Sydney Girls High School

2012

HSC Assessment Task 3

Mathematics Extension 2

Topics Assessed: Polynomials, Integration, Volumes.

General Instructions:

- Reading time – 5 minutes.
- Working time – 90 minutes.
- There are 6 questions which **are not** of equal value.
- Write using black or blue pen.
- Board-approved calculators may be used.
- All necessary working should be shown in every question.
- Write your student number clearly at the top of each question and clearly number each question.

- **Total Marks: 80**

Marks

Question 2 (13 marks) Start question 2 on a new page.

(a) Find $\int \sin^4 \theta \cos \theta \, d\theta$. **2**

(b) Find $\int \dfrac{dx}{\sqrt{5+4x-x^2}}$. **2**

(c) Evaluate $\int_{-2}^{2} \dfrac{\sin x}{x^2+3} dx$. **1**

(d) Evaluate $\int_0^{\frac{2}{3}} \sqrt{4-9x^2}\, dx$. **4**

(e) Evaluate $\int_0^{\frac{\pi}{2}} \dfrac{dx}{1+\sin x}$. **4**

Marks

Question 3 (13 marks) Start question 3 on a new page.

(a) Find the equation of a monic polynomial with real coefficients of degree 2, given that it has a root at $2+3i$. **1**

(b) Factorise x^2+4x+5 over the complex field. **2**

(c) Find the value of a such that x^2+2 is a factor of x^4-6x^2+a. **2**

(d) The polynomial $P(x)=3x^3+4x^2-2x+1$ has roots α, β and γ. Find the equation of a polynomial with roots:

(i) $2\alpha, 2\beta$ and 2γ **2**

(ii) α^2, β^2 and γ^2 **3**

(iii) $\alpha\beta, \alpha\gamma$ and $\beta\gamma$ **3**

Marks

Question 4 (13 marks) Start question 4 on a new page.

(a) If α, β, γ are the roots of $x^3+3x^2-6x+1=0$ find:

(i) $\sum\alpha$ **1**

(ii) $\sum\alpha^2$ **2**

(iii) $\sum\alpha^3$ **2**

(iv) $\sum\alpha^2\beta^2$ **2**

(b) The cubic equation $x^3+ax^2+bx+15=0$ (where a and b are real) has $2+i$ as one root. Find a and b and solve the equation. **3**

(c) The equation $2x^3+3x^2-72x+a=0$ has a double root. Find the possible values of a. **3**

Marks

Question 5 (13 marks) Start question 5 on a new page.

(a) Find the volume of a solid whose base is a circle with equation $x^2+y^2=36$. Cross sections perpendicular to the y axis are equilateral triangles. **4**

(b) Find the exact volume of the solid formed if the circle $x^2+(y-2)^2=1$ is rotated about the x axis. **4**

(c) The area enclosed between the curve $y=(x-3)^2$ the x axis and the line $x=4$ is rotated about the y axis.

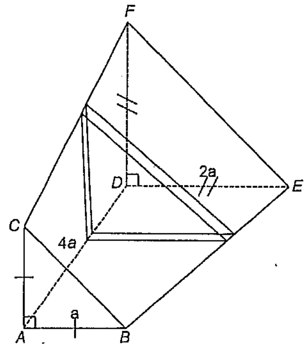
(i) Draw a diagram to show the area. **1**

(ii) <u>Using the method of cylindrical shells</u>, find the volume of the solid formed. **4**

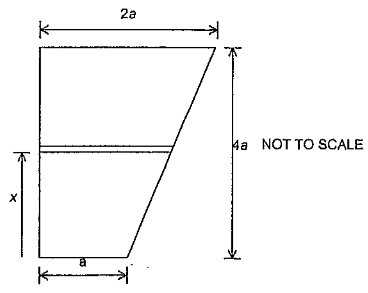
Marks

Question 6 (15 marks) Start question 6 on a new page.

(a)



The diagram above shows a solid that is $4a$ units long. The end ABC is a right angled isosceles triangle whose equal sides are a units long. The end DFE is a right angled isosceles triangle whose equal sides are $2a$ units long. Both ends are perpendicular to the base. The diagram below shows the top view of the solid.



