Diagnostic Topic Tests:
NSW Mathematics Extension 1 (Preliminary & HSC)

# Test 4: Inverse Functions

Total 40 marks (Suggested time: 45 minutes)

**Directions to students**

- Attempt ALL questions.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- The marks for each question are indicated at the start of the question.

**QUESTION 1.** (10 marks) — Marks

(a) For each of the following functions, find its inverse and sketch the function and its inverse on the same number plane. Clearly label each graph. **8**

(i) $f(x) = \dfrac{5-3x}{2}$

(ii) $g(x) = \sqrt{2x-3}$

(b) Evaluate $\cos^{-1}\left(\dfrac{1}{2}\right) + \tan^{-1}(-1)$ without using the calculator. **2**

**QUESTION 2.** (10 marks)

(a) If $f(x) = x^2 \sin^{-1}x$, show that $f'\left(\dfrac{1}{2}\right) = \dfrac{1}{6}(\pi + \sqrt{3})$. **4**

(b) Consider the function $y = 3\sin^{-1}(2x)$. **6**

(i) State the domain and range.

(ii) Sketch the graph, showing the important features.

(iii) Find $\dfrac{dy}{dx}$.

(iv) State the values of $x$ for which $\dfrac{dy}{dx}$ is defined.





**QUESTION 3.** (10 marks) — Marks

(a) Show that $\displaystyle\int_0^{\frac{3}{2}} \frac{dt}{\sqrt{9-2t^2}} = \frac{\pi}{4\sqrt{2}}$. **3**

(b) The region bounded by the curve $y = \dfrac{1}{\sqrt{9+x^2}}$, the lines $x = -3$ and $x = 3$ and the $x$-axis is rotated about the $x$-axis. Find the volume of the solid formed. **4**

(c) Without using a calculator, show that $\tan^{-1}\left(\dfrac{2}{3}\right) + \tan^{-1}\left(\dfrac{1}{5}\right) = \dfrac{\pi}{4}$. **3**

**QUESTION 4.** (10 marks)

(a) Consider the function $y = \cos^{-1}(2x-1)$. **3**

(i) State the domain and range.

(ii) Sketch the curve.

(b) Find the exact value of $\sin\left(2\cos^{-1}\dfrac{15}{17}\right)$. **3**

(c) Consider the functions $y = -\cos^{-1}x$ and $y = 2\tan^{-1}(x-1)$. **4**

(i) Show that the graphs of these functions intersect on the $y$-axis.

(ii) Show that the graphs have a common tangent at this point of intersection.

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# Test 4: Inverse Functions

## Suggested Solutions

### QUESTION 1.

(a) (i) Let $y = \dfrac{5-3x}{2}$ — Note: Always write in the form $y = f(x)$

Inverse is $x = \dfrac{5-3y}{2}$ — Note: Interchange $x$ and $y$.

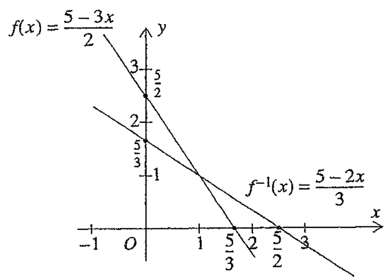
$2x = 5 - 3y$

$3y = 5 - 2x$

$y = \dfrac{5-2x}{3}$ — Note: Solve for $y$.

$f^{-1}(x) = \dfrac{5-2x}{3}$ — Note: Replace $y$ by $f^{-1}(x)$.

Note: The domain and range of both $f$ and $f^{-1}$ consist of all real numbers.




4

(ii) Let $y = \sqrt{2x-3}$, $x \geq \frac{3}{2}$, $y \geq 0$. — Note: Always write in the form $y = f(x)$

Inverse is $x = \sqrt{2y-3}$, $y \geq \frac{3}{2}$, $x \geq 0$ — Note: Interchange $x$ and $y$.

$x^2 = 2y - 3$

$2y = x^2 + 3$

$y = \dfrac{x^2+3}{2}$

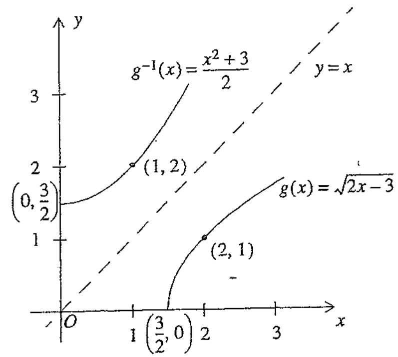
$g^{-1}(x) = \dfrac{x^2+3}{2}$ for $x \geq 0$.

Note: Replace $y$ by $g^{-1}(x)$

Note: In the original function $y \geq 0$. Therefore $x \geq 0$ in the inverse function.








