CRANBROOK

Extension 2 Assessment Integration 2-6-11

Name_______________

[Begin a new booklet]. RDS

1. Find the following indefinite and or evaluate the definite integrals:

(a) $\int \frac{1}{4x\sqrt{1-x^2}}dx$ using a suitable substitution. 3

(b) $\int_4^5 \frac{-dx}{x^2-3x}$ 3

(c) $\int \frac{x\sin^{-1}x}{2}dx$ 3

(d) $\int \sqrt{\frac{x+7}{x-3}}dx$ 3

(e) $\int \frac{2}{1-\cos x+\sin x}dx$ 3

[Begin a new booklet]. SKB

2. (a) Prove by integration: $\int \sec^3 4x\ dx = \frac{1}{8}[\sec 4x \tan 4x + \ln|\sec 4x + \tan 4x|] + c$ 4

Hence evaluate: $\int_0^{\frac{\pi}{4}} \sec^3 4x\ dx$ 1

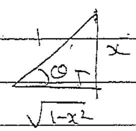
(b) Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ 2

Hence evaluate $\int_0^{\frac{\pi}{2}} \frac{\operatorname{cosec}^3 x}{\operatorname{cosec}^3 x+\sec^3 x}dx$ 3

(c) If $I_n = \int_0^a (a^2+x^2)^n dx$ show that $I_n = \frac{1}{2n+1}[a^{2n+1}.2^n + 2a^2 n I_{n-1}], n \geq 1$. 5

① (a) $I = \int \frac{1}{4x\sqrt{1-x^2}}\, dx$

let $x = \sin\theta \quad \therefore \frac{dx}{d\theta} = \cos\theta$



$\therefore I = \int \frac{\cancel{\cos\theta}\, d\theta}{4\sin\theta\, \cancel{\sqrt{1-\sin^2\theta}}}$

$= \frac{1}{4}\int \operatorname{cosec}\theta\, d\theta$

$= -\frac{1}{4}\ln\left|\operatorname{cosec}\theta + \cot\theta\right| + c$

$= -\frac{1}{4}\ln\left|\frac{1+\sqrt{1-x^2}}{x}\right| + c$

(b) $I = \int_4^5 \frac{dx}{x^2-3x}$

$= \int_4^5 \frac{dx}{\left(x-\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^2}$

$= \frac{1}{2\left(\frac{3}{2}\right)}\left[\ln\left|\frac{x-\frac{3}{2}-\frac{3}{2}}{x-\frac{3}{2}+\frac{3}{2}}\right|\right]_4^5$

$= -\frac{1}{3}\left[\ln\left|\frac{x-3}{x}\right|\right]_4^5$

$= -\frac{1}{3}\left[\ln\frac{2}{5} - \ln\frac{1}{4}\right]$

$= -\frac{1}{3}\ln\left|\frac{2}{5}\times\frac{4}{1}\right|$

$= -\frac{1}{3}\ln\frac{8}{5}$

(c) $I = \int \frac{x\sin^{-1}x}{2}\, dx$

$= \frac{1}{2}\int x\sin^{-1}x\, dx$

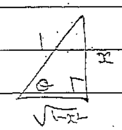
Let $u = \sin^{-1}x \qquad dv = x\, dx$

$\therefore \frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad v = \frac{x^2}{2}$

$\therefore I = \frac{1}{2}\left[\frac{x^2\sin^{-1}x}{2} - \frac{1}{2}\int\frac{x^2}{\sqrt{1-x^2}}\, dx\right]$

Let $x = \sin\theta$

$\therefore \frac{dx}{d\theta} = \cos\theta$



$I = \frac{1}{2}\left[\frac{x^2\sin^{-1}x}{2} - \frac{1}{2}\int\frac{\sin^2\theta \cdot \cancel{\cos\theta}\, d\theta}{\cancel{\sqrt{1-\sin^2\theta}}}\right]$

$= \frac{1}{2}\left[\frac{x^2\sin^{-1}x}{2} - \frac{1}{2}\int\sin^2\theta\, d\theta\right]$

$= \frac{1}{2}\left[\frac{x^2\sin^{-1}x}{2} - \frac{1}{2}\int\frac{1}{2}\left[1-\cos 2\theta\right] d\theta\right]$

$= \frac{1}{2}\left[\frac{x^2\sin^{-1}x}{2} - \frac{1}{4}\left[\theta - \frac{\sin 2\theta}{2}\right]\right] + c$

$= \frac{1}{2}\left[\frac{x^2\sin^{-1}x}{2} - \frac{1}{4}\left[\sin^{-1}x - x\sqrt{1-x^2}\right]\right] + c$

$= \frac{x^2\sin^{-1}x}{4} - \frac{\sin^{-1}x}{8} + \frac{x\sqrt{1-x^2}}{8} + c$

(d) $I = \int\sqrt{\frac{x+7}{x-3}}\, dx$

$= \int\frac{\sqrt{x+7}}{\sqrt{x-3}}\cdot\frac{\sqrt{x+7}}{\sqrt{x+7}}\, dx$

$= \int\frac{x+7}{\sqrt{x^2+4x-21}}\, dx$

$= \int\frac{\frac{1}{2}(2x+4)+5}{\sqrt{x^2+4x-21}}\, dx$

$= \frac{1}{2}\int\frac{2x+4}{\sqrt{x^2+4x-21}}\, dx + 5\int\frac{dx}{\sqrt{(x+2)^2-5^2}}$

$= \frac{1}{2}\cdot 2\sqrt{x^2+4x-21} + 5\ln\left(x+2+\sqrt{x^2+4x-21}\right) + c$