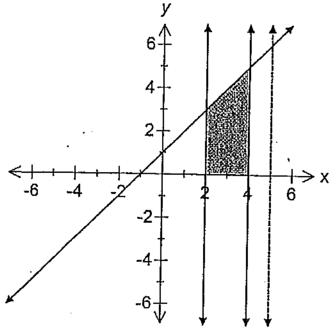
(i) Show that the volume of a thin cross sectional slice taken x units from the front of the solid is given by:

$$\delta V=\frac{1}{32}\left(x^2+8ax+16a^2\right)\delta x.$$ **3**

(ii) Find the exact volume of the solid in terms of a. **2**

Marks

(b) The area bounded by the lines $y=x+1$, $x=2$, $x=4$ and the x axis is rotated about the line $x=5$. By taking slices perpendicular to the y axis, find the volume of the solid formed. **4**



(c) (i) If $I_n = \int_0^{\frac{\pi}{2}} \cos^n x\,dx$, $n \geq 2$ show that $I_n = (n-1)(I_{n-2} - I_n)$. **3**

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} \cos^5 x\,dx$. **3**

Question 1

(a) $$\int \frac{dx}{\sqrt{x^2-2}} = \ln\left(x+\sqrt{x^2-2}\right) + c$$

(b) $$\int \frac{x+5}{x-5}\,dx = \int\left(\frac{x-5}{x-5} + \frac{10}{x-5}\right)dx$$
$$= x + 10\ln(x-5) + C$$

(c) $u = \sec x$ $\quad \dfrac{du}{dx} = \sec x \tan x$

$$\int \sec x \tan^3 x\,dx = \int \tan^2 x \sec x \tan x\,dx$$
$$= \int (\sec^2 x - 1)\sec x \tan x\,dx$$
$$= \int (u^2-1)\,du$$
$$= \frac{u^3}{3} - u$$
$$\therefore\ I = \frac{\sec^3 x}{3} - \sec x + C$$

(d) (i) $3 = a(2x^2+3) + x(bx+c)$

let $x=0 \Rightarrow$ $3 = 3a$ $\therefore a = 1$

coeff. of $x^2 \Rightarrow$ $0 = 2a + b$ $\quad b = -2a = -2$

coeff. of $x \Rightarrow$ $0 = c$ $\therefore c = 0$

Question 1 (continued)

(d) (ii) $$\int \frac{3}{x(2x^2+3)}\,dx = \int\left(\frac{1}{x} - \frac{2x}{2x^2+3}\right)dx$$
$$\therefore\ I = \ln x - \frac{1}{2}\ln(2x^2+3) + C$$

(e) $$\int x \sin 3x\,dx$$ $\quad u = x \quad v' = \sin 3x$

$u' = 1 \quad v = -\frac{1}{3}\cos 3x$

$$I = -\frac{x}{3}\cos 3x + \int \frac{1}{3}\cos 3x\,dx$$
$$\therefore\ I = -\frac{x}{3}\cos 3x + \frac{1}{9}\sin 3x + C$$

2012 HSC TASK 3
EXTENSION 2
MATHEMATICS
SOLUTIONS

Task 3 Extn 2 2012 Solutions Q2

a) $I = \int \sin^4\theta \cos\theta \, d\theta$

let $u = \sin\theta \rightarrow du = \cos\theta \, d\theta$ ✓

$I = \int u^4 \, du$

$= \frac{u^5}{5}$

$= \frac{\sin^5\theta}{5} + C$ ✓ (2)

b) $I = \int \frac{dx}{\sqrt{5+4x-x^2}}$

$= \int \frac{dx}{\sqrt{9-(x^2-4x+4)}}$ ✓

$= \int \frac{dx}{\sqrt{3^2-(x-2)^2}}$

$= \sin^{-1}\frac{x-2}{3} + C$ ✓ (2)

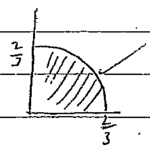
c) $I = \int_{-2}^{2} \frac{\sin x}{x^2+3} \, dx$ (1)

$= 0$ [$f(x)$ odd] ✓

d) $I = \int_0^{\frac{2}{3}} \sqrt{4-9x^2} \, dx$ (the short way)

$= \int_0^{\frac{2}{3}} \sqrt{9\left(\frac{4}{9}-x^2\right)} \, dx$ ✓

$= 3\int_0^{\frac{2}{3}} \sqrt{\frac{4}{9}-x^2}$ ✓



$= 3 \times \frac{1}{4}\pi\left(\frac{2}{3}\right)^2$ (4)

$= \frac{\pi}{3}$ ✓

OR

d) $I = \int_0^{\frac{2}{3}} \sqrt{4-9x^2} \, dx$ the long way

$= \int_0^{\frac{2}{3}} \sqrt{9}\sqrt{\frac{4}{9}-x^2} \, dx$

$= 3\int_0^{\frac{2}{3}} \sqrt{\frac{4}{9}-x^2} \, dx$ ✓

let $x = \frac{2}{3}\sin\theta$ ✓
$dx = \frac{2}{3}\cos\theta \, d\theta$
when $x = \frac{2}{3}$, $\theta = \frac{\pi}{2}$
$x = 0$, $\theta = 0$

