$.

ii. Hence use integration by parts to evaluate $\int_4^9 \frac{1}{\sqrt{x}}\log_e(x-1)\,dx$.†

74) Find

i. $\int \frac{1}{3+2x-x^2}dx$.

ii. $\int \frac{1}{e^x+e^{-x}}dx$.†

75) i. Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{1+\cos x+\sin x}dx$ using the substitution $t=\tan\frac{x}{2}$.

ii. Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{x}{1+\cos x+\sin x}dx$ using the substitution $u=\frac{\pi}{2}-x$.†

76) i. Let $I_n = \int_1^e x(\text{Ln}\,x)^n dx$, $n = 0, 1, 2, 3, \ldots$.

Use integration by parts to show that $I_n = \frac{e^2}{2} - \frac{n}{2}I_{n-1}$, $n = 1, 2, 3, \ldots$.

ii. The area bounded by the curve $y=\sqrt{x}(\text{Ln}\,x)^2$, $x \le 1$, the x-axis and the line $x = e$ is rotated through 2π radians about the x axis. Find the exact value of the volume of the solid of revolution so formed.†

77) i. By expressing $x^2 - 2x - 1$ as the difference of two squares, or otherwise, show that $x^2 - 2x - 1 = (x - 1 - \sqrt{2})(x - 1 + \sqrt{2})$.

ii. Hence express $\frac{1}{x^2-2x-1}$ in the form $\frac{A}{(x-1-\sqrt{2})}+\frac{B}{(x-1+\sqrt{2})}$, where A and B are real numbers, and find $\int \frac{1}{x^2-2x-1}dx$.†

78) Use the substitution u = -x to show that: $\int_{-2}^{2} \frac{x^2}{e^x+1}dx = \int_{-2}^{2} \frac{x^2e^x}{e^x+1}dx$.

Hence evaluate $\int_{-2}^{2} \frac{x^2}{e^x+1}dx$.†

79) i. If $I_n = \int_0^1 (1-x^2)^n dx$ show that: $I_n = \frac{2n}{2n+1}I_{n-1}$ for all positive integers $n \ge 1$.

ii. Hence or otherwise ,show that: $I_n = \frac{2^{2n}(n!)^2}{(2n+1)!}$ for all positive integers $n \ge 1$.†

80) a. If $y = \tan^{-1}e^x$ show that $\frac{d^2y}{dx^2} = 2\left(\frac{dy}{dx}\right)^2 \cot 2y$.

b. Find $\int \frac{e^{2x}}{e^x+1}dx$.

c. i. By using partial fractions show that $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{x(\pi-2x)}dx = \frac{2}{\pi}\log_e 2$.

ii. By using the substitution u = a + b - x show that $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$.

iii. Hence evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\cos^2 x}{x(\pi - 2x)} dx$.†

81) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{4 + 5\sin x} dx$.†

82) Using the substitution $x = 4\sin^2\theta$ or otherwise show that $\int_0^2 \sqrt{x(4-x)}dx = \pi$.†

83) Find $\int \frac{\log_e x}{\sqrt{x}} dx$.†

84) Use the substitution $u = \pi - x$ to show that $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x} dx$.

Deduce that $\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x} dx = \frac{\pi^2}{4}$.†

85) Find $\int \frac{\sin^3\theta}{\cos^4\theta} d\theta$.†

86) a. Show that $\int_{-a}^{0} f(x)dx = \int_0^a f(-x)dx$.

b. Deduce that $\int_{-a}^{a} f(x)dx = \int_0^a [f(x) + f(-x)]dx$.

c. Hence evaluate $\int_{\frac{-\pi}{4}}^{\frac{\pi}{4}} \frac{1}{1+\sin x} dx$.†

87) Use integration by parts to evaluate $\int_0^1 \frac{\sin^{-1} x}{\sqrt{1+x}} dx$.†

88) Use partial fractions to find the integral $\int \frac{x^2}{(x+1)(x+2)} dx$.†

89) Use the substitution $x = \frac{\pi}{2} - u$ to show that $\int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x + \cos x} dx = \int_0^{\frac{\pi}{2}} \frac{\cos x}{\sin x + \cos x} dx$ and hence determine their value.†