$= \sqrt{x^2+4x-21} + 5\ln\left(x+2+\sqrt{x^2+4x-21}\right) + c$

(e) $I = \int \frac{2\,dx}{1-\cos x+\sin x}$

Let $t = \tan\frac{x}{2}$ $\therefore dx = \frac{2dt}{1+t^2}$

$\cos x = \frac{1-t^2}{1+t^2}$ $\sin x = \frac{2t}{1+t^2}$

$\therefore I = 2\int \frac{2\frac{dt}{1+t^2}}{1-\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}}$

$= 2\int \frac{2dt}{1+t^2-1+t^2+2t}$

$= 2\int \frac{dt}{t^2+t}$

$= 2\int \frac{dt}{(t+\frac{1}{2})^2-(\frac{1}{2})^2}$

$= 2\frac{1}{2\cdot\frac{1}{2}} \ln\left|\frac{t+\frac{1}{2}-\frac{1}{2}}{t+\frac{1}{2}+\frac{1}{2}}\right| + c$

$= 2\ln\left|\frac{t}{t+1}\right| + c$

$= 2\ln\left|\frac{\tan\frac{x}{2}}{\tan\frac{x}{2}+1}\right| + c$

2(a) $I = \int \sec^3 4x\,dx$

$= \int \sec 4x \sec^2 4x\,dx$

let $u = \sec 4x$ $dv = \sec^2 4x\,dx$

$\frac{du}{dx} = 4\sec 4x \tan 4x$ $v = \frac{\tan 4x}{4}$ ✓

$\therefore I = \frac{1}{4}\sec 4x \tan 4x - \int \sec 4x \tan^2 4x\,dx$

$= \frac{1}{4}\sec 4x \tan 4x - \int \sec 4x(\sec^2 4x - 1)\,dx$ ✓

$= \frac{1}{4}\sec 4x \tan 4x - \int \sec^3 4x\,dx + \int \sec 4x\,dx$ ✓

$\therefore 2I = \frac{1}{4}\sec 4x \tan 4x + \frac{1}{4}\ln|\sec 4x + \tan 4x|$

$\therefore I = \frac{1}{8}\left[\sec 4x \tan 4x + \ln|\sec 4x + \tan 4x|\right] + c$ ✓

Now $\int_0^{\frac{\pi}{4}} \sec^3 4x\,dx = \frac{1}{8}\left[\sec 4x \tan 4x + \ln|\sec 4x + \tan 4x|\right]_0^{\frac{\pi}{4}}$

$= \frac{1}{8}\left[(1.0 + \ln|1+0|) - (0 + \ln|1+0|)\right]$

$= \frac{1}{8}(0)$ ✓

$= 0.$

(b) TO PROVE: $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

PROOF: $LHS = \int_0^a f(x)\,dx$

let $t = a-x$ $\therefore \frac{dt}{dx} = -1$

when $x=0$ $t=a$, when $x=a$ $t=0$

$\therefore LHS = \int_a^0 f(a-t) - dt$ ✓

$= \int_0^a f(a-t)\,dt$ ✓

$= \int_0^a f(a-x)\,dx$, reverting to variable x.

$= RHS.$

Now $I = \int_0^{\frac{\pi}{2}} \frac{\operatorname{cosec}^3 x}{\operatorname{cosec}^3 x + \sec^3 x}\,dx$

$= \int_0^{\frac{\pi}{2}} \frac{\operatorname{cosec}^3(\frac{\pi}{2}-x)}{\operatorname{cosec}^3(\frac{\pi}{2}-x) + \sec^3(\frac{\pi}{2}-x)}\,dx$ ✓

$= \int_0^{\frac{\pi}{2}} \frac{\sec^3 x}{\sec^3 x + \operatorname{cosec}^3 x}\,dx$

$$\therefore 2I = \int_0^{\frac{\pi}{2}} \frac{\operatorname{cosec}^3 x + \sec^3 x}{\operatorname{cosec}^3 x + \sec^3 x}\, dx$$

$$= \int_0^{\frac{\pi}{2}} 1 \, dx$$

$$= \Big[x \Big]_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{2} - 0$$

$$\therefore I = \frac{\pi}{4}$$

(c) $$I_n = \int_0^a (a^2+x^2)^n \, dx$$

$$= \int_0^a (a^2+x^2)^n \cdot 1 \, dx$$

let $u = (a^2+x^2)^n$ $\quad dv = 1\, dx$

$\therefore \frac{du}{dx} = n(a^2+x^2)^{n-1} \cdot 2x \quad v = x$

$$\therefore I_n = \Big[x(a^2+x^2)^n \Big]_0^a - 2n\int_0^a x^2 (a^2+x^2)^{n-1}\, dx$$

$$= \Big[a(2a^2)^n - 0 \Big] - 2n\int_0^a (a^2+x^2-a^2)(a^2+x^2)^{n-1}\, dx$$

$$= 2^n \cdot a \cdot a^{2n} - 2n\int_0^a (a^2+x^2)^n dx + 2na^2\int_0^a (a^2+x^2)^{n-1} dx$$

$$= a^{2n+1} \cdot 2^n - 2n I_n + 2na^2 I_{n-1}$$

$$\therefore I_n(1+2n) = a^{2n+1} \cdot 2^n + 2a^2 n\, I_{n-1}$$

$$\therefore I_n = \frac{1}{2n+1}\Big[a^{2n+1} \cdot 2^n + 2a^2 n\, I_{n-1} \Big], \; n \geq 1$$