$I = 3 \times \frac{2}{3}\int_0^{\frac{\pi}{2}} \sqrt{\frac{4}{9}\cos^2\theta} \, \cos\theta \, d\theta$

$= \frac{4}{3}\int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta$

$= \frac{4}{3} \times \frac{1}{2}\int_0^{\frac{\pi}{2}} (1+\cos 2\theta) \, d\theta$ ✓

$= \frac{2}{3}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$ (4)

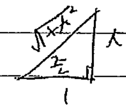
$= \frac{2}{3}\left[\left(\frac{\pi}{2}+0\right)-(0+0)\right]$ ✓

$= \frac{\pi}{3}$

e) $I = \int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$

let $t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$ ✓



$\frac{dt}{dx} = \frac{1}{2}(1+t^2)$

$dx = \frac{2\,dt}{1+t^2}$

when $x = \frac{\pi}{2}$, $t = 1$ ✓
$x = 0$, $t = 0$

$\sin x = \frac{2t}{1+t^2}$

$I = \int_0^1 \frac{2\,dt}{1+t^2} \div \left(1+\frac{2t}{1+t^2}\right)$

$= \int_0^1 \frac{2\,dt}{1+t^2} \div \frac{1+t^2+2t}{1+t^2}$

$= 2\int_0^1 \frac{dt}{1+t^2} \times \frac{1}{(1+t)^2}$

$= 2\int_0^1 (1+t)^{-2} \, dt$ ✓

$= -2\left[(1+t)^{-1}\right]_0^1$

$= 2\left[\frac{1}{1+t}\right]_0^1$

$= -2\left[\frac{1}{2}-1\right]$ ✓

$= 1$

Note [Can also be done by multiplying by $\frac{1-\sin x}{1-\sin x}$]

Question 5

a) $x^2 - (2+3i+2-3i)x + (2+3i)(2-3i) = 0$

$x^2 - 4x + 13 = 0$

b) $x^2+4x+5 = x^2+4x+4+1$

$= (x+2)^2 + 1$

$= (x+2+i)(x+2-i)$

c) $P(\sqrt{2}i) = 0$

$(\sqrt{2}i)^4 - 6(\sqrt{2}i)^2 + a = 0$

$4 + 12 = a$

$a = -16$

d) i) $P(x) = 3x^3 + 4x^2 - 2x + 1$

$x = \alpha, \beta, \gamma$

$y = 2\alpha, 2\beta, 2\gamma$

$y = 2x$

$x = \frac{y}{2}$

$3\left(\frac{y}{2}\right)^3 + 4\left(\frac{y}{2}\right)^2 - 2\left(\frac{y}{2}\right) + 1 = 0$

$3y^3 + 8y^2 - 8y + 8 = 0$

ii) $x = \alpha, \beta, \gamma$

$y = \alpha^2, \beta^2, \gamma^2$

$y = x^2$

$x = \sqrt{y}$

$3(\sqrt{y})^3 + 4(\sqrt{y})^2 - 2\sqrt{y} + 1 = 0$

$3y\sqrt{y} + 4y - 2\sqrt{y} + 1 = 0$

$\sqrt{y}(3y-2) = -(4y+1)$

$y(9y^2 - 12y + 4) = 16y^2 + 8y + 1$

$9y^3 - 28y^2 - 4y - 1 = 0$

iii) $\alpha\beta\gamma = -d/a$

$= -1/3$

$x = \alpha, \beta, \gamma$

$y = -1/3\alpha, -1/3\beta, -1/3\gamma$

$y = -1/3x$

$x = -1/3y$

$3\left(-1/3y\right)^3 + 4\left(-1/3y\right)^2 - 2\left(-1/3y\right) + 1 = 0$

$-1/9y^3 + 4/9y^2 + 2/3y + 1 = 0$

$-1 + 4y + 6y^2 + 9y^3 = 0$

Question 4.

a) i) $\Sigma\alpha = -b/a$

$= -3$

ii) $\Sigma\alpha^2 = (\Sigma\alpha)^2 - 2\Sigma\alpha\beta$

$= 9 - 2\times -6$

$= 21$

iii) $P(\alpha) = \alpha^3 + 3\alpha^2 - 6\alpha + 1 = 0$ ①

$P(\beta) = \beta^3 + 3\beta^2 - 6\beta + 1 = 0$ ②

$P(\gamma) = \gamma^3 + 3\gamma^2 - 6\gamma + 1 = 0$ ③

① + ② + ③ :