90) Find the integral $\int \frac{e^x + e^{2x}}{1+e^{2x}} dx$.†

91) Find $\int_0^{\frac{\pi}{4}} x\sec^2 x dx$.†

92) Find $\int \frac{dx}{x\sqrt{x^2 - 1}}$.†

93) Evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{3+5\cos x} dx$.†

94) Show that $\int_0^1 \frac{\sqrt{x}}{(1+x)} dx = 2 - \frac{\pi}{2}$.†

95) Given that $I_n = \int \sec^n x\, dx$, where $n \geq 2$, show that $(n-1)I_n = \tan x \sec^{n-2} x + (n-2)I_{n-2}$.

Hence evaluate $\int_0^{\frac{\pi}{4}} \sec^6 x\, dx$.†

6) Find: $\int \operatorname{cosec} x \, dx$ by using the substitution $t = \tan\frac{x}{2}$.†

97) Find: $\int \frac{dx}{x(1+x^2)}$.†

98) Find:

a. $\int x\sqrt{x^2-1}\,dx$;

b. $\int_1^2 x\sqrt{3x-2}\,dx$.†

99) If $I_n = \int_0^1 x^n e^x dx$, where n is a positive integer, show that $I_{n-1} = e - (n+1)I_n$.

Hence evaluate $\int_0^{0.2} t^3 e^{5t} dt$, leaving your answer in terms of e.†

[[End Of Qns]]

9. $I = \int x \tan^{-1} x \, dx$

let $v' = x$ $\quad u = \tan^{-1} x$

$v = \frac{x^2}{2}$ $\quad \frac{du}{dx} = \frac{1}{x^2+1}$

$$I = \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \frac{x^2}{x^2+1} \, dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int \frac{x^2+1-1}{x^2+1} \, dx \quad ✓$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \int 1 - \frac{1}{x^2+1} \, dx$$

$$= \frac{x^2 \tan^{-1} x}{2} - \frac{1}{2} \left[x - \tan^{-1} x \right] + c \quad ✓$$

$$= \frac{x^2 \tan^{-1} x - x + \tan^{-1} x}{2} + c$$

11. i) $\int_0^{\frac{\pi}{2}} (\sin x)^{2k} \cos x \, dx$

let $u = \sin x$, $\frac{du}{dx} = \cos x$

$x = \frac{\pi}{2}, u = 1$ $\quad x = 0, u = 0$ ✓

$$I = \int_0^1 u^{2k} \frac{du}{dx} x$$

$$= \left[\frac{u^{2k+1}}{2k+1} \right]_0^1 \quad ✓$$

$$= \frac{1}{2k+1} \quad ✓$$

ii) $(\cos x)^{2n} = (1-\sin^2 x)^n$

$$\int_0^{\frac{\pi}{2}} (\cos x)^{2n} (\cos x) \, dx$$

$$= \int_0^{\frac{\pi}{2}} (1-\sin^2 x)^n \cos x \, dx$$

let $u = \sin x$ $\Rightarrow x = \frac{\pi}{2}, u = 1$ $x = 0, u = 0$

$\frac{du}{dx} = \cos x$ ✓

$$I = \int_0^1 (1-u^2)^n \frac{du}{dx} dx$$

$$= \int_0^1 \binom{n}{0} 1 - \binom{n}{n-1} u^2 + \binom{n}{n-2} u^4 - \binom{n}{n-3} u^6 + \dots + (-1)^n (u^2)^n \, du$$ ✓

$$= \left[\binom{n}{0} u - \binom{n}{1} \frac{u^3}{3} + \dots + (-1)^n \frac{u^{2n+1}}{2n+1} \right]_0^1$$

$$= \binom{n}{0} 1 - \binom{n}{1} \frac{1^3}{3} + \dots + (-1)^n \frac{1^{2n+1}}{2n+1}$$ ✓

$$= \sum_{k=0}^{n} \frac{(-1)^k}{2k+1} \binom{n}{k}$$ ✓

iii) let $n = 2$

$$I = \sum_{k=0}^{2} \frac{(-1)^k}{2k+1} \binom{2}{k}$$ ✓

$$= \frac{(-1)^0}{1}\binom{2}{0} + \frac{(-1)^1}{3}\binom{2}{1} + \frac{(-1)^2}{5}\binom{2}{2}$$