4

Note: The graph of $y = g^{-1}(x)$ is the reflection of the graph of $y = g(x)$ in the line $y = x$.

Note: The domain and range for both the function and its inverse have restrictions as shown in the diagram.

(b) $\cos^{-1}\left(\frac{1}{2}\right) + \tan^{-1}(-1) = \frac{\pi}{3} + \left(-\frac{\pi}{4}\right)$

$= \frac{\pi}{12}$

2

Note: $\tan^{-1}(-x) = -\tan^{-1}x$

### QUESTION 2.

(a) $f(x) = x^2 \sin^{-1} x$

$f'(x) = 2x \times \sin^{-1} x + \dfrac{1}{\sqrt{1-x^2}} \times x^2$ — Note: Using product rule.

$= 2x \sin^{-1} x + \dfrac{x^2}{\sqrt{1-x^2}}$

$f'\left(\frac{1}{2}\right) = 2 \times \frac{1}{2} \times \sin^{-1}\frac{1}{2} + \dfrac{\left(\frac{1}{2}\right)^2}{\sqrt{1-\left(\frac{1}{2}\right)^2}}$

$= \dfrac{\pi}{6} + \dfrac{\frac{1}{4}}{\sqrt{\frac{3}{4}}}$

$= \dfrac{\pi}{6} + \dfrac{\frac{1}{4}}{\frac{\sqrt{3}}{2}}$

$= \dfrac{\pi}{6} + \dfrac{1}{2\sqrt{3}}$

$= \dfrac{\pi}{6} + \dfrac{\sqrt{3}}{6}$ — Note: Rationalise the denominator.

$= \frac{1}{6}(\pi + \sqrt{3})$

4

(b) $y = 3\sin^{-1}(2x)$

(i) Domain: $-1 \le 2x \le 1$

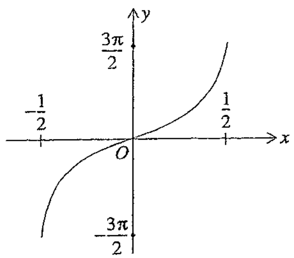
i.e. $-\frac{1}{2} \le x \le \frac{1}{2}$

Range: $-\frac{\pi}{2} \le \frac{y}{3} \le \frac{\pi}{2}$

i.e. $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$ **2**

Note: For $y = \sin^{-1}x$, domain is $-1 \le x \le 1$, range is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

(ii)




**2**

(iii) $\frac{dy}{dx} = 3 \times \frac{1}{\sqrt{1-4x^2}} \times 2$

$= \frac{6}{\sqrt{1-4x^2}}$ **1**

(iv) $\frac{dy}{dx}$ is defined for $-\frac{1}{2} < x < \frac{1}{2}$ as the tangents are vertical at $x = -\frac{1}{2}$ and $x = \frac{1}{2}$ and so the derivative does not exist at the endpoints of the domain. **1**

Note: If this is not recognised then it can be shown by solving the inequality $1 - 4x^2 > 0$ giving $-\frac{1}{2} < x < \frac{1}{2}$.





## QUESTION 3.

(a) $\int_0^{\frac{3}{2}} \frac{1}{\sqrt{9-2t^2}}dt = \int_0^{\frac{3}{2}} \frac{1}{\sqrt{2\left(\frac{9}{2}-t^2\right)}}dt$

$= \frac{1}{\sqrt{2}}\int_0^{\frac{3}{2}} \frac{1}{\sqrt{\left(\frac{3}{\sqrt{2}}\right)^2 - t^2}}dt$

$= \frac{1}{\sqrt{2}}\left[\sin^{-1}\left(\frac{t}{\frac{3}{\sqrt{2}}}\right)\right]_0^{\frac{3}{2}}$

$= \frac{1}{\sqrt{2}}\left[\sin^{-1}\frac{t\sqrt{2}}{3}\right]_0^{\frac{3}{2}}$

$= \frac{1}{\sqrt{2}}\left[\sin^{-1}\frac{1}{\sqrt{2}} - \sin^{-1}0\right]$

$= \frac{1}{\sqrt{2}} \times \left(\frac{\pi}{4} - 0\right)$

$= \frac{\pi}{4\sqrt{2}}$ **3**

Note: The formula in the HSC table of standard integrals is:

$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\frac{x}{a}$.

Here $a^2 = \frac{9}{2}$, $a = \frac{3}{\sqrt{2}}$.