$\Sigma\alpha^3 + 3\Sigma\alpha^2 - 6\Sigma\alpha + 3 = 0$

$\Sigma\alpha^3 = -3\Sigma\alpha^2 + 6\Sigma\alpha - 3$

$= -3\times 21 + 6\times -3 - 3$

$= -84$

iv) $\Sigma\alpha^2\beta^2 = (\Sigma\alpha\beta)^2 - 2\alpha\beta\gamma(\Sigma\alpha)$

$= (-6)^2 - 2\times -1\times -3$

$= 36 - 6$

$= 30$

b) $x^3 + ax^2 + bx + 15 = 0$

$x^3 + ax^2 + bx + 15 = (x^2 - 4x + 5)(x + c)$

equating constants :

$15 = 5c$

$c = 3$

$\therefore$ other roots are $x = -3,\ 2-i$.

coeff of x^2: $a = 3 - 4$

$= -1$

coeff of x: $b = -4c + 5$

$= -7$

c) The double root of $P(x)$ is a single root of $P'(x)$

$P(x) = 2x^3 + 3x^2 - 72x + a$

$P'(x) = 6x^2 + 6x - 72$

$6x^2 + 6x - 72 = 0$

$x^2 + x - 12 = 0$

$(x+4)(x-3) = 0$

$\therefore x = -4,\ 3$

$\therefore$ Possible double roots are $x = -4$ or $x = 3$.

If $x = -4$

$2(-4)^3 + 3(-4)^2 - 72(-4) + a = 0$

$208 + a = 0$

$a = -208$

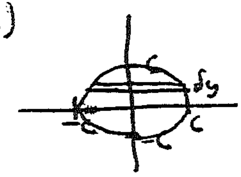
If $x = 3$

$2(3)^3 + 3(3)^2 - 72(3) + a = 0$

$-135 + a = 0$

$a = 135$

(a)



$V_{slice} = Ah$

$= \frac{1}{2} \times 2x \times 2x \times \sin 60° \, \delta y$

$= 2x^2 \times \frac{\sqrt{3}}{2} \, \delta y$

$= \sqrt{3} x^2 \, \delta y$

$V_{solid} = \lim_{\delta y \to 0} \sum_{y=-6}^{6} \sqrt{3} x^2 \, \delta y$

$= 2\sqrt{3} \int_0^6 x^2 \, dy$

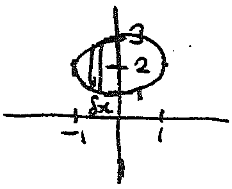
$= 2\sqrt{3} \int_0^6 (36 - y^2) \, dy$

$= 2\sqrt{3} \left[36y - \frac{y^3}{3} \right]_0^6$

$= 2\sqrt{3} \left(36 \times 6 - \frac{6^3}{3} \right)$

$= 288\sqrt{3} \ u^3$

(b)



$V_{slice} = \pi (R^2 - r^2) h$

$= \pi (y_2^2 - y_1^2) \delta x$

$V_{solid} = \lim_{\delta x \to 0} \sum_{y=-1}^{1} \pi (y_2^2 - y_1^2) \delta x$

$= \pi \int_{-1}^{1} (y_2^2 - y_1^2) \, dx$

$= \pi \int_{-1}^{1} \left\{ (2 + \sqrt{1-x^2})^2 - (2 - \sqrt{1-x^2})^2 \right\} dx$

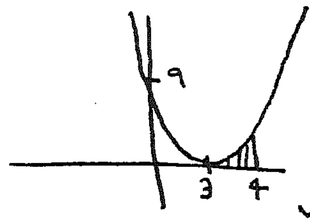
$= \pi \int_{-1}^{1} 4 \times 2\sqrt{1-x^2} \, dx$

$= 16\pi \int_0^1 \sqrt{1-x^2} \, dx$

$= 16\pi \times \frac{1}{4} \times \pi \times 1^2$

$= 4\pi^2 \ u^3$

(c) (i)