$$= 1 - \frac{2}{3} + \frac{1}{5} = \frac{8}{15}$$ ✓

19) i) $I_n = \int_0^{\frac{\pi}{2}} (\sin x)^n \, dx$

Let $v' = \sin x$ $\quad u = \sin^{n-1} x$

$v = -\cos x$ $\quad \frac{du}{dx} = (n-1)(\sin x)^{n-2} \cos x$

$$I_n = \left[-\cos x \sin^{n-1} x \right]_0^{\frac{\pi}{2}} + (n-1) \int_0^{\frac{\pi}{2}} \cos^2 x \, (\sin x)^{n-2} \, dx$$ ✓

$$= (n-1) \int_0^{\frac{\pi}{2}} (1 - \sin^2 x) \sin^{n-2} x \, dx$$

$$= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x - \sin^n x \, dx$$ ✓

$$I_n = (n-1) I_{n-2} - (n-1) I_n$$

~~$I_n (n+2)$~~

$$n I_n = (n-1) I_{n-2}$$ ✓

$$I_n = \left(\frac{n-1}{n} \right) I_{n-2}$$

ii) $I_{2n} = \left(\frac{2n-1}{2n} \right) I_{2n-2}$

$$I_{2n-2} = \frac{2n-3}{2n-2} I_{2n-4}$$ ✓

" etc....

$$\therefore I_{2n} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \int_0^{\frac{\pi}{2}} 1 \, dx$$

$$= \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}$$ ✓

$$I_{2n+1} = \frac{2n}{2n+1} \cdot I_{2n-1}$$

$$I_{2n-1} = \frac{2n-2}{2n-1} \cdot I_{2n-3} \checkmark$$

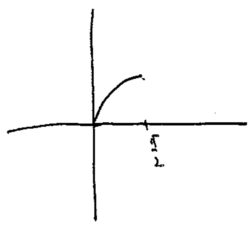
etc...

$$\therefore I_{2n+1} = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{2}{3} \cdot \int_0^{\frac{\pi}{2}} \sin x \, dx$$

$$= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{2}{3} \cdot \left[-\cos x \right]_0^{\frac{\pi}{2}} \checkmark$$

$$= \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{2}{3} \cdot 1 \checkmark$$

iii)



For $0 \le x \le \frac{\pi}{2}$, $0 \le \sin x \le 1$

$\therefore \sin^k x > \sin^{k+1} x$ $\quad \therefore \int_0^{\pi/2} \sin^k x \, dx > \int_0^{\pi/2} \sin^{k+1} x \, dx$.

$\therefore I_k > I_{k+1} \checkmark$

~~(iv) $I_{2n+1} < I_{2n} < I_{2n-1}$~~

~~$(2n-1)(2n-3)\ldots 3 \cdot 1 \pi$~~

iv) $I_{2n} > I_{2n+1}$

$$\frac{(2n-1)(2n-3)\ldots(3)(1)\pi}{2n(2n-2)\ldots 4.2.2} > \frac{2n(2n-2)\ldots 4.2.1}{(2n+1)(2n-1)\ldots 5.3.1}$$

$$(2n+1)(2n-1)^2(2n-3)^2\ldots(3^2)(1)^2\pi > 2(2n)^2(2n-2)^2\ldots 4^2.2^2.1^2$$

$$\frac{\pi}{2} > \frac{(2n)^2(2n-2)^2\ldots 2^2.1^2}{1.3^2.5^2\ldots(2n-1)^2(2n+1}$$