(b) $y = \frac{1}{\sqrt{9+x^2}}$, from $x = -3$ to $x = 3$

$V = \pi\int_a^b y^2 dx$

$V = \pi\int_{-3}^{3}\left(\frac{1}{\sqrt{9+x^2}}\right)^2 dx$

$V = 2\pi\int_0^3 \frac{1}{9+x^2}dx$

$V = 2\pi\left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^3$

$V = \frac{2\pi}{3}\left[\tan^{-1}\frac{x}{3}\right]_0^3$

$V = \frac{2\pi}{3}[\tan^{-1}(1) - \tan^{-1}(0)]$

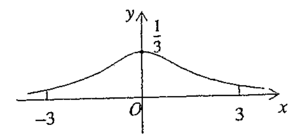
$V = \frac{2\pi}{3}\left[\frac{\pi}{4} - 0\right]$

$V = \frac{\pi^2}{6} \doteq 1.64$

$\therefore$ required volume is $\frac{\pi^2}{6}$ units$^3$.

(or 1.64 units$^3$ correct to two decimal places). **4**

Note: It is often important to draw a diagram.




Note: This is an even function (i.e. $f(x) = f(-x)$) which means the area on both sides of the $y$ axis is the same. We can change the limits as shown.

Note: The formula in the HSC table of standard integrals is:

$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\frac{x}{a}$

Here $a^2 = 9$, $a = 3$

(c) To prove $\tan^{-1}\left(\frac{2}{3}\right)+\tan^{-1}\left(\frac{1}{5}\right)=\frac{\pi}{4}$.

Let $A=\tan^{-1}\frac{2}{3}$ and $B=\tan^{-1}\frac{1}{5}$.

$\therefore\ \tan A=\frac{2}{3}$ and $\tan B=\frac{1}{5}$

$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$

Note: Multiply numerator and denominator by 15.

$$=\frac{\frac{2}{3}+\frac{1}{5}}{1-\frac{2}{3}\times\frac{1}{5}}$$

$$=\frac{10+3}{15-2}$$

$$=1$$

$$A+B=\frac{\pi}{4}$$

Note: Since $A$ and $B$ are acute.

$$\therefore\ \tan^{-1}\left(\frac{2}{3}\right)+\tan^{-1}\left(\frac{1}{5}\right)=\frac{\pi}{4}$$

**3**

**QUESTION 4.**

(a) $y=\cos^{-1}(2x-1)$

(i) Domain: $-1\le 2x-1\le 1$

$0\le 2x\le 2$

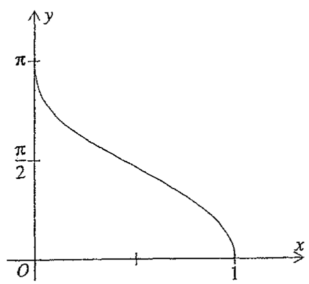
$0\le x\le 1$

Range: $0\le y\le\pi$

**1**

Note: For $f(x)=\cos^{-1}x$, domain is $-1\le x\le 1$, range is $0\le\cos^{-1}x\le\pi$.

(ii)




**2**



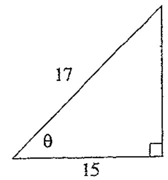


(b) Let $\theta=\cos^{-1}\frac{15}{17}$

$\therefore\cos\theta=\frac{15}{17}$

Using a right-angle triangle and Pythagoras' theorem,

Note: $\theta$ is acute.




$\therefore\ \sin\theta=\frac{8}{17}$

$$\sin2\theta=2\sin\theta\cos\theta$$

$$=2\times\frac{8}{17}\times\frac{15}{17}$$

$$=\frac{240}{289}$$

$$\therefore\ \sin\left(2\cos^{-1}\frac{15}{17}\right)=\frac{240}{289}$$

**3**

(c) $y=-\cos^{-1}x$ ....... (1)

$y=2\tan^{-1}(x-1)$ ... (2)

(i) Substitute $x=0$ into (1): $y=-\cos^{-1}0$

$y=-\frac{\pi}{2}$

Note: All points on the $y$-axis have an $x$-coordinate of 0.

Substitute $x=0$ into (2): $y=2\tan^{-1}(-1)$

$y=2\left(-\frac{\pi}{4}\right)$

$y=-\frac{\pi}{2}$

Hence the curves intersect at the same point $\left(0,-\frac{\pi}{2}\right)$ on the $y$ axis.

**2**

(ii) For $y = -\cos^{-1}x$

$$\frac{dy}{dx} = -\left(\frac{-1}{\sqrt{1-x^2}}\right)$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

When $x = 0$, $\frac{dy}{dx} = \frac{1}{\sqrt{1-0^2}}$

$$= 1$$

For $y = 2\tan^{-1}(x-1)$

$$\frac{dy}{dx} = 2\frac{1}{(x-1)^2+1} \times 1$$

$$\frac{dy}{dx} = \frac{2}{(x-1)^2+1}$$

When $x = 0$, $\frac{dy}{dx} = \frac{2}{(0-1)^2+1} = 1$

$\therefore$ the tangents at $\left(0, -\frac{\pi}{2}\right)$ have the same gradient.

Hence the curves have a common tangent at this point. **2**