(ii) $V_{shell} = \pi (R^2 - r^2) h$

$= \pi \left\{ (x + \delta x)^2 - x^2 \right\} y$

$= 2\pi xy \, \delta x$

$V_{solid} = \lim_{\delta x \to 0} \sum_{x=3}^{4} 2\pi xy \, \delta x$

$= 2\pi \int_3^4 xy \, dx$

$= 2\pi \int_3^4 x (x-3)^2 \, dx$

$= 2\pi \int_3^4 (x^3 - 6x^2 + 9x) \, dx$

$= 2\pi \left[\frac{x^4}{4} - 2x^3 + \frac{9x^2}{2} \right]_3^4$

$= 2\pi \left\{ \left(\frac{4^4}{4} - 2 \times 4^3 + \frac{9 \times 4^2}{2} \right) - \left(\frac{3^4}{4} - 2 \times 3^3 + \frac{9 \times 3^2}{2} \right) \right\}$

$= \frac{5\pi}{2} \ u^3$

6 (a) (i) $b = mx + c$

$x = 0,\ b = a$

$\therefore c = a$

$x = 4a,\ b = 2a$

$2a = 4am + a$

$a = 4am$

$m = \frac{1}{4}$

$b = \frac{x}{4} + a$

$V_{slice} = Ah$

$= \frac{1}{2}\left(\frac{x}{4}+a\right)^2 \delta x$

$= \left(\frac{x^2}{32} + \frac{ax}{4} + \frac{a^2}{2}\right)\delta x$

$= \frac{1}{32}\left(x^2 + 8ax + 16a^2\right)\delta x$

(ii) $V = \frac{1}{32}\int_0^{4a} (x^2 + 8ax + 16a^2)\, dx$

$= \frac{1}{32}\left[\frac{x^3}{3} + 4ax^2 + 16a^2x\right]_0^{4a}$

$= \frac{1}{32}\left(\frac{64a^3}{3} + 64a^3 + 64a^3\right)$

$= \frac{7 \times 64 a^3}{96}$

$= \frac{14}{3}a^3$

(b) $V_{slice} = \pi(R^2 - r^2)h$

$= \pi\left((5-x)^2 - 1^2\right)\delta y$

$= \pi\left(25 - 10x + x^2 - 1\right)\delta y$

$= \pi\left(24 - 10x + x^2\right)\delta y$

$V_{solid} = \pi(3^2 - 1^2) \times 3 + \sum_{y=3}^{5} \pi(24 - 10x + x^2)\delta y$

$= 24\pi + \pi\int_3^5 (24 - 10x + x^2)\, dy$

$= 24\pi + \pi\int_3^5 (24 - 10(y-1) + (y-1)^2)\, dy$

$= 24\pi + \pi\int_3^5 (24 - 10y + 10 + y^2 - 2y + 1)\, dy$

$= 24\pi + \pi\int_3^5 (y^2 - 12y + 35)\, dy$

$= 24\pi + \pi\left[\frac{y^3}{3} - \frac{12y^2}{2} + 35y\right]_3^5$

$= 24\pi + \pi\left(\frac{5^3}{3} - 6 \times 5^2 + 35 \times 5 - \frac{3^3}{3} + 6 \times 3^2 - 35 \times 3\right)$

$= \frac{92\pi}{3}\ u^3$

(c) (i) Let $u = \cos^{n-1}x \qquad v' = \cos x$

$u' = (n-1)\cos^{n-2}x \times -\sin x \qquad v = \sin x$

$I_n = \left[\cos^{n-1}x \cdot \sin x\right]_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} (n-1)\cos^{n-2}x \sin^2 x\, dx$

$= 0 + (n-1)\int_0^{\frac{\pi}{2}} \cos^{n-2}x(1 - \cos^2 x)\, dx$

$= 0 + (n-1)\left\{\int_0^{\frac{\pi}{2}} \cos^{n-2}x\, dx - \int_0^{\frac{\pi}{2}} \cos^n x\, dx\right\}$

$= (n-1)(I_{n-2} - I_n)$

(ii) $\int_0^{\frac{\pi}{2}} \cos^5 x\, dx = I_5$

$I_5 = 4(I_3 - I_5)$

$I_3 = 2(I_1 - I_3)$

$3I_3 = 2I_1$

$I_3 = \frac{2}{3}I_1$

$I_1 = \int_0^{\frac{\pi}{2}} \cos x\, dx$

$= \left[\sin x\right]_0^{\frac{\pi}{2}}$

$= \sin\frac{\pi}{2} - \sin 0$

$= 1$

$I_3 = \frac{2}{3} \times 1 = \frac{2}{3}$

$5I_5 = 4I_3$

$I_5 = \frac{4}{5}I_3$

$= \frac{4}{5} \times \frac{2}{3}$

$= \frac{8}{15}$