$I_{2n-1} > I_{2n}$

$$I_{2n+1} = \frac{2n}{2n+1} I_{2n-1}, \quad I_{2n-1} = I_{2n+1} \div \frac{2n}{2n+1}$$

$$\frac{(2n-2)(2n-4)\ldots 4.2.1}{(2n-1)(2n-3)\ldots(5)(3)} > \frac{(2n-1)(2n-3)\ldots(3)(1)(\pi)}{(2n)(2n-2)\ldots(4)(2)(2)}$$

$$2n(2n-2)^2(2n-4)^2\ldots 4^2.2^2.2 > (2n-1)^2(2n-3)^2\ldots 5^2.3^2.\pi$$

$$\frac{2n(2n-2)^2(2n-4)^2\ldots 2^2.2}{(2n-1)^2(2n-3)^2\ldots 5^3.3^3} > \pi$$

$$\frac{(2n^2)(2n-2)^2\ldots 2^2.1}{(2n+1)(2n-1)^2\ldots 5^2.3^2.1^2} > \frac{\pi}{2}\left(\frac{2n}{2n+1}\right)$$

$$\therefore \frac{\pi}{2}\left(\frac{2n}{2n+1}\right) < \frac{2^2.4^2\ldots(2n^2)}{1^2.3^2\ldots(2n-1)^2(2n+1)} < \frac{\pi}{2}$$

40) $I_n = \int_0^x (1+t^2)^n \, dt$

let $v' = 1$ $\quad$ $u = (1+t^2)^n$

$v = ~~x~~ t$ $\quad$ $\frac{du}{dx} = n(1+t^2)^{n-1} \cdot 2t$ ✓

$I_n = \left[t(1+t^2)^n \right]_0^x - 2n \int_0^x t^2 (1+t^2)^{n-1} \, dt$ ✓

$= x(1+x^2)^n - 2n \int_0^x (t^2+1)(1+t^2)^{n-1} - (1+t^2)^{n-1} \, dt$

$I_n = x(1+x^2)^n - 2n \left[I_n - I_{n-1} \right]$ ✓

$2nI_n + I_n = x(1+x^2)^n + 2nI_{n-1}$

$I_n = \frac{1}{2n+1}(1+x^2)^n x + \frac{2n}{2n+1} I_{n-1}$ ✓

59) i)

$RHS = \frac{x^{n-1}(x+1)}{\sqrt{x+1}}$ ✓

$= x^{n-1}\sqrt{x+1}$ ✓ $= LHS$

~~ii) $I_n = \int_0^1 x^{n-1}\sqrt{x+1} - \frac{x^{n-1}}{\sqrt{x+1}} \, dx$~~

~~$I_n = \int_0^1 \frac{x^{n-1}}{\sqrt{x+1}} \left((x+1) - 1 \right)$~~

ii) $I_n = \int_0^1 \frac{x^n}{\sqrt{x+1}}\,dx$

let $u = x^n$ $\quad v' = \frac{1}{\sqrt{x+1}}$

$\frac{du}{dx} = nx^{n-1}$ $\quad v = 2\sqrt{x+1}$

$I_n = \left[2x^n\sqrt{x+1}\right]_0^1 - 2n\int_0^1 x^{n-1}\sqrt{x+1}\,dx$ ✓

$= 2\sqrt{2} - 2n\int_0^1 \frac{x^n}{\sqrt{x+1}} + \frac{x^{n-1}}{\sqrt{x+1}}\,dx$

$I_n = 2\sqrt{2} - 2n\left[I_n + I_{n-1}\right]$ ✓

$(2n+1)I_n = 2\sqrt{2} - 2nI_{n-1}$

iii) $I_3 = 2\sqrt{2} - 6I_2$

$I_2 = 2\sqrt{2} - 4I_1$

$I_1 = 2\sqrt{2} - 2I_0$ ✓

$I_0 = \int_0^1 \frac{1}{\sqrt{x+1}}\,dx$

$= \left[2\sqrt{x+1}\right]_0^1 = 2\sqrt{2} - 2$ ✓

$I_1 = 2\sqrt{2} - 4\sqrt{2} + 4 = 4 - 2\sqrt{2}$

$I_2 = 2\sqrt{2} - 4(4 - 2\sqrt{2})$

$= 2\sqrt{2} - 16 + 8\sqrt{2}$ ✓

$= 10\sqrt{2} - 16$

$I_3 = 2\sqrt{2} - 6(10\sqrt{2} - 16)$

$= 2\sqrt{2} - 60\sqrt{2} + 96$ ✓

$= \underline{96 - 58\sqrt{2